T-duality with categorified principal bundles

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T-duality: Motivation

- String theories on backgrounds with $\mathsf{U}(1)$-isometries: exchange of winding/momentum modes $\Rightarrow$ a T-dual partner
- This duality qualitatively separates strings from particles
- Many reasons for studying T-duality
  - Better understanding of strings
  - Higher bundles/gerbes with connection
  - Non-geometric backgrounds
  - Mathematics: relation to Fourier–Mukai transform
  - ...
- But: T-duality begs to be studied in non-trivial topologies

Approximations in the following

- String theories on backgrounds with $\mathsf{U}(1)$-isometries:
- Low-energy limit: corresponding supergravity contains B-field:
  $\Rightarrow$ connective structure on a gerbe

Geometric string background:

- A (Riemannian) manifold X
- A principal/affine torus bundle $\pi : P \to X$ (with connection)
- An abelian gerbe (with connection) $\mathscr{G}$ on the total space of P

Ignore dynamics, i.e. no equations of motion imposed

Topological T-duality

Geometric string background:

- A topological manifold X
- A principal/affine torus bundle $\pi : P \to X$
- An abelian gerbe $\mathscr{G}$ on the total space of P

Topological T-duality from exactness of the Gysin sequence

For example, for principal circle bundle:

$$\ldots \to \mathrm{H}^3(X,\mathbb{Z}) \xrightarrow{\pi^*} \mathrm{H}^3(P,\mathbb{Z}) \xrightarrow{\pi_*} \mathrm{H}^2(X,\mathbb{Z}) \xrightarrow{F\cup} \mathrm{H}^4(X,\mathbb{Z}) \to \ldots$$

- Gerbe over P: 3-form $H \in \mathrm{H}^3(P,\mathbb{Z})$
- Fiber integration $\pi_* H = \hat{F} \in \mathrm{H}^2(X,\mathbb{Z})$ with $F \cup \hat{F} = 0$
- $\Rightarrow$ There is $\hat{H} \in \mathrm{H}^3(\hat{P},\mathbb{Z})$ with $\pi_* \hat{H} = F$.
- Top. T-duality: $(F,H) = (\pi_* \hat{H}, H) \longleftrightarrow (\hat{F},\hat{H}) = (\pi_* H, \hat{H})$
  Note: possibility of topology change!

Bouwknegt, Evslin, Hannabuss, Mathai (2004)

Topological T-duality, geometrically

T-correspondence:

$$\mathscr{G}_{\rm C} = \check{\mathsf{p}}^*\check{\mathscr{G}} \otimes \hat{\mathsf{p}}^*\hat{\mathscr{G}}^{-1} \cong \mathcal{I}$$

$$\begin{array}{ccccccc} & & & \downarrow & & & \\ & & & \check{P} \times_X \hat{P} & & & \\ & & \overset{\check{\mathsf{p}}}{\swarrow} & & \overset{\hat{\mathsf{p}}}{\searrow} & & \\ \check{\mathscr{G}} \longrightarrow & \check{P} & & & & \hat{P} & \longleftarrow \hat{\mathscr{G}} \\ & & \underset{\check{\pi}}{\searrow} & & \underset{\hat{\pi}}{\swarrow} & & \\ & & & X & & & \end{array}$$

Bunke, Rumpf, Schick (2005, 2006)

Principal 2-bundles (without connections) over X:

$$\begin{array}{ccc} & \mathscr{P}_{\rm C} & \\ \overset{\check{\mathsf{p}}}{\swarrow} & & \overset{\hat{\mathsf{p}}}{\searrow} \\ \check{\mathscr{P}} & & \hat{\mathscr{P}} \end{array}$$

Nikolaus, Waldorf (2018)

Two open problems

I. T-duality can lead to non-geometric backgrounds:

F^3: H has no legs along fiber
T-duality: identity

F^2: H has 1 leg along fiber
T-duality $\rightarrow$ geometric string background

F^1: H has 2 legs along fiber
T-duality $\rightarrow$ Q-space, (e.g. T-folds) locally geometric

F^0: H has all legs along fiber
T-duality $\rightarrow$ R-space, non-geometric

Nikolaus/Waldorf cover $F^2 \leftrightarrow F^2$ and $F^2 \leftrightarrow F^1$ T-dualities

What about the general case?

II. Differential refinement of this picture

Why is this interesting/hard?

I. need to use suitable groupoids and augmented groupoids

II. connections on principal 2-bundles often require adjustment

Outline

- Categorified parallel transport
- Adjusted connections on principal 2-bundles
- F^k, $k \geq 2$: Geometric T-duality with principal 2-bundles
- Explicit example throughout: Nilmanifold
- The T-duality group from Kaluza–Klein reduction
- Non-geometric T-dualities: Q-spaces and R-spaces

Principal 2-bundles or Non-Abelian Gerbes

with Adjusted Connections

Parallel transport

Principal bundles define parallel transport.

Parallel transport with gauge group G :

- Assignment $\gamma \mapsto g$ for path γ and group elt. $g \in \mathsf{G}$.
- Gluing paths together leads to multiplication of the group elts.
- holonomy functor for points: $\mathsf{hol}(\gamma) = P\exp(\int_\gamma A) \in \mathsf{G}$
  A: gauge potential, γ: surface, P: path ordering

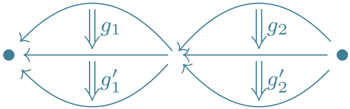
Higher parallel transport

Higher principal bundles define higher parallel transport.

Parallel transport with gauge group G?

- Assignment $\sigma \mapsto g$ for surface σ and group elt. $g \in \mathsf{G}$?
- Gluing surfaces together leads to multiplication of group elts.?
- holonomy functor for surfaces: $\mathsf{hol}(\sigma) = P\exp(\int_\sigma B) \in \mathsf{G}$?

  B: gauge potential, σ: surface, P: does not exist:



  Consistency of parallel transport requires:

$$(g_1' g_2')(g_1 g_2) = (g_1' g_1)(g_2' g_2)$$

  This renders group G abelian. Eckmann and Hilton, 1962

- Way out: higher categories, categorification:

$$(g_1' \otimes g_2') \circ (g_1 \otimes g_2) = (g_1' \circ g_1) \otimes (g_2' \circ g_2)\ .$$

A mathematical structure ("Bourbaki-style") consists of

- Sets
- Structure Functions
- Structure Equations

"Categorification":

Sets → Categories

Structure Functions → Structure Functors

Structure Equations → Structure Isomorphisms

Example: Group → 2-Group

- Set G → Category $\mathscr{G}$
- product, identity ($\mathbb{1} : * \to \mathsf{G}$), inverse → Functors
- $a(bc) = (ab)c$ → Associator $\mathsf{a} : a \otimes (b \otimes c) \Rightarrow (a \otimes b) \otimes c$
- $\mathbb{1}a = a\mathbb{1} = a$ → Unitors $\mathsf{l}_a : a \otimes \mathbb{1} \Rightarrow a$, $\mathsf{r}_a : \mathbb{1} \otimes a \Rightarrow a$
- $aa^{-1} = a^{-1}a = \mathbb{1}$ → weak inv. $\mathsf{inv}(x) \otimes x \Rightarrow \mathbb{1} \Leftarrow x \otimes \mathsf{inv}(x)$

Note: Process not unique, variants: weak/strict/...

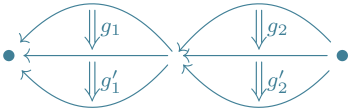
Categorification: Higher dimensional algebra

Higher groups: we are doing higher dimensional algebra.

- In a group, we can multiply ordered elements in one dimension:

$$a \cdot b \cdot \ldots \cdot d$$

- In a 2-group, we can multiply "vertically" and "horizontally", i.e. in two dimensions:



$\vdots$

- In an n-group, we can multiply in n dimensions

Example: The Lie 2-group $\underline{\mathsf{TD}}_n$

Lie 2-group:

- Strict monoidal category
- Vertical product: $\circ$, composition of morphisms
- Horizontal product: $\otimes$

$\underline{\mathsf{TD}}_n$:

$$\mathbb{R}^{2n} \times \mathbb{Z}^{2n} \times \mathsf{U}(1) \rightrightarrows \mathbb{R}^{2n}$$

$$\xi \xleftarrow{(\xi, m_1, \phi_1)} \xi - m_1 \xleftarrow{(\xi - m_1, m_2, \phi_2)} \xi - m_1 - m_2$$

$$\xi \xleftarrow{(\xi, m_1 + m_2, \phi_1 + \phi_2)} \xi - m_1 - m_2$$

$$\mathsf{id}_\xi \coloneqq (\xi, 0, 0)~, \quad (\xi, m, \phi)^{-1} \coloneqq (\xi - m, -m, -\phi)$$

$$(\xi_1, m_1, \phi_1) \otimes (\xi_2, m_2, \phi_2) \coloneqq (\xi_1 + \xi_2, m_1 + m_2, \phi_1 + \phi_2 - \langle \xi_1, m_2 \rangle)$$

$$\mathsf{inv}(\xi, m, \phi) \coloneqq (-\xi, -m, -\phi - \langle \xi, m \rangle)$$

Example: The Lie 2-group $\underline{\mathsf{TD}}_n$

Lie 2-groups are equivalently crossed modules of Lie groups:

- Pair of Lie groups (G, H)
- Group homomorphism $\mathsf{t} : \mathsf{H} \to \mathsf{G}$
- Action $\mathsf{G} \curvearrowright \mathsf{H}$ by automorphisms.

TD_n:

$$\mathsf{TD}_n \;:=\; \big(\mathbb{Z}^{2n} \times \mathsf{U}(1) \xrightarrow{\mathsf{t}} \mathbb{R}^{2n}\big)$$
$$\mathsf{t}(m, \phi) := m$$
$$\xi \triangleright (m, \phi) := (m, \phi - \langle \xi, m \rangle)$$

Principal 2-bundles, topologically

Essentially, all definitions of principal bundles have higher version.

Here: Čech cocycle description subordinate to a cover.

- Surjective submersion $\sigma : Y \twoheadrightarrow X$, e.g. $Y = \sqcup_a U_a$
- Čech groupoid:

$$\check{\mathscr{C}}(\sigma) \;:\; Y \times_X Y \rightrightarrows Y \;, \quad (y_1, y_2) \circ (y_2, y_3) = (y_1, y_3) \;.$$

- Principal G-bundle:
  Transition functions are functor $g : \check{\mathscr{C}}(\sigma) \to (\mathsf{G} \rightrightarrows *)$

$$\begin{array}{ccc} Y \times_X Y & \xrightarrow{\;g\;} & \mathsf{G} \\ \downarrow\downarrow & & \downarrow\downarrow \\ Y & \xrightarrow{\;*\;} & * \end{array} \qquad g(y_1, y_2) g(y_2, y_3) = g(y_1, y_3)$$

  Equivalences/bundle isomorphisms: natural isomorphisms.

Principal 2-bundles, topologically

- Principal G-bundle:
  Trans. fncs.: weak 2-functors $g : \check{\mathscr{C}}(\sigma) \to (\mathsf{G} \ltimes \mathsf{H} \rightrightarrows \mathsf{G} \rightrightarrows *)$

$$
\begin{array}{ccc}
Y \times_X Y & \xrightarrow{(g,h)} & \mathsf{G} \ltimes \mathsf{H} \\
\downarrow\downarrow\downarrow & & \downarrow\downarrow\downarrow \\
Y \times_X Y & \xrightarrow{g} & \mathsf{G} \\
\downarrow\downarrow & & \downarrow\downarrow \\
Y & \xrightarrow{*} & *
\end{array}
$$

- Special case: $\mathsf{H} = \mathsf{U}(1)$, $\mathsf{G} = *$: abelian gerbes.
- Similarly: groupoid bundles, 2-groupoid bundles, ..., n-groupoid bundles.

Connections on principal 2-bundles: work a bit more...

Breen, Messing (2005), Aschieri, Cantini, Jurčo (2005)

Data obtained for 2-group $\mathsf{G} \ltimes \mathsf{H} \rightrightarrows \mathsf{G}$ and Lie 2-algebra $\mathfrak{g} \ltimes \mathfrak{h} \rightrightarrows \mathfrak{g}$:

$$h \in \Omega^0(Y^{[3]}, \mathsf{H}) \quad \Lambda \in \Omega^1(Y^{[2]}, \mathfrak{h}) \quad B \in \Omega^2(Y, \mathfrak{h}) \quad \delta \in \Omega^2(Y^{[2]}, \mathfrak{h})$$

$$g \in \Omega^0(Y^{[2]}, \mathsf{G}) \quad A \in \Omega^1(Y, \mathfrak{g})$$

- Note that δ sticks out unnaturally.
- It was dropped in most later work (Baez, Schreiber, ...)
- Price to pay: part of curvature must vanish
- Otherwise, problems with composition of gauge transformations

Principal 2-Bundles

Object	Principal G-bundle	Principal $(\mathsf{H} \xrightarrow{\mathsf{t}} \mathsf{G})$-bundle
Cochains	(g_{ab}) valued in G	(g_{ab}) valued in G, (h_{abc}) valued in H
Cocycle	$g_{ab}g_{bc} = g_{ac}$	$\mathsf{t}(h_{abc})g_{ab}g_{bc} = g_{ac}$ $h_{acd}h_{abc} = h_{abd}(g_{ab} \rhd h_{bcd})$
Coboundary	$g_a g'_{ab} = g_{ab}g_b$	$g_a g'_{ab} = \mathsf{t}(h_{ab})g_{ab}g_b$ $h_{ac}h_{abc} = (g_a \rhd h'_{abc})h_{ab}(g_{ab} \rhd h_{bc})$
gauge pot.	$A_a \in \Omega^1(U_a) \otimes \mathfrak{g}$	$A_a \in \Omega^1(U_a) \otimes \mathfrak{g}$, $B_a \in \Omega^2(U_a) \otimes \mathfrak{h}$
Curvature	$F_a = \mathrm{d}A_a + \frac{1}{2}[A_a, A_a]$	$\mathcal{F}_a = \mathrm{d}A_a + \frac{1}{2}[A_a, A_a] - \mathsf{t}(B_a) \stackrel{!}{=} 0$ $H_a = \mathrm{d}B_a + A_a \rhd B_a$
Gauge trafos	$\tilde{A}_a := g_a^{-1}A_a g_a + g_a^{-1}\mathrm{d}g_a$	$\tilde{A}_a := g_a^{-1}A_a g_a + g_a^{-1}\mathrm{d}g_a + \mathsf{t}(\Lambda_a)$ $\tilde{B}_a := g_a^{-1} \rhd B_a + \tilde{A}_a \rhd \Lambda_a + \mathrm{d}\Lambda_a - \Lambda_a \wedge \Lambda_a$

Remarks:

- A principal $(1 \xrightarrow{\mathsf{t}} \mathsf{G})$-bundle is a principal G-bundle.
- A principal $(\mathsf{U}(1) \xrightarrow{\mathsf{t}} 1) = \mathsf{BU}(1)$-bundle is an abelian gerbe.

Why should the fake curvature(s) vanish?

$$\mathcal{F} := \mathrm{d}A + \tfrac{1}{2}[A,A] + \mathsf{t}(B) \stackrel{!}{=} 0$$

Without this condition:

- Closure of gauge transformations generically requires $\mathcal{F} = 0$
- Composition of cocycles generically requires $\mathcal{F} = 0$
- Higher parallel transport is not reparameterization invariant
- 6d Self-duality equation $H = \star H$ is not gauge-covariant:

$$H \rightarrow \tilde{H} = g \rhd H - \mathcal{F} \rhd \Lambda$$

With this condition:

- Principal $(1 \xrightarrow{\mathsf{t}} \mathsf{G})$-bundle is flat principal G-bundle.
- Higher connections are locally abelian!

Gastel (2019), CS, Schmidt (2020)

Solution: Adjustment

Many (not all!) higher gauge groups come with

Adjustment of higher group $\mathcal{G}$:

CS, Schmidt (2020), Rist, CS, Wolf (2022)

- Additional map $\kappa : \mathcal{G} \times \mathsf{Lie}(\mathcal{G}) \to \mathsf{Lie}(\mathcal{G})$ + condition
- Necessary for consistent definition of invariant polynomials.
- From Alternator ($\Rightarrow EL_\infty$-algebras, Borsten, Kim, CS (2021))

For connections on principal $\mathcal{G}$-bundles:

- specifies $\delta \in \Omega^2(Y^{[2]}, \mathfrak{h})$ in terms of A and F
- Adjustment of curvature/cocycle/coboundary relations
- Can drop fake flatness condition

Physics example: Heterotic supergravity

Archetypal example: string Lie 2-algebra

$$\mathfrak{string}(n) = \mathbb{R}[1] \to \mathfrak{spin}(n)$$

$$\mu_2(x_1, x_2) = [x_1, x_2] \ , \quad \mu_3(x_1, x_2, x_3) = (x_1, [x_2, x_3])$$

Gauge potentials:

$$(A, B) \ \in \ \Omega^1(U) \otimes \mathfrak{spin}(n) \ \oplus \ \Omega^2(U)$$

Curvatures:

$$\begin{aligned} F &:= \mathrm{d}A + \tfrac{1}{2}[A, A] \\ H &:= \mathrm{d}B - \tfrac{1}{3!}(A, [A, A]) + (A, F) \\ &= \mathrm{d}B + \underbrace{(A, \mathrm{d}A) + \tfrac{1}{3}(A, [A, A])}_{\mathrm{cs}(A)} \end{aligned}$$

Bianchi identities:

$$\mathrm{d}F + [A, F] = 0 \ , \quad \mathrm{d}H - (F, F) = 0$$

Geometric T-duality

Geometric T-duality: General Picture

$$\check{\mathscr{P}} \xleftarrow{\ \check{\mathsf{p}}\ } \mathscr{P}_{\mathrm{C}} \xrightarrow{\ \hat{\mathsf{p}}\ } \hat{\mathscr{P}}$$

- Nikolaus/Waldorf: Topological part:
  - Gerbe and circle fibration combined into 2-bundles $\check{\mathscr{P}}$ and $\hat{\mathscr{P}}$
  - $\check{\mathscr{P}}$ and $\hat{\mathscr{P}}$ are principal $\mathsf{TB}_n^{\mathsf{F2}}$-bundles
  - $\mathscr{P}_C$ is a principal TD_n-bundle
  - $\check{\mathsf{p}}$ is a projection induced by strict morphism $\hat{\phi} : \mathsf{TD}_n \to \mathsf{TB}_n^{\mathsf{F2}}$
  - $\hat{\mathsf{p}}$ induced by $\check{\phi} = \hat{\phi} \circ \phi_{\mathsf{flip}}$, flip morphism $\phi_{\mathsf{flip}} : \mathsf{TD}_n \to \mathsf{TD}_n$

Geometric T-duality: General Picture

$$\check{\mathscr{P}} \xleftarrow{\ \check{\mathsf{p}}\ } \mathscr{P}_{\mathrm{C}} \xrightarrow{\ \hat{\mathsf{p}}\ } \hat{\mathscr{P}}$$

- Nikolaus/Waldorf: Topological part:
  - $\check{\mathscr{P}}$ and $\hat{\mathscr{P}}$ are principal $\mathsf{TB}_n^{\mathsf{F2}}$-bundles
  - $\mathscr{P}_C$ is a principal TD_n-bundle
- Differential refinement: (i.e. B-field+metric) Kim, CS (2022)
  - $\mathsf{TB}_n^{\mathsf{F2}}$ does not come with adjustment, but
  - TD_n comes with very natural adjustment map
  - Have topological and full connection data on $\mathscr{P}_C$
  - Can reconstruct gerbe and bundle data on $\check{\mathscr{P}}$ and $\hat{\mathscr{P}}$
- Reproduces Buscher rules Waldorf (2022)
- Generalization to affine torus bundles: use $\mathsf{GL}(n,\mathbb{Z}) \ltimes \mathsf{TD}_n$

Example: 3d Nilmanifolds

Geometry of string background $\check{\mathscr{G}}_\ell \to N_k$:

- Principal circle bundle over T^2 with $c_1 = k$
- Subordinate to $\mathbb{R}^2 \to T^2$ and with $\mathsf{U}(1) \cong \mathbb{R}/\mathbb{Z}$

$$(x,y,z) \sim (x,y+1,z) \sim (x,y,z+1) \sim (x+1,y,z-ky)$$

- Local connection form: $A(x,y) = kx\,\mathrm{d}y \in \Omega^1(\mathbb{R}^2)$
- Kaluza–Klein metric: $g(x,y,z) = \mathrm{d}x^2 + \mathrm{d}y^2 + (\mathrm{d}z + kx\,\mathrm{d}y)^2$
- Gerbes on N_k characterized by element of $H^3(N_k,\mathbb{Z}) \cong \mathbb{Z}$

T-duality:

$$(\check{\mathscr{G}}_\ell \to N_k) \quad \longleftrightarrow \quad (\hat{\mathscr{G}}_k \to N_\ell)$$

Explicit T-duality example with principal 2-bundles

Kim, CS (2022)

$$\check{\mathscr{P}} \xleftarrow{\ \check{\mathsf{p}}\ } \mathscr{P}_{\mathrm{C}} \xrightarrow{\ \hat{\mathsf{p}}\ } \hat{\mathscr{P}}$$

Lie 2-group:

$$\mathsf{TD}_1 \;:=\; \big(\mathbb{Z}^2 \times \mathsf{U}(1) \xrightarrow{\ \mathsf{t}\ } \mathbb{R}^2\big)$$

Topological cocycle data:

$$g = \begin{pmatrix} \hat{\xi} \\ \check{\xi} \end{pmatrix}~, \quad \begin{aligned} \hat{\xi}(x,y;x',y') &= \ell(x'-x)y~, \\ \check{\xi}(x,y;x',y') &= k(x'-x)y~, \end{aligned}$$

$$h = \begin{pmatrix} \hat{m} \\ \check{m} \\ \phi \end{pmatrix}~, \quad \begin{aligned} \hat{m}(x,y;x',y';x'',y'') &= -\ell(x''-x')(y'-y) \\ \check{m}(x,y;x',y';x'',y'') &= -k(x''-x')(y'-y) \\ \phi &= \tfrac{1}{2}k\ell\big(y'(xx''-xx'-x'x'')-(x''-x')(y'^2-y^2)x\big) \end{aligned}$$

Cocycle data of differential refinement:

$$A = \begin{pmatrix} \check{A} \\ \hat{A} \end{pmatrix} = \begin{pmatrix} kx\,\mathrm{d}y \\ \ell x\,\mathrm{d}y \end{pmatrix}~, \quad B = 0~, \quad \Lambda = \tfrac{1}{2}k\ell(xx'\,\mathrm{d}y + (xy + x'y' + y^2(x'-x))\,\mathrm{d}x)$$

Can reconstruct both string backgrounds fully.

The T-duality group from Kaluza–Klein Reduction

The group TD_n from Kaluza–Klein reduction

Observation:

T-duality is intimately linked to Kaluza–Klein reduction:

- Gysin sequence contains fiber integration
- Metric on total space given by Kaluza–Klein metric
- Literature: e.g. Berman (2019), Alfonsi (2019), ...

- Geometric objects from maps into classifying spaces $\mathcal{C}$.
- Note: currying $C^0(X \times T^n, \mathcal{C}) \cong C^0(X, C^0(T^n, \mathcal{C}))$
- Non-trivial fibrations: cyclic torus space: $C^0(T^n, \mathcal{C})//\mathsf{U}(1)^n$
  cf. Fiorenza, Sati, Schreiber (2016a,2016b)
- Kaluza–Klein reduction:
  - Principal G-bundle over circle fibration $P \to X$
  - Classifying space BG
  - Cyclic loop space $L\mathsf{BG}//\mathsf{U}(1) \cong \mathsf{BH}$
  - Work with principal H-bundles over X

The 2-group TD_n from Kaluza–Klein reduction

Abstract nonsense: KK-reduction along circle fibers:

- $\mathsf{BBU}(1) \quad \to \quad L\mathsf{BBU}(1)/\!/\mathsf{U}(1) \cong \mathsf{B}(\mathsf{BU}(1) \times \mathsf{U}(1) \times \mathsf{U}(1))$
- $\mathsf{BU}(1) \quad \to \quad L\mathsf{BU}(1)/\!/\mathsf{U}(1) \cong \mathsf{BU}(1) \times \mathsf{U}(1) \times \mathsf{BU}(1)$

TD_1 from KK-reduction of gerbe on circle bundle

- Gerbe: $C^0(P, \mathcal{C})$ with $\mathcal{C} = \mathsf{BBU}(1) \sim (\mathsf{U}(1) \rightrightarrows * \rightrightarrows *)$
- Replace $\mathsf{U}(1)$ with $\mathbb{Z} \to \mathbb{R}$: $\mathsf{TD}_1 \ := \ \big(\mathsf{U}(1) \times \mathbb{Z}^2 \xrightarrow{\mathsf{t}} \mathbb{R}^2\big)$

TD_2 from KK-reduction of principal TD_1-bundle on circle bundle

- Principal 2-bundle: $C^0(P, \mathcal{C})$ with $\mathcal{C} = \mathsf{BTD}_1$
- Replace $\mathsf{U}(1)$ with $\mathbb{Z} \to \mathbb{R}$: $\mathsf{TD}_2 \ := \ \big(\mathsf{U}(1) \times \mathbb{Z}^4 \xrightarrow{\mathsf{t}} \mathbb{R}^4\big)$
- Here, we dropped parts, we actually get a 2-groupoid: $\mathscr{TD}_2 \cong \mathsf{BBU}(1) \times \mathsf{BU}(1)^{\times 4} \times \mathsf{U}(1)^{\times 4}$
- Clear that g, B dim reduced on T^2 yield four scalar modes.

Iterate: $\mathsf{TD}_n \ := \ \big(\mathsf{U}(1) \times \mathbb{Z}^{2n} \xrightarrow{\mathsf{t}} \mathbb{R}^{2n}\big)$ and $\mathscr{TD}_n$.

Automorphisms of TD_n

Abstract nonsense:

- Natural definition of morphism of 2-groups
- Automorphisms of 2-group form naturally a 2-group
- 2-group action $\mathscr{G} \curvearrowright \mathscr{H}$: morphism $\mathscr{G} \to \mathsf{Aut}(\mathscr{H})$

Automorphisms of the 2-group TD_n:

- Can be computed to be weak (unital) Lie 2-group

$$\mathscr{GO}(n,n;\mathbb{Z}) := \Big(\mathsf{GO}(n,n;\mathbb{Z}) \times \mathbb{Z}^{2n} \Longrightarrow \mathsf{GO}(n,n;\mathbb{Z}) \Big)$$

see also Waldorf (2022)

- While $\mathsf{GO}(n,n;\mathbb{Z})$ does not act on TD_n, $\mathscr{GO}(n,n;\mathbb{Z})$ does.
- Recover T-duality group for affine torus bundles
- Explicit: geometric subgroup, B- and β-trafos, T-dualities as endo-2-functors on TD_n
- $\Rightarrow$ arrange everything based on $\mathscr{GO}(n,n;\mathbb{Z})$

Non-geometric T-dualities: Q-spaces and R-spaces

The 2-groupoid $\mathscr{TD}_n$

- Two T-dualities yield scalars from metric and 2-form.
- Scalars live on the Narain moduli space for affine torus bundles:

$$\begin{aligned} GM_n &= \mathsf{GO}(n,n;\mathbb{Z}) \setminus \mathsf{O}(n,n;\mathbb{R}) \,/\, \big(\mathsf{O}(n;\mathbb{R}) \times \mathsf{O}(n;\mathbb{R})\big) \\ &=: \mathsf{GO}(n,n;\mathbb{Z}) \setminus Q_n \end{aligned}$$

- Note: $Q_n \cong \mathbb{R}^{n^2}$ is a nice space
- Resolve into action groupoid:

$$\mathsf{GO}(n,n;\mathbb{Z}) \ltimes Q_n \;\rightrightarrows\; Q_n$$

- Extend to $\mathscr{GO}(n,n;\mathbb{Z})$-action ($\mathscr{GO}(n,n;\mathbb{Z}) \cong \mathsf{Aut}(\mathsf{TD}_n)$)
- Place TD_n-fiber over every point in Q_n
- Include action of $\mathscr{GO}(n,n;\mathbb{Z})$ on TD_n
- The result is the Lie 2-groupoid $\mathscr{TD}_n$

T-duality as $\mathscr{T}\mathscr{D}_n$-bundles

A non-geometric T-duality is simply a $\mathscr{T}\mathscr{D}_n$-bundle.

Remarks:

- The T-duality group $\mathscr{G}\mathscr{O}(n,n;\mathbb{Z}) \supset \mathsf{GO}(n,n;\mathbb{Z})$ is gauged!
- Explicitly visible: $\mathsf{GO}(n,n;\mathbb{Z})$-gluing of local data
- Matches topological discussion in Nikolaus, Waldorf (2018)
- Differential refinement imposes restriction on top. cocycles
- This describes all T-dualities between pairs of T-folds
- Concrete conditions for "half-geometric" T-dualities
- Concrete cocycles of the T-fold in the nilmanifold example

To describe Q-spaces/T-folds:
(can) use higher instead of noncommutative geometry.

Example: T-folds

Consider again the nilmanifold example, this time $X = S^1$.

- Gauge groupoid $\mathscr{T}\mathscr{D}_2$
- General cocycle data:

$$
\begin{aligned}
(g, z, \xi, m, \phi, q) &\in C^\infty(Y^{[3]}, \mathsf{GO}(2,2;\mathbb{Z}) \times \mathbb{Z}^4 \times \mathbb{R}^4 \times \mathbb{Z}^4 \times \mathsf{U}(1) \times Q_2) \\
(g, \xi, q) &\in C^\infty(Y^{[2]}, \mathsf{GO}(2,2;\mathbb{Z}) \times \mathbb{R}^4 \times Q_2) \\
q &\in C^\infty(Y, Q_2)
\end{aligned}
$$

- Topology: all data over $Y^{[3]}$ are trivial.
- Topology: no T^n-bundles over S^1: ξ is trivial
- Remaining: $q : Y \to Q_2 \cong \mathbb{R}^4$, $g : Y^{[2]} \to \mathsf{GO}(2,2;\mathbb{Z})$ s.t.:

$$
q(y_1) = g(y_1, y_2) q(y_2)~, \quad g(y_1, y_2) g(y_2, y_3) = g(y_1, y_3)
$$

- $\mathbb{R}^4$: scalar modes g_{yy}, g_{yz}, g_{zz}, B_{yz}
- Well-known T-fold is the special case where

$$
g_{x+1,x} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & \ell & 1 & 0 \\ -\ell & 0 & 0 & 1 \end{pmatrix}
$$

What about R-spaces?

- T-folds/Q-spaces relatively harmless, as locally geometric
- R-spaces are not even locally geometric
- But perhaps higher description still works?

Note:

- One T-duality direction: B-field $\rightarrow$ 2-, 1-forms
  $\Rightarrow$ Lie 2-group TD_n-bundles with connection
- Two T-duality directions: B-field $\rightarrow$ 2-, 1-, 0-forms
  $\Rightarrow$ Lie 2-groupoid $\mathscr{T}\mathscr{D}_n$-bundles with connection
- Three T-duality directions: B-field $\rightarrow$ 2-, 1-, 0-, "(-1)-forms"
  (Note: (-1)-forms have global "curvature" 0-forms)
  $\Rightarrow$ Augmented Lie 2-groupoid $\mathscr{T}\mathscr{D}_n^{\mathsf{aug}}$-bundles with connection

Augmented groupoid bundles

Need to switch to simplicial picture:

- (Higher) groupoids are Kan simplicial manifolds
- Higher groupoid 1-morphisms are simplicial maps
- Higher groupoid 2-morphisms are simplicial homotopies
- "quasi-groupoids" or "$(\infty, 1)$-groupoids"

Augmented $\mathscr{G}$-groupoid bundles subordinate to $\sigma : Y \twoheadrightarrow X$:

$$
\begin{array}{ccc}
Y \times_X Y \times_X Y & \xrightarrow{\quad g_2 \quad} & \mathscr{G}_2 \\
\downarrow\!\downarrow\!\downarrow & & \downarrow\!\downarrow\!\downarrow \\
Y \times_X Y & \xrightarrow{\quad g_1 \quad} & \mathscr{G}_1 \\
\downarrow\!\downarrow & & \downarrow\!\downarrow \\
Y & \xrightarrow{\quad g_0 \quad} & \mathscr{G}_0 \\
\downarrow{\scriptstyle \sigma} & & \downarrow \\
X & \xrightarrow{\quad g_{-1} \quad} & \mathscr{G}_{-1}
\end{array}
$$

T-duality as $\mathscr{T}\mathscr{D}_n^{\mathsf{aug}}$-bundles

Construction of $\mathscr{T}\mathscr{D}_n^{\mathsf{aug}}$:

- Augmentation by suitable space of R-fluxes
- Determined by finite version of tensor hierarchy
- Finite embedding tensor $\mathbb{R}^{2n} \to \mathsf{GO}(n,n;\mathbb{Z}) \subset \mathscr{G}\mathscr{O}(n,n;\mathbb{Z})$
- plus some standard consistency conditions
- Beyond this, augmentation fairly trivial

Remarks on T-duality with $\mathscr{T}\mathscr{D}_n^{\mathsf{aug}}$-bundles:

- Explicit examples, e.g. from nilmanifolds
- Yields consistency conditions between Q- and R-fluxes
- All previously discussed cases included
- All previously discussed also for affine $\mathsf{U}(1)$-bundles

To describe R-spaces:
(can) use higher instead of nonassociative geometry.

Summary

What has been done:

- Top. T-duality can be described using principal 2-bundles
- Differential refinement with adjusted curvatures
- Explicit description of geometric T-duality with nilmanifolds
- T-duality group is really a 2-group derived from KK-reduction
- Extended to Q-spaces or T-folds using 2-groupoid bundles
- Extended to R-spaces using augmented 2-groupoid bundles

Future work:

- Link some mathematical results to physical expectations
- Link to pre-NQ-manifold pictures, DFT, and similar
- Non-abelian T-duality?
- U-duality

Thank You!