Non-relativistic Geometry And Why You Might Care

Group Seminar University of Hertfordshire

Julian Kupka

15.11.2023

Goals/Structure

Goals/Structure

1. Motivate Non-relativistic geometry

Goals/Structure

1. Motivate Non-relativistic geometry
2. Properly explain “non-relativistic” symmetries

Goals/Structure

1. Motivate Non-relativistic geometry
2. Properly explain “non-relativistic” symmetries
3. Derive corresponding geometry by gauge procedure

Goals/Structure

1. Motivate Non-relativistic geometry
2. Properly explain “non-relativistic” symmetries
3. Derive corresponding geometry by gauge procedure
4. Understand how this encodes Newtonian physics

Goals/Structure

1. Motivate Non-relativistic geometry
2. Properly explain "non-relativistic" symmetries
3. Derive corresponding geometry by gauge procedure
4. Understand how this encodes Newtonian physics
5. Understand how this relates to null Reductions

Goals/Structure

1. Motivate Non-relativistic geometry
2. Properly explain “non-relativistic” symmetries
3. Derive corresponding geometry by gauge procedure
4. Understand how this encodes Newtonian physics
5. Understand how this relates to null Reductions
6. Hint at why you might encounter these

Motivation

- Motion of free falling observer $x(s), s \in I \subseteq \mathbb{R}$:

$$\ddot{x}^\mu + \Gamma^\mu{}_{\alpha\beta}\dot{x}^\alpha\dot{x}^\beta = 0$$

with $\mu, \alpha, \beta = 0, \dots d-1$

Motivation

- Motion of free falling observer $x(s), s \in I \subseteq \mathbb{R}$:

$$\ddot{x}^{\mu} + \Gamma^{\mu}{}_{\alpha\beta}\dot{x}^{\alpha}\dot{x}^{\beta} = 0$$

with $\mu, \alpha, \beta = 0, \dots d-1$

- Compare to particle in Newtonian potential ϕ:

$$\frac{\mathrm{d}^2 x^a}{\mathrm{d}t^2} + \delta^{ab}\partial_b \phi = 0,$$

with $a = 1, \dots d-1$ spacial

Motivation

- Motion of free falling observer $x(s), s \in I \subseteq \mathbb{R}$:

$$\ddot{x}^\mu + \Gamma^\mu{}_{\alpha\beta}\dot{x}^\alpha\dot{x}^\beta = 0$$

  with $\mu, \alpha, \beta = 0, \ldots d-1$

- Compare to particle in Newtonian potential ϕ:

$$\frac{\mathrm{d}^2 x^a}{\mathrm{d}t^2} + \delta^{ab}\partial_b\phi = 0,$$

  with $a = 1, \ldots d-1$ spacial

- Deviation from straight line due to:
  Gravitational force $\leftrightarrow$ Curvature in *Newtonian Spacetime*

Affine Connection

- Assume: special spacetime coordinates: $(x^\mu(t)) = (t, x^a(t))$

Affine Connection

- Assume: special spacetime coordinates: $(x^\mu(t)) = (t, x^a(t))$
- Read off Christoffel Symbols from

$$\frac{\mathrm{d}^2 x^a}{\mathrm{d}t^2} + \delta^{ab}\partial_b \phi = 0 \tag{1}$$

Affine Connection

- Assume: special spacetime coordinates: $(x^\mu(t)) = (t, x^a(t))$
- Read off Christoffel Symbols from

$$\frac{\mathrm{d}^2 x^a}{\mathrm{d}t^2} + \delta^{ab}\partial_b\phi = 0 \tag{1}$$

$\implies$ only non-zero Christoffels

$$\Gamma^a{}_{00} = \delta^{ab}\partial_b\phi$$

Affine Connection

- Assume: special spacetime coordinates: $(x^\mu(t)) = (t, x^a(t))$
- Read off Christoffel Symbols from

$$\frac{\mathrm{d}^2 x^a}{\mathrm{d}t^2} + \delta^{ab}\partial_b\phi = 0 \tag{1}$$

$\implies$ only non-zero Christoffels

$$\Gamma^a{}_{00} = \delta^{ab}\partial_b\phi$$

$\implies$ only non-zero Curvature

$$R^a{}_{0b0} = \delta^{ac}\partial_c\partial_b\phi$$

EOM + Interpretation

- Impose EOM

$$R \coloneqq {R^a}_{0a0} \equiv \Delta\phi = 4\pi G_N \rho, \tag{2}$$

$G_N \ldots$ Newton constant, $\rho \ldots$ mass density

EOM + Interpretation

- Impose EOM

$$R \coloneqq {R^a}_{0a0} \equiv \Delta\phi = 4\pi G_N \rho, \tag{2}$$

$G_N \dots$ Newton constant, $\rho \dots$ mass density

$\implies$ Recover Newtonian gravity:

Poisson equation eq. 2 + *Newtonian force* eq. 1

EOM + Interpretation

- Impose EOM

$$R \coloneqq R^{a}{}_{0a0} \equiv \Delta\phi = 4\pi G_N \rho, \qquad (2)$$

$G_N \ldots$ Newton constant, $\rho \ldots$ mass density

$\Longrightarrow$ Recover Newtonian gravity:

Poisson equation eq. 2 + *Newtonian force* eq. 1

Question:

How to interpret this geometrically?

Non-relativistic symmetries

- Newtonian $\implies$ At least *Galilei symmetries*

$$t' = t + \zeta^0$$
$$x'^a = A^a{}_b x^b + v^a t + \zeta^a,$$

  - $\zeta^0 \leftrightarrow$ time translation
  - $\zeta^a \leftrightarrow$ space translation
  - $A^a{}_b \leftrightarrow SO(d-1)$, i.e. spacial rotations
  - $v^a \leftrightarrow$ boosts

Non-relativistic symmetries

- Newtonian $\implies$ At least *Galilei symmetries*

$$t' = t + \zeta^0$$
$$x'^a = A^a{}_b x^b + v^a t + \zeta^a,$$

  - $\zeta^0 \leftrightarrow$ time translation
  - $\zeta^a \leftrightarrow$ space translation
  - $A^a{}_b \leftrightarrow SO(d-1)$, i.e. spacial rotations
  - $v^a \leftrightarrow$ boosts

- Note: [Boosts, Translations] $= 0$

Non-relativistic symmetries

- Newtonian $\implies$ At least *Galilei symmetries*

$$t' = t + \zeta^0$$
$$x'^a = A^a{}_b x^b + v^a t + \zeta^a,$$

  - $\zeta^0 \leftrightarrow$ time translation
  - $\zeta^a \leftrightarrow$ space translation
  - $A^a{}_b \leftrightarrow SO(d-1)$, i.e. spacial rotations
  - $v^a \leftrightarrow$ boosts

- Note: [Boosts, Translations] $= 0$
- Spacetime version: $x'^\mu = \Lambda^\mu{}_\nu x^\nu + \zeta^\mu$ with

$$\Lambda = \begin{pmatrix} 1 & 0 \\ v & A \end{pmatrix}$$

Defining tensors

- Galilei preserves two *degenerate* “metrics”

Defining tensors

- Galilei preserves two *degenerate* “metrics”
- Temporal metric

$$(t_{\mu\nu}) = \begin{pmatrix} 1 & 0 \\ 0 & 0_{d-1} \end{pmatrix}$$

Defining tensors

- Galilei preserves two *degenerate* "metrics"
- Temporal metric

$$(t_{\mu\nu}) = \begin{pmatrix} 1 & 0 \\ 0 & 0_{d-1} \end{pmatrix}$$

- Spacial co-metric

$$(h^{\mu\nu}) = \begin{pmatrix} 0 & 0 \\ 0 & \mathbb{1}_{d-1} \end{pmatrix}$$

Defining tensors

- Galilei preserves two *degenerate* “metrics”
- Temporal metric

$$(t_{\mu\nu}) = \begin{pmatrix} 1 & 0 \\ 0 & 0_{d-1} \end{pmatrix}$$

- Spacial co-metric

$$(h^{\mu\nu}) = \begin{pmatrix} 0 & 0 \\ 0 & \mathbb{1}_{d-1} \end{pmatrix}$$

- Mutually orthogonal

$$t_{\mu\nu} h^{\nu\rho} = 0$$

Non-relativistic Vielbeine

- Introduce vielbein formalism
- *clock form*

$$t_{\mu\nu} = \tau_\mu \tau_\nu$$

Non-relativistic Vielbeine

- Introduce vielbein formalism
- *clock form*

$$t_{\mu\nu} = \tau_\mu \tau_\nu$$

- *spacial vielbein*

$$\delta^{ab} = h^{\mu\nu} e_\mu{}^a e_\nu{}^b$$

Non-relativistic Vielbeine

- Introduce vielbein formalism
- *clock form*

$$t_{\mu\nu} = \tau_\mu \tau_\nu$$

- *spacial vielbein*

$$\delta^{ab} = h^{\mu\nu} e_\mu{}^a e_\nu{}^b$$

- Read off variations

$$\delta \tau_\mu = \mathcal{L}_\xi \tau_\mu,$$
$$\delta e_\mu{}^a = \mathcal{L}_\xi e_\mu{}^a + \lambda^a{}_b e_\mu{}^b + \lambda^a \tau_\mu,$$

spacial rotation $\lambda^a{}_b$, boost λ^a and diffeomorphism ξ

Splitting Tensors

- Projective inverses:

$$e_\mu{}^a \tau^\mu = 0, \qquad\qquad e^\mu{}_a \tau_\mu = 0,$$

Splitting Tensors

- Projective inverses:

$$e_\mu{}^a \tau^\mu = 0, \qquad e^\mu{}_a \tau_\mu = 0,$$

$$e_\mu{}^a e^\mu{}_b = \delta^a{}_b, \qquad \tau_\mu \tau^\mu = 1,$$

Splitting Tensors

- Projective inverses:

$$e_\mu{}^a \tau^\mu = 0, \qquad e^\mu{}_a \tau_\mu = 0,$$

$$e_\mu{}^a e^\mu{}_b = \delta^a{}_b, \qquad \tau_\mu \tau^\mu = 1,$$

$$\delta^\mu{}_\nu = \tau^\mu \tau_\nu + e^\mu{}_a e_\nu{}^a$$

Splitting Tensors

- Projective inverses:

$$e_\mu{}^a \tau^\mu = 0, \qquad e^\mu{}_a \tau_\mu = 0,$$

$$e_\mu{}^a e^\mu{}_b = \delta^a{}_b, \qquad \tau_\mu \tau^\mu = 1,$$

$$\delta^\mu{}_\nu = \tau^\mu \tau_\nu + e^\mu{}_a e_\nu{}^a$$

- Split tensors into flat parts

$$T_\mu = \delta^\nu{}_\mu T_\nu = (\tau^\nu T_\nu)\tau_\mu + (e^\nu{}_a T_\nu) e_\mu{}^a =: T_0 \tau_\mu + T_a e_\mu{}^a$$

Simple Example

Question:

Did we find all relevant symmetries?

Simple Example

Question:

Did we find all relevant symmetries?

- Study simplest representation: Free Newtonian particle $(x^\mu) = (x^0, x^a)$ with action

$$S[x^\mu] = \frac{m}{2} \int d\tau \frac{\delta_{ab} \dot{x}^a \dot{x}^b}{\dot{x}^0}$$

Simple Example

Question:
Did we find all relevant symmetries?

- Study simplest representation: Free Newtonian particle $(x^\mu) = (x^0, x^a)$ with action

$$S[x^\mu] = \frac{m}{2} \int d\tau \frac{\delta_{ab}\dot{x}^a\dot{x}^b}{\dot{x}^0}$$

- S inert under rotations and translations

Simple Example

Question:

Did we find all relevant symmetries?

- Study simplest representation: Free Newtonian particle $(x^\mu) = (x^0, x^a)$ with action

$$S[x^\mu] = \frac{m}{2} \int d\tau \frac{\delta_{ab} \dot{x}^a \dot{x}^b}{\dot{x}^0}$$

- S inert under rotations and translations
- S only *quasi-invariant* under boosts $\delta_B x^a = \lambda^a x^0$, i.e.

$$\delta_B S = \int d\tau \frac{\mathrm{d}}{\mathrm{d}\tau} \left(m \lambda_i x^i \right)$$

Simple Example

Question:

Did we find all relevant symmetries?

- Study simplest representation: Free Newtonian particle $(x^\mu) = (x^0, x^a)$ with action

$$S[x^\mu] = \frac{m}{2} \int d\tau \frac{\delta_{ab} \dot{x}^a \dot{x}^b}{\dot{x}^0}$$

- S inert under rotations and translations
- S only *quasi-invariant* under boosts $\delta_B x^a = \lambda^a x^0$, i.e.

$$\delta_B S = \int d\tau \frac{\mathrm{d}}{\mathrm{d}\tau} \left(m \lambda_i x^i \right)$$

$\implies$ modifies boost Noether charge

Commutation Relations

- Compute algebra of Noether charges

Commutation Relations

- Compute algebra of Noether charges
- Agrees with Galilei symmetries except for boosts and translations

$$\left.\begin{aligned} Q_P &= -p_a \zeta^a, \\ Q_B &= m\lambda_a x^a - p_a \lambda^a x^0 \end{aligned}\right\} \implies \{Q_B, Q_P\} = -m\lambda_a \zeta^a \neq 0$$

Commutation Relations

- Compute algebra of Noether charges
- Agrees with Galilei symmetries except for boosts and translations

$$\left.\begin{aligned} Q_P &= -p_a\zeta^a, \\ Q_B &= m\lambda_a x^a - p_a\lambda^a x^0 \end{aligned}\right\} \implies \{Q_B, Q_P\} = -m\lambda_a\zeta^a \neq 0$$

Conclusion

Need central extension of Galilei: The *Bargmann algebra* $\mathfrak{barg}$

Interpretation

- Extra charge $\leftrightarrow$ Mass conservation

Interpretation

- Extra charge $\leftrightarrow$ Mass conservation
- Introduce coordinate s s.t. $\delta s = -\lambda_a x^a$

$$S[x, s] = \frac{m}{2} \int d\tau \left(\frac{\delta_{ab} \dot{x}^a \dot{x}^b}{\dot{x}^0} + 2\dot{s} \right)$$

Interpretation

- Extra charge $\leftrightarrow$ Mass conservation
- Introduce coordinate s s.t. $\delta s = -\lambda_a x^a$

$$S[x,s] = \frac{m}{2}\int d\tau \left(\frac{\delta_{ab}\dot{x}^a\dot{x}^b}{\dot{x}^0} + 2\dot{s}\right)$$

- conjugate momentum to $s \leftrightarrow$ mass m

$$p_s = \frac{\partial \mathcal{L}}{\partial \dot{s}} \equiv m$$

Interpretation

- Extra charge $\leftrightarrow$ Mass conservation
- Introduce coordinate s s.t. $\delta s = -\lambda_a x^a$

$$S[x, s] = \frac{m}{2} \int d\tau \left(\frac{\delta_{ab} \dot{x}^a \dot{x}^b}{\dot{x}^0} + 2\dot{s} \right)$$

- conjugate momentum to $s \leftrightarrow$ mass m

$$p_s = \frac{\partial \mathcal{L}}{\partial \dot{s}} \equiv m$$

- s cyclic $\implies$ m conserved

The Algebra Of Newtonian Physics

- Full definition of $\mathfrak{barg}$:

$$[J_{ab}, J_{cd}] = 4\delta_{[a[c}J_{d]b]},$$
$$[P_a, J_{bc}] = 2\delta_{a[b}P_{c]},$$
$$[B_a, J_{bc}] = 2\delta_{a[b}B_{c]},$$
$$[H, B_a] = P_a,$$
$$[P_a, B_b] = \delta_{ab}M,$$

with $a = 1, \dots d-1$

- $J_{ab} \leftrightarrow$ rotations
- $P_a \leftrightarrow$ spacial translations
- $H \leftrightarrow$ time translations
- $B_a \leftrightarrow$ non-relativistic boosts
- $M \leftrightarrow$ central extension ($U(1)$) generator

The Gauge Fields

- Geometry of $\mathfrak{barg} \leftrightarrow$ *Newton-Cartan geometry*

The Gauge Fields

- Geometry of $\mathfrak{barg} \leftrightarrow$ *Newton-Cartan geometry*
- Usual gauging procedure:

Generator	Parameter	Gauge field
H	ξ^0	τ_μ
P_a	ξ^a	$e_\mu{}^a$
$J_{[ab]}$	λ^{ab}	$\omega_\mu{}^{ab}$
B_a	λ^a	$\omega_\mu{}^a$
M	σ	m_μ

How to Include Diffeomorphisms

- Variations:

$$\begin{aligned}\delta\tau_\mu &= \partial_\mu\xi^0, \\ \delta e_\mu{}^a &= \partial_\mu\xi^a - \omega_\mu{}^a{}_b\xi^b + \lambda^a{}_b e_\mu{}^b + \lambda^a\tau_\mu - \omega_\mu{}^a\xi^0, \\ \delta m_\mu &= \partial_\mu\sigma - \omega_\mu{}^a\xi_a + \lambda_a e_\mu{}^a\end{aligned}$$

- Curvature corresponding to generator $T \leftrightarrow R(T)$

How to Include Diffeomorphisms

- Variations:

$$\begin{aligned} \delta\tau_\mu &= \partial_\mu \xi^0, \\ \delta e_\mu{}^a &= \partial_\mu \xi^a - \omega_\mu{}^a{}_b \xi^b + \lambda^a{}_b e_\mu{}^b + \lambda^a \tau_\mu - \omega_\mu{}^a \xi^0, \\ \delta m_\mu &= \partial_\mu \sigma - \omega_\mu{}^a \xi_a + \lambda_a e_\mu{}^a \end{aligned}$$

- Diffeomorphism ξ expressed via above variations

$$\begin{aligned} \delta_\xi \tau_\mu \equiv& \mathcal{L}_\xi \tau_\mu = \partial_\mu \left(\xi^\alpha \tau_\alpha\right) - \xi^\alpha R_{\mu\alpha}(H), \\ \delta_\xi e_\mu{}^a =& \partial_\mu \left(\xi^\alpha e_\alpha{}^a\right) - \omega_\mu{}^a{}_b \xi^\alpha e_\alpha{}^b + \xi^\alpha \omega_\alpha{}^a{}_b e_\mu{}^b + \xi^\alpha \omega_\alpha{}^a \tau_\mu \\ & - \omega_\mu{}^a \xi^\alpha \tau_\alpha - \xi^\alpha R_{\mu\alpha}(P^a), \\ \delta_\xi m_\mu =& \partial_\mu \left(\xi^\alpha m_\alpha\right) - \omega_\mu{}^a \xi^\alpha e_{\mu a} + \xi^\alpha \omega_{\alpha a} e_\mu{}^a - \xi^\alpha R_{\mu\alpha}(M) \end{aligned}$$

- Curvature corresponding to generator $T \leftrightarrow R(T)$

Vielbeine and Connections

- $\tau_\mu, e_\mu{}^a \leftrightarrow$ NR-Vielbeine:

$$\xi^0 \equiv \xi^\mu \tau_\mu, \qquad \delta\tau_\mu = \mathcal{L}_\xi \tau_\mu,$$
$$\xi^a \equiv \xi^\mu e_\mu{}^a, \qquad \delta e_\mu{}^a = \mathcal{L}_\xi e_\mu{}^a + \lambda^a{}_b e_\mu{}^b + \lambda^a \tau_\mu$$

Vielbeine and Connections

- τ_μ, ${e_\mu}^a \leftrightarrow$ NR-Vielbeine:

$$\xi^0 \equiv \xi^\mu \tau_\mu, \qquad \delta\tau_\mu = \mathcal{L}_\xi \tau_\mu,$$
$$\xi^a \equiv \xi^\mu {e_\mu}^a, \qquad \delta {e_\mu}^a = \mathcal{L}_\xi {e_\mu}^a + {\lambda^a}_b {e_\mu}^b + \lambda^a \tau_\mu$$

- Central charge "mass" vielbein $\delta m_\mu = \mathcal{L}_\xi m_\mu + \partial_\mu \sigma + \lambda_a {e_\mu}^a$

Vielbeine and Connections

- $\tau_\mu, e_\mu{}^a \leftrightarrow$ NR-Vielbeine:

$$\xi^0 \equiv \xi^\mu \tau_\mu, \qquad \delta\tau_\mu = \mathcal{L}_\xi \tau_\mu,$$
$$\xi^a \equiv \xi^\mu e_\mu{}^a, \qquad \delta e_\mu{}^a = \mathcal{L}_\xi e_\mu{}^a + \lambda^a{}_b e_\mu{}^b + \lambda^a \tau_\mu$$

- Central charge "mass" vielbein $\delta m_\mu = \mathcal{L}_\xi m_\mu + \partial_\mu \sigma + \lambda_a e_\mu{}^a$
- Torsion

$$R_{\mu\nu}(H) = 2\partial_{[\mu}\tau_{\nu]},$$
$$R_{\mu\nu}(P^a) = 2\partial_{[\mu}e_{\nu]}{}^a - 2\omega_{[\mu}{}^{ab}e_{\nu]b} - 2\omega_{[\mu}{}^a\tau_{\nu]},$$
$$R_{\mu\nu}(M) = 2\partial_{[\mu}m_{\nu]} - 2\omega_{[\mu}{}^a e_{\nu]a}$$

Vielbeine and Connections

- $\tau_\mu, e_\mu{}^a \leftrightarrow$ NR-Vielbeine:

$$\xi^0 \equiv \xi^\mu \tau_\mu, \qquad \delta\tau_\mu = \mathcal{L}_\xi \tau_\mu,$$
$$\xi^a \equiv \xi^\mu e_\mu{}^a, \qquad \delta e_\mu{}^a = \mathcal{L}_\xi e_\mu{}^a + \lambda^a{}_b e_\mu{}^b + \lambda^a \tau_\mu$$

- Central charge "mass" vielbein $\delta m_\mu = \mathcal{L}_\xi m_\mu + \partial_\mu \sigma + \lambda_a e_\mu{}^a$
- Torsion

$$R_{\mu\nu}(H) = 2\partial_{[\mu}\tau_{\nu]},$$
$$R_{\mu\nu}(P^a) = 2\partial_{[\mu} e_{\nu]}{}^a - 2\omega_{[\mu}{}^{ab} e_{\nu]b} - 2\omega_{[\mu}{}^a \tau_{\nu]},$$
$$R_{\mu\nu}(M) = 2\partial_{[\mu} m_{\nu]} - 2\omega_{[\mu}{}^a e_{\nu]a}$$

- See $\omega_\mu{}^{ab}, \omega_\mu{}^a$ appear only algebraically
  $\implies$ solve for in terms of (τ, e, m)

Intrinsic torsion

- ω does not appear in $R(H) \implies$ cannot be absorbed into ω

Intrinsic torsion

- ω does not appear in $R(H) \implies$ cannot be absorbed into ω
  $\implies R(H) \equiv$ *intrinsic torsion*

Intrinsic torsion

- ω does not appear in $R(H) \implies$ cannot be absorbed into ω $\implies R(H) \equiv$ *intrinsic torsion*
- Classifies geometries:
    1. Torsionless $R(H) \equiv d\tau = 0$:

Intrinsic torsion

- ω does not appear in $R(H) \implies$ cannot be absorbed into ω
  $\implies R(H) \equiv$ *intrinsic torsion*
- Classifies geometries:
  1. Torsionless $R(H) \equiv d\tau = 0$:
     - *Absolute time*: $\exists\ t : M \to \mathbb{R}$ s.t. $\tau = dt$

$$\implies T := \int_\gamma \tau \equiv t(x) - t(y)$$

Intrinsic torsion

- ω does not appear in $R(H) \implies$ cannot be absorbed into ω
  $\implies R(H) \equiv$ *intrinsic torsion*
- Classifies geometries:
  1. Torsionless $R(H) \equiv d\tau = 0$:
     - *Absolute time*: $\exists\ t : M \to \mathbb{R}$ s.t. $\tau = dt$

$$\implies T := \int_\gamma \tau \equiv t(x) - t(y)$$

     - *Absolute simultaneity*: Ker $\tau \equiv$ Im e defines foliation into spaces of simultaneity

Intrinsic torsion

- ω does not appear in $R(H) \implies$ cannot be absorbed into ω
  $\implies R(H) \equiv$ *intrinsic torsion*
- Classifies geometries:
  1. Torsionless $R(H) \equiv d\tau = 0$:
     - *Absolute time*: $\exists\ t : M \to \mathbb{R}$ s.t. $\tau = dt$

$$\implies T := \int_\gamma \tau \equiv t(x) - t(y)$$

     - *Absolute simultaneity*: Ker $\tau \equiv$ Im e defines foliation into spaces of simultaneity
  2. Twistless torsional: $d\tau \neq 0$ but $\tau \wedge d\tau = 0$
     $\implies$ Absolute simultaneity but not absolute time

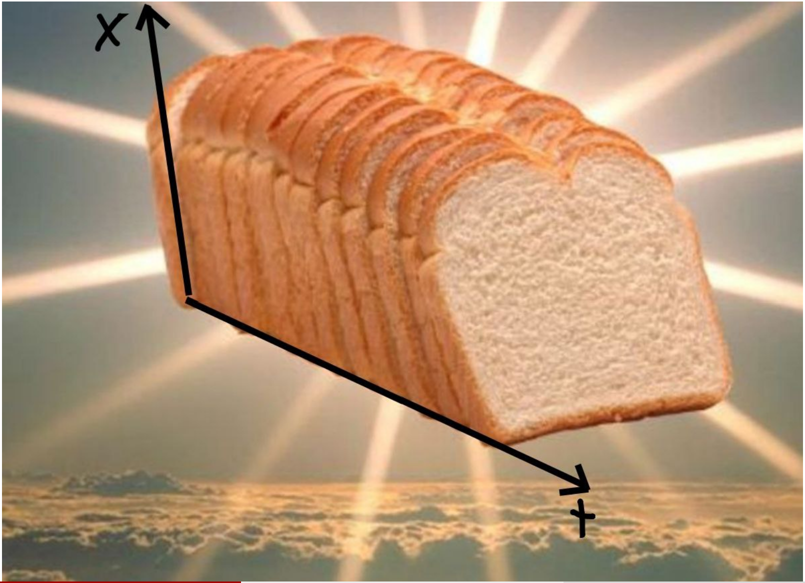
Intrinsic torsion

- ω does not appear in $R(H) \implies$ cannot be absorbed into ω
  $\implies R(H) \equiv$ *intrinsic torsion*
- Classifies geometries:
  1. Torsionless $R(H) \equiv d\tau = 0$:
     - *Absolute time*: $\exists\ t : M \to \mathbb{R}$ s.t. $\tau = dt$

$$\implies T := \int_\gamma \tau \equiv t(x) - t(y)$$

     - *Absolute simultaneity*: Ker $\tau \equiv$ Im e defines foliation into spaces of simultaneity
  2. Twistless torsional: $d\tau \neq 0$ but $\tau \wedge d\tau = 0$
     $\implies$ Absolute simultaneity but not absolute time
  3. Torsional: No constraints

Visualization Of Foliation



Galilean Gravity

Question:

How to connect to Newtonian gravity we had in beginning?

Galilean Gravity

Question:

How to connect to Newtonian gravity we had in beginning?

- Impose curvature/torsion constraints:

$$R(P) = R(H) = R(M) = R(J) \equiv 0$$

Galilean Gravity

Question:
How to connect to Newtonian gravity we had in beginning?

- Impose curvature/torsion constraints:

$$R(P) = R(H) = R(M) = R(J) \equiv 0$$

- Keep only curvature of boosts

$$R_{\mu\nu}(B^a) = 2\partial_{[\mu}\omega_{\nu]}{}^a - 2\omega_{[\mu}{}^{ab}\omega_{\nu]b}$$

$\Longrightarrow$ Encodes gravitational dynamics

Galilean Gravity

- Flat Galilean manifold has special frames

$$\tau_\mu = \delta^0{}_\mu, \quad e_\mu{}^a = \delta^a{}_\mu, \quad m_a = 0, \quad \omega_\mu{}^{ab} = 0,$$

corresponding to Galilean coordinates $(x^\mu) = (t, x^a)$

Galilean Gravity

- Flat Galilean manifold has special frames

$$\tau_\mu = \delta^0{}_\mu, \quad e_\mu{}^a = \delta^a{}_\mu, \quad m_a = 0, \quad \omega_\mu{}^{ab} = 0,$$

corresponding to Galilean coordinates $(x^\mu) = (t, x^a)$

- Constrains variations to known Galilean ones

$$\xi^0(x) = \xi^0, \quad \lambda^{ab}(x) = \lambda^{ab}, \quad \sigma = 0,$$
$$\xi^a(x) = \xi^a(t) - \lambda^a{}_b \delta^b{}_\mu x^\mu, \quad \lambda^a(x) = -\dot{\xi}^a(t)$$

Galilean Gravity

- Flat Galilean manifold has special frames

$$\tau_\mu = \delta^0{}_\mu, \quad e_\mu{}^a = \delta^a{}_\mu, \quad m_a = 0, \quad \omega_\mu{}^{ab} = 0,$$

corresponding to Galilean coordinates $(x^\mu) = (t, x^a)$

- Constrains variations to known Galilean ones

$$\xi^0(x) = \xi^0, \quad \lambda^{ab}(x) = \lambda^{ab}, \quad \sigma = 0,$$
$$\xi^a(x) = \xi^a(t) - \lambda^a{}_b \delta^b{}_\mu x^\mu, \quad \lambda^a(x) = -\dot{\xi}^a(t)$$

- Only independent field left is a static scalar field

$$m_0((x^a)) =: \phi((x^a))$$

Galilean Gravity

- Only non-zero boost connection

$$\omega_0{}^a(x) = -\partial^a \phi(x)$$

Galilean Gravity

- Only non-zero boost connection

$$\omega_0{}^a(x) = -\partial^a \phi(x)$$

- Impose equation of motion

$$R_{0a}(B^a) \equiv \partial_a \partial^a \phi = 0$$

$\implies$ Recover Poisson eq. 2

Galilean Gravity

- Only non-zero boost connection

$$\omega_0{}^a(x) = -\partial^a \phi(x)$$

- Impose equation of motion

$$R_{0a}(B^a) \equiv \partial_a \partial^a \phi = 0$$

$\implies$ Recover Poisson eq. 2

- Geodesic equation for this connection $\implies$ Newtonian force eq. 1

Galilean Gravity

- Only non-zero boost connection

$$\omega_0{}^a(x) = -\partial^a \phi(x)$$

- Impose equation of motion

$$R_{0a}(B^a) \equiv \partial_a \partial^a \phi = 0$$

$\implies$ Recover Poisson eq. 2

- Geodesic equation for this connection $\implies$ Newtonian force eq. 1

Conclusion

Newton-Cartan geometry encodes Newtonian gravity

Scherk-Schwarz Reductions

- Basis of dimensional reduction: $M_{d+1} = M_d \times S(1)$

Scherk-Schwarz Reductions

- Basis of dimensional reduction: $M_{d+1} = M_d \times S(1)$
- Adapted coordinates (x, z)

Scherk-Schwarz Reductions

- Basis of dimensional reduction: $M_{d+1} = M_d \times S(1)$
- Adapted coordinates (x, z)
- Scherk-Schwarz ansatz: Global symmetry $g(z) \in G$ allows for

$$\phi(x, z) = g(z)(\psi(x)) \tag{3}$$

Flat Null Reduction

- Choose lightcone coordinates (x^+, x^-, x^a), $a = 1, \ldots, d-1$

Flat Null Reduction

- Choose lightcone coordinates (x^+, x^-, x^a), $a = 1, \ldots, d-1$

  $\implies$ Minkowski metric $\eta_{+-} = -1$, $\eta_{ab} = \delta_{ab}$

Flat Null Reduction

- Choose lightcone coordinates (x^+, x^-, x^a), $a = 1, \ldots, d-1$
  $\implies$ Minkowski metric $\eta_{+-} = -1$, $\eta_{ab} = \delta_{ab}$
- Assume x^+ compactified null direction and ansatz eq. 3 with $U(1)$-symmetry

$$\phi(x, x^+, x^-) = g(x^+)(\psi(x, x^-)) = e^{-imx^+}\psi(x, x^-),$$

with $m \in \mathbb{R}$

Flat Null Reduction

- Choose lightcone coordinates (x^+, x^-, x^a), $a = 1, \dots, d-1$
  $\implies$ Minkowski metric $\eta_{+-} = -1$, $\eta_{ab} = \delta_{ab}$
- Assume x^+ compactified null direction and ansatz eq. 3 with $U(1)$-symmetry

$$\phi(x, x^+, x^-) = g(x^+)(\psi(x, x^-)) = e^{-imx^+}\psi(x, x^-),$$

  with $m \in \mathbb{R}$
- ϕ is a massless Klein-Gordon scalar

$$\Box_{d+1}\phi = \Delta_d\phi - 2\partial_+\partial_-\phi = 0$$

Flat Null Reduction

- Choose lightcone coordinates (x^+, x^-, x^a), $a = 1, \dots, d-1$
  $\implies$ Minkowski metric $\eta_{+-} = -1$, $\eta_{ab} = \delta_{ab}$
- Assume x^+ compactified null direction and ansatz eq. 3 with $U(1)$-symmetry

$$\phi(x, x^+, x^-) = g(x^+)(\psi(x, x^-)) = e^{-imx^+}\psi(x, x^-),$$

  with $m \in \mathbb{R}$
- ϕ is a massless Klein-Gordon scalar

$$\Box_{d+1}\phi = \Delta_d\phi - 2\partial_+\partial_-\phi = 0$$

- Renaming $x^- \equiv t$, we find NR Schrödinger equation for ψ

$$i\partial_t\psi(x, t) = -\frac{1}{2m}\Delta_x\psi(x, t)$$

Non-flat Null Reduction

- Assume null Killing vector $\hat{\chi}$ in $d+1$ dimensions

Non-flat Null Reduction

- Assume null Killing vector $\hat{\chi}$ in $d+1$ dimensions
- Adapted coordinates $(x^{\hat{\mu}}) = (x^\mu, v)$ s.t. $\partial_v = \hat{\chi}$

Non-flat Null Reduction

- Assume null Killing vector $\hat{\chi}$ in $d+1$ dimensions
- Adapted coordinates $(x^{\hat{\mu}}) = (x^\mu, v)$ s.t. $\partial_v = \hat{\chi}$
- Write vielbein index with null directions $\hat{A} = (a, +, -)$, $a = 1, \ldots d-1$

Non-flat Null Reduction

- Assume null Killing vector $\hat{\chi}$ in $d+1$ dimensions
- Adapted coordinates $(x^{\hat{\mu}}) = (x^\mu, v)$ s.t. $\partial_v = \hat{\chi}$
- Write vielbein index with null directions $\hat{A} = (a, +, -)$, $a = 1, \dots d-1$
- Vielbeine can be written as Newton-Cartan fields

$$(E_{\hat{\mu}}{}^{A}) = \begin{array}{c} \\ \mu \\ v \end{array} \begin{array}{c} \begin{array}{ccc} a & - & + \end{array} \\ \begin{pmatrix} e_\mu{}^a & \tau_\mu & m_\mu \\ 0 & 0 & 1 \end{pmatrix} \end{array}$$

Non-flat Null Reduction

- Assume null Killing vector $\hat{\chi}$ in $d+1$ dimensions
- Adapted coordinates $(x^{\hat{\mu}}) = (x^\mu, v)$ s.t. $\partial_v = \hat{\chi}$
- Write vielbein index with null directions $\hat{A} = (a, +, -)$, $a = 1, \dots d-1$
- Vielbeine can be written as Newton-Cartan fields

$$(E_{\hat{\mu}}{}^A) = \begin{array}{c} \\ \mu \\ v \end{array} \begin{array}{c} \begin{array}{ccc} a & - & + \end{array} \\ \begin{pmatrix} e_\mu{}^a & \tau_\mu & m_\mu \\ 0 & 0 & 1 \end{pmatrix} \end{array}$$

- with NR variations

$$\begin{aligned} \delta\tau_\mu &= \mathcal{L}_\xi \tau_\mu \\ \delta e_\mu{}^a &= \mathcal{L}_\xi e_\mu{}^a + \lambda^a{}_b e_\mu{}^b + \lambda^a \tau_\mu \\ \delta m_\mu &= \mathcal{L}_\xi m_\mu + \partial_\mu \sigma + \lambda_a e_\mu{}^a \end{aligned}$$

Conclusion

- Newton-Cartan geometry appears as “geometry perpendicular to null direction”

Conclusion

- Newton-Cartan geometry appears as “geometry perpendicular to null direction”
- Includes null horizons of black holes

Conclusion

- Newton-Cartan geometry appears as “geometry perpendicular to null direction”
- Includes null horizons of black holes
- Solutions of SUGRA/string theory such as *pp-wave* or *fundamental NS string*

Conclusion

- Newton-Cartan geometry appears as “geometry perpendicular to null direction”
- Includes null horizons of black holes
- Solutions of SUGRA/string theory such as *pp-wave* or *fundamental NS string*
- Gives interpretation for compactified null directions