Higher symmetries and homotopy algebras: scattering amplitudes, colour–kinematics duality, $\mathrm{BV}^{\square}_{\infty}$-algebras, the double copy and M2-branes models.

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Work with A. Anastasiou, M.J. Duff, M. Hughes, B. Jurco, Hyungrok Kim, A. Marrani, S. Nagy, T. Macrelli, C. Saemann, M. Wolf, M. Zoccali

Introduction: Homotopy algebras, quantum field theory and colour-kinematic duality

Introduction: theme I

Perturbative quantum field theory	Homotopy algebraic perspective
classical BV action S	metric L_∞-algebra $\mathfrak{L}_S$
tree-level scattering amplitude for S	minimal model for $\mathfrak{L}_S$
choice of gauge fixing	embedding of minimal model into $\mathfrak{L}_S$
integrating out fields	homotopy transfer to smaller L_∞-algebra
semi-classical equivalence $S \sim \tilde{S}$	quasi-isomorphism $\mathfrak{L}_S \cong \mathfrak{L}_{\tilde{S}}$
Feynman diagram expansion	homological perturbation lemma
Berends–Giele recursion relation	homological geometric series
special properties of amplitudes	homotopy algebraic refinement of L_∞-algebra $\mathfrak{L}_S$
colour-stripping of amplitudes	factorisation $\mathfrak{L}_S \cong \mathfrak{g} \otimes \mathfrak{C}$ with $\mathfrak{C}$ a C_∞-algebra
colour–kinematics duality	the C_∞-algebra is a $\mathrm{BV}^{\blacksquare}_{\infty}$-algebra $\mathfrak{B}$
manifest colour–kinematics duality	$\mathfrak{L}_S \cong \mathfrak{g} \otimes \mathfrak{B}$ with $\mathfrak{B}$ a $\mathrm{BV}^{\blacksquare}$-algebra
double copy	tensor product of metric $\mathrm{BV}^{\blacksquare}$-algebras

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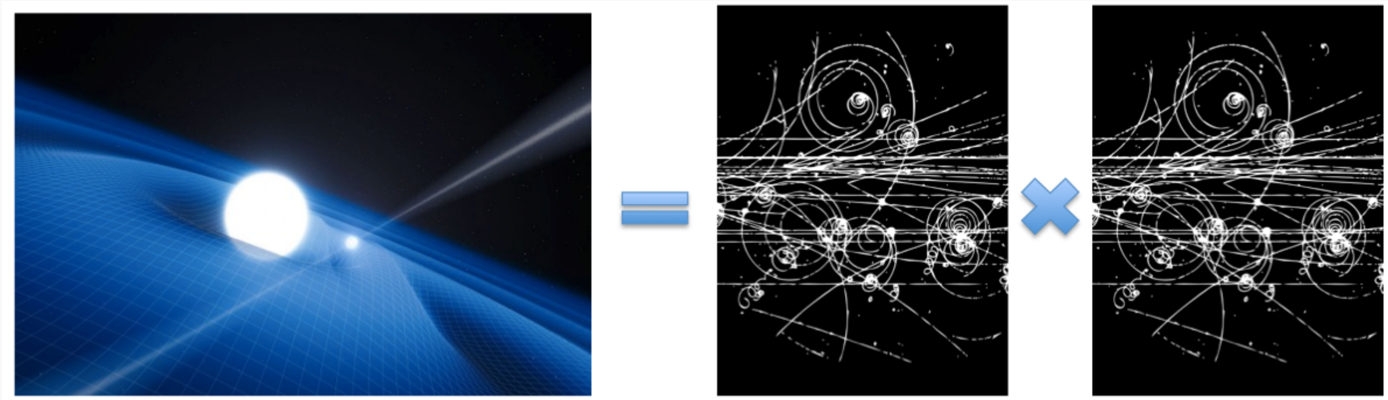

$$g_{\mu\nu} \qquad A_\mu{}^a \qquad A_\nu{}^b$$

- Is gravity the double copy of the other fundamental forces of Nature?
  [Feynman; Papini; Kawai, Lewellen, Tye; Berends, Giele, Kuijf; Bern, Dixon, Dunbar, Perelstein , Rozowsky. . .]

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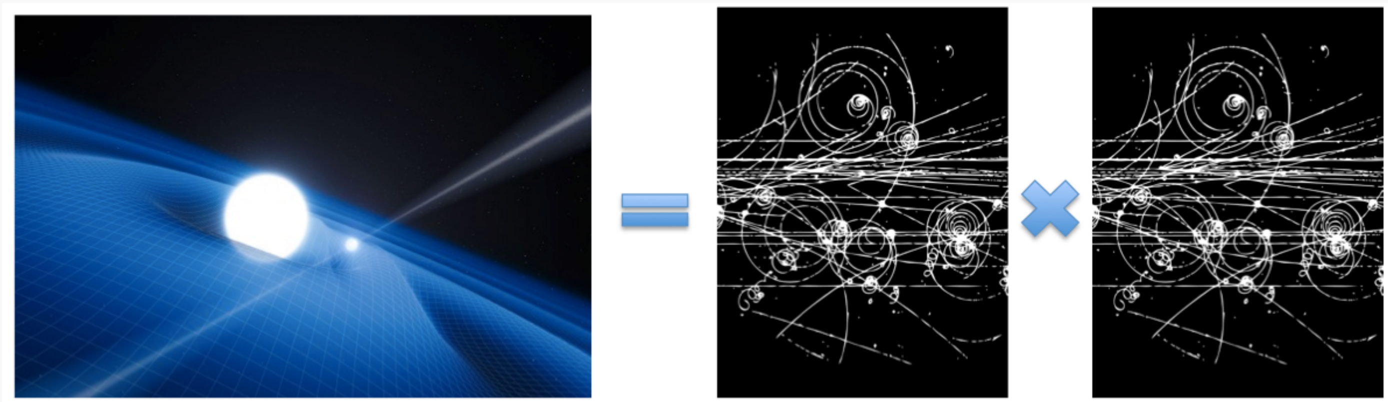

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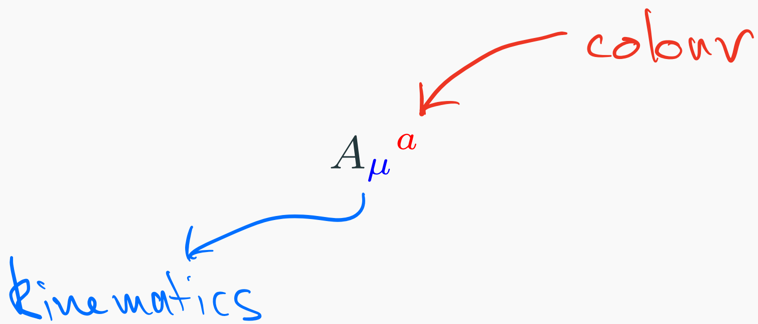
- Renaissance: Bern–Carrasco–Johansson Colour–Kinematics (CK) duality conjecture and double copy of gauge theory and gravity scattering amplitudes
  [Bern, Carrasco, Johansson ’08, ’10; Bern, Dennen, Huang, Kiermaier ’10]

Colour–kinematics duality: mysterious property of scattering amplitudes



Punch line I

Colour–kinematics duality: mysterious property of scattering amplitudes

$$A_{\mu}{}^{a}$$

Conventional (possibly anomalous) symmetry of BV/BRST action with kinematic (homotopy) Lie algebra derived from underlying (homotopy) $\mathrm{BV}^{\blacksquare}$-algebra

[Borsten, Jurčo, Kim, Macrelli, Saemann, Wolf (BJKMSW) '20, '21, '22]

Punch line II

Assuming colour-kinematics duality we can double copy scattering amplitudes

“gravity $=$ gauge $\times$ gauge”

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Action double copy and tensor product of $\mathrm{BV}^{\blacksquare}$-algebras: gravitational L_∞-algebra

[BJKMSW ‘20, ‘23; see also Bonezzi, Chiaffrino, Díaz–Jaramillo, and Hohm ’23]

Corollaries

Closed-form BV/BRST actions that manifest/establish colour-kinematics duality up to possible colour-kinematics duality anomalies [BJKMSW '22, '23]:

- Self-dual (super) Yang–Mills theories in $D = 4$ (twistors, susy $\Rightarrow$ anomaly free)
- (Super) Yang–Mills theories in all dimensions (twistors or pure spinors)
- M2-brane world–volume theories (pure spinors)

New double copy actions:

- Bi-form gravity in $D = 2 + 1$ (from double copy of Chern-Simons)
- Cubic pure spinor action for supergravity (from double copy of super Yang–Mills)

The Plan For Today

1. Homotopy algebras, quantum field theory and scattering amplitudes
   Batalin–Vilkovisky formalism, homotopy Lie algebras and minimal models

2. Colour-kinematics duality and the double copy
   Hidden property of gluon amplitudes: "gravity $=$ gauge $\times$ gauge"

3. Manifesting colour-kinematics duality in the Batalin–Vilkovisky formalism
   Colour-kinematics duality as conventional (possibly anomalous) symmetry

4. Colour-kinematics duality, double copy and (homotopy) $\mathrm{BV}^{\square}$ algebras
   Confluence and conclusion: colour-kinematics was always there (up to homotopy!)

5. Examples: Chern-Simons, (self-dual) super Yang–Mills and M2-branes
   New formulations: simple proofs of tree-level colour-kinematics duality

Homotopy algebras, quantum field theory and scattering amplitudes

Homotopy algebras

Consider a cochain complex $(C^\bullet, \mathsf{d})$

$$\cdots \xrightarrow{\quad \mathsf{d} \quad} C^i \xrightarrow{\quad \mathsf{d} \quad} C^{i+1} \xrightarrow{\quad \mathsf{d} \quad} C^{i+2} \xrightarrow{\quad \mathsf{d} \quad} \cdots$$

$\mathsf{d}^2 = 0$ with some compatible algebraic structure ("mulitplication" map m)

$$\mathsf{m} : C^i \times C^j \to C^{i+j}; \quad (x, y) \mapsto \mathsf{m}(x, y)$$

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$$\mathsf{dm}(x, y) = \mathsf{m}(\mathsf{d}x, y) + (-)^x \mathsf{m}(x, \mathsf{d}y)$$

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Example: Hodge–de Rham complex $\Omega^\bullet(M)$ of i-forms with exterior derivative

$$\mathsf{m}(A_i, A_j) = A_i \wedge A_j = (-)^{ij} A_j \wedge A_i, \quad \mathsf{d}(A_i \wedge A_j) = \mathsf{d}A_i \wedge A_j + (-)^i A_i \wedge \mathsf{d}A_j$$

is a differential graded commutative algebra (dgca)

Homotopy algebras

Given a morphism $\varphi : (C^\bullet, \mathrm{d}) \to (\tilde{C}^\bullet, \tilde{\mathrm{d}})$

$$
\begin{array}{ccccccccc}
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 & & \downarrow{\scriptstyle \varphi_i} & & \downarrow{\scriptstyle \varphi_{i+1}} & & \downarrow{\scriptstyle \varphi_{i+2}} & & \\
\cdots & \xrightarrow[\tilde{\mathrm{d}}]{} & \tilde{C}^i & \xrightarrow[\tilde{\mathrm{d}}]{} & \tilde{C}^{i+1} & \xrightarrow[\tilde{\mathrm{d}}]{} & \tilde{C}^{i+2} & \xrightarrow[\tilde{\mathrm{d}}]{} & \cdots
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Q: Can the algebraic structure m on $(C^\bullet, \mathrm{d})$ also be transferred to an algebraic structure $\tilde{\mathrm{m}}$ on $(\tilde{C}^\bullet, \tilde{\mathrm{d}})$?

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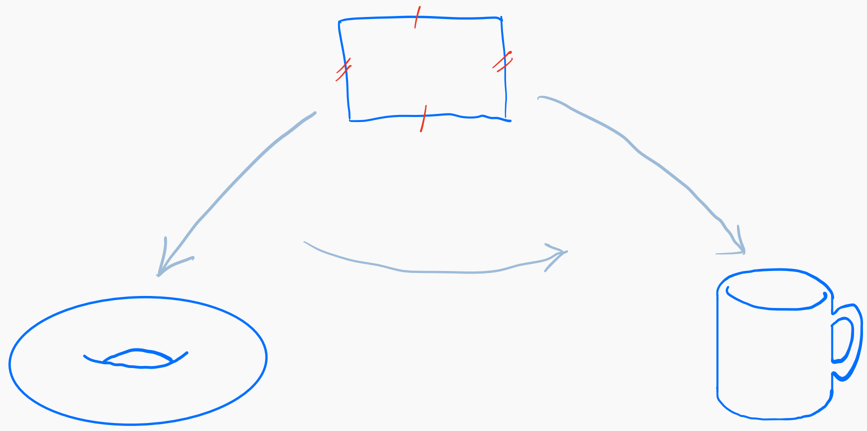
A: Yes, if we allow for a richer homotopy algebraic structure

Homotopy algebras

Algebraic identities (e.g. associativity, commutativity or Jacobi) hold only up to cochain homotopies



$\longrightarrow$ tower of higher products $\mathsf{d}(x) = \mathsf{m}_1(x), \mathsf{m}_2(x,y), \mathsf{m}_3(x,y,z), \ldots$

$$\mathsf{m}_n : C^{i_1} \times C^{i_2} \times \cdots \times C^{i_n} \to C^{i_1+i_2+\cdots i_n - n+2}$$

Informally: generalise familiar algebras to include higher products satisfying higher relations up to homotopies:

Associative algebras	$\longrightarrow$	homotopy associative A_∞-algebras **[Stasheff '63]**
Commutative algebras	$\longrightarrow$	homotopy commutative C_∞-algebras **[Kadeishvili '82]**
Lie algebras	$\longrightarrow$	homotopy Lie L_∞-algebras **[Zwiebach '93; Hinich, Schechtman '93]**

Homotopy Lie algebras

Lie algebra	L_∞-algebra
Vector space $\mathfrak{g} = V_0$	Graded vector space $\mathfrak{L} = \bigoplus_n V_n$
Bracket $\mu_2 = [-,-]$	Higher brackets $\mu_1 = [-],\ \mu_2 = [-,-],\ \mu_3 = [-,-,-], \ldots$
Relations Antisymmetry + Jacobi	Homotopy relations Graded antisymmetry + homotopy Jacobi

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Example: Semistrict Lie 2-algebra is a 2-term L_∞-algebra $\mathfrak{L} \cong V_{-1} \oplus V_0$ with

Differential $\mu_1 = [-]$; Lie bracket $\mu_2 = [-,-]$; Jacobiator $\mu_3 = [-,-,-]$.

$$[[x,y],z] + (-1)^{x(y+z)}[[y,z],x] + (-1)^{y(x+z)}[[x,z],y] = -[[x,y,z]]$$

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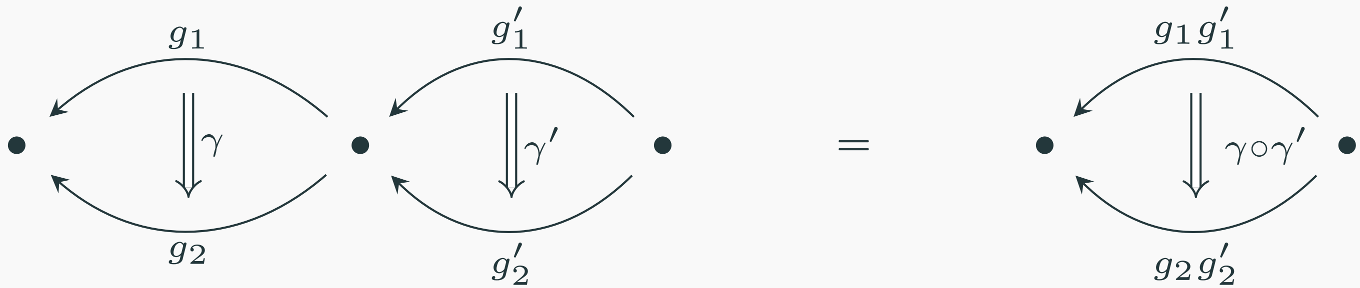
Integrate to Lie groups	Integrate to ∞-Lie Groups

Generalised symmetries increasingly prevalent in CMT, TQFT, QI, AdS/CFT ...

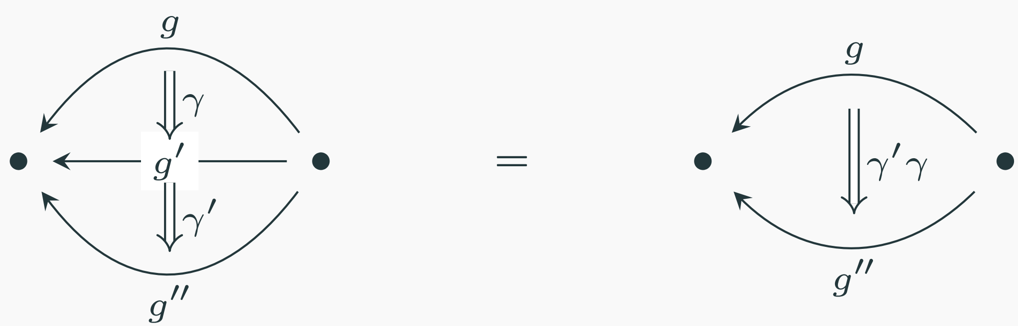
[Das,Gregory,Iqbal '21; Del Zotto,García Etxebarria,Schafer-Nameki '22; Etxebarria,Iqbal '22; Bhardwaj, Bullimore,Ferrari,Schafer-Nameki '22; Bartsch, Bullimore,Ferrari,Pearson '22; Bartsch,Bullimore,Grigoletto '23...]

Lie 2-Group

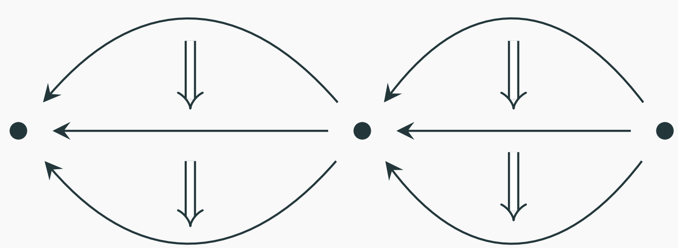
2-arrows form a group under horizontal composition



2-arrows form a groupoid under vertical composition



Interchange law: horizontal and vertical composition are coherent



Lie 2-group $\rightarrow$ Lie 2-algebra $\rightarrow$ strict 2-term L_∞-algebra

Homotopy Lie algebras: higher products and relations

Operadic definition: L_∞-algebras are given degree one differential derivations on $\mathcal{L}ie^!((V[1])^*)$ for some graded vector space V

Operads are the appropriate mathematical arena for constructing homotopy algebras

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Unpacking this definition: an L_∞-algebra $\mathfrak{L}$ is a graded vector space $V \cong \bigoplus_i V_i$ together with graded anti-symmetric i-linear maps

$$\mu_i : V \times \cdots \times V \to V$$

of degree $2-i$ that satisfy the homotopy Jacobi identities

$$\sum_{\substack{i=j+k \\ \sigma \in \overline{\mathrm{Sh}}(j,k;i)}} (-1)^k \chi_{(\sigma;v_1,\ldots,v_i)} \mu_{k+1}(\mu_j(v_{\sigma(1)},\ldots,v_{\sigma(j)}), v_{\sigma(j+1)},\ldots,v_{\sigma(i)}) = 0$$

Homotopy Lie algebras: higher products and relations

The first three homotopy Jacobi identities are

$$\mu_1(\mu_1(v_1)) = 0$$

$$\mu_1(\mu_2(v_1, v_2)) = \mu_2(\mu_1(v_1), v_2) + (-1)^{|v_1|}\mu_2(v_1, \mu_1(v_2))$$

$$\begin{aligned}\mu_2(\mu_2(v_1, v_2), v_3) + (-1)^{|v_1|\,|v_2|}\mu_2(v_2, \mu_2(v_1, v_3)) - \mu_2(v_1, \mu_2(v_2, v_3)) \\ = \mu_1(\mu_3(v_1, v_2, v_3)) + \mu_3(\mu_1(v_1), v_2, v_3) + (-1)^{|v_1|}\mu_3(v_1, \mu_1(v_2), v_3) \\ + (-1)^{|v_1|+|v_2|}\mu_3(v_1, v_2, \mu_1(v_3))\end{aligned}$$

- The unary product μ_1 is a differential and a derivation with respect to the binary product μ_2
- The ternary product μ_3 captures the failure of the binary product μ_2 to satisfy the standard Jacobi identity

Homotopy Lie algebras: quasi-isomorhisms

Morphisms of L_∞-algebras are families of i-linear maps

$$\phi : \mathfrak{L} \to \tilde{\mathfrak{L}}, \qquad \phi_i : \mathfrak{L} \times \cdots \times \mathfrak{L} \to \tilde{\mathfrak{L}}$$

that are functorial, e.g. $\phi_1 \circ \mu_1 = \tilde{\mu}_1 \circ \phi_1$

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Quasi-isomorphisms are morphisms that induce isomorphisms on the μ_1-cohomologies

$$\phi_1 : H^\bullet_{\mu_1}(V) \xrightarrow{\sim} H^\bullet_{\tilde{\mu}_1}(\tilde{V})$$

Homotopy Lie algebras: structure theorems

Strictification (retification) theorem:

Every $\mathfrak{L}$ is quasi-isomorphic to an $\tilde{\mathfrak{L}}$ with $\tilde{\mu}_i = 0$ for all $i > 2$

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Special deformation retract of complexes

$$\mathsf{h} \circlearrowright (V, \mu_1) \underset{\mathsf{e}}{\overset{\mathsf{p}}{\rightleftarrows}} (H^\bullet_{\mu_1}(V), \tilde{\mu}_1 = 0)\ , \qquad 1 = \mu_1 \mathsf{h} + \mathsf{h}\mu_1 + \mathsf{e} \circ \mathsf{p}$$

Homological perturbation lemma (we can perturb the differential to include nonlinear terms) determines higher products of minimal model recursively from $\phi_1 = \mathsf{e}$

$$\begin{aligned}
\tilde{\mu}_1(\tilde{v}) &= 0 \\
\tilde{\mu}_2(\tilde{v}_1, \tilde{v}_2) &= \mathsf{p}\mu_2(\mathsf{e}(\tilde{v}_1), \mathsf{e}(\tilde{v}_2)) \\
\tilde{\mu}_3(\tilde{v}_1, \tilde{v}_2, \tilde{v}_3) &\sim \mathsf{p}\mu_3(\mathsf{e}(\tilde{v}_1), \mathsf{e}(\tilde{v}_2), \mathsf{e}(\tilde{v}_2)) + \mathsf{p}\mu_2(\mathsf{h}\mu_2(\mathsf{e}(\tilde{v}_1), \mathsf{e}(\tilde{v}_2)), \mathsf{e}(\tilde{v}_3)) + \cdots
\end{aligned}$$

Homotopy Lie algebras and the BV formalism

Given differential graded Lie algebra $(\mathfrak{g}, \mathsf{d})$ with inner product (cf. Cartan-Killing form)

$$\langle x, \mathsf{d}y \rangle = (-)^{1+x+y+xy} \langle y, \mathsf{d}x \rangle, \qquad \langle x, [y, z] \rangle = (-)^{z(x+y)} \langle z, [x, y] \rangle$$

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$$f = da + \tfrac{1}{2}[a, a] = 0, \qquad S_{\mathrm{MC}} = \tfrac{1}{2}\langle a, \mathrm{d}a \rangle + \tfrac{1}{3!}\langle a, [a, a] \rangle$$

Covariant derivative, Bianchi identity and gauge transformations:

$$Dx = \mathsf{d}x + [a, x], \quad Df = 0, \quad \delta_c a = Dc$$

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Given L_∞-algebra $(\mathfrak{L}, \mu_i)$ with cyclic structure

$$\langle x_1, \mu_i(x_2, \dots x_{i+1})\rangle = (-)^{i+i(x_1+x_{i+1})+x_{i+1}\sum_{j=1}^{i} x_j}\langle x_{i+1}, \mu_i(x_1, \dots x_i)\rangle$$

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$$F = \sum_k \tfrac{1}{k!}\mu_k(a,a,\ldots,a) = 0, \qquad S_{\mathrm{hMC}} = \sum_k \tfrac{1}{(k+1)!}\langle a, \mu_k(a,a,\ldots,a)\rangle$$

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$$Dx = \sum_k \frac{(-1)^k}{k!}\mu_{k+1}(x, a, \dots, a), \quad DF = 0, \quad \delta_c a = Dc$$

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- Yang-Mills theory $\mathfrak{L}^{\mathsf{YM}} = (V^{\mathsf{YM}}, \mu_i)$

$$
\begin{array}{ccccccc}
V_0^{\mathsf{YM}} & \oplus & V_1^{\mathsf{YM}} & \oplus & V_2^{\mathsf{YM}} & \oplus & V_3^{\mathsf{YM}} \\
c & \xrightarrow{\mu_1 = d} & A & \xrightarrow{d^\dagger d} & A^+ & \xrightarrow{d^\dagger} & c^+ \\
 & & b & \xrightarrow{\mathsf{Id}} & \bar{c} & & \\
 & & \bar{c}^+ & \xrightarrow{-\mathsf{Id}} & b^+ & &
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$$S_{\mathsf{BV}}^{\mathsf{YM}} = \mathrm{tr} \int A \wedge \star\, d^\dagger dA + A \wedge \star \mu_2(A, A) + \cdots$$

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- Colour-stripping: $\mathfrak{L}^{\mathsf{YM}} = \mathfrak{g} \otimes \mathfrak{C}^{\mathsf{YM}} \leftarrow$ Yang–Mills C_∞-algebra

- Physical equivalence (field redefinitions etc): L_∞ quasi-isomorphisms
  See [Jurčo-Raspollini-Saemann-Wolf '18; Jurčo-Macrelli-Saemann-Wolf '19; BJKMSW '20 '23]

Homotopy Lie algebras and scattering amplitudes

Now consider the minimal model $(V^{\mathsf{theory}}, \mu_i) \cong (H^\bullet_{\mu_1}(V^{\mathsf{theory}}), \tilde{\mu}_1 = 0, \tilde{\mu}_i)$

$$\mathsf{h} \circlearrowright (V, \mu_1) \underset{\mathsf{e}}{\overset{\mathsf{p}}{\rightleftarrows}} (H^\bullet_{\mu_1}(V), \tilde{\mu}_1 = 0) \ , \qquad 1 = \mu_1 \mathsf{h} + \mathsf{h}\mu_1 + \Pi$$

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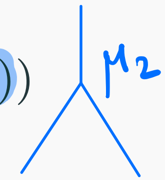
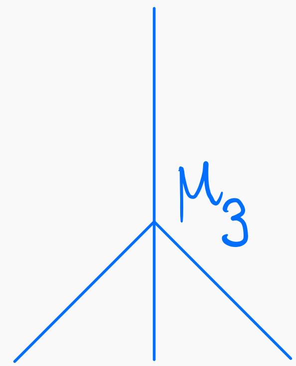
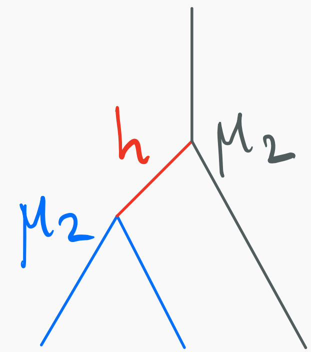
$$\Rightarrow \mathsf{h} = \text{propergator}$$

Homological perturbation lemma yields higher brackets as Feynman diagram expansion

$$\tilde{\mu}_2(\tilde{v}_1, \tilde{v}_2) = \mathsf{p}\mu_2(\mathsf{e}(\tilde{v}_1), \mathsf{e}(\tilde{v}_2))$$



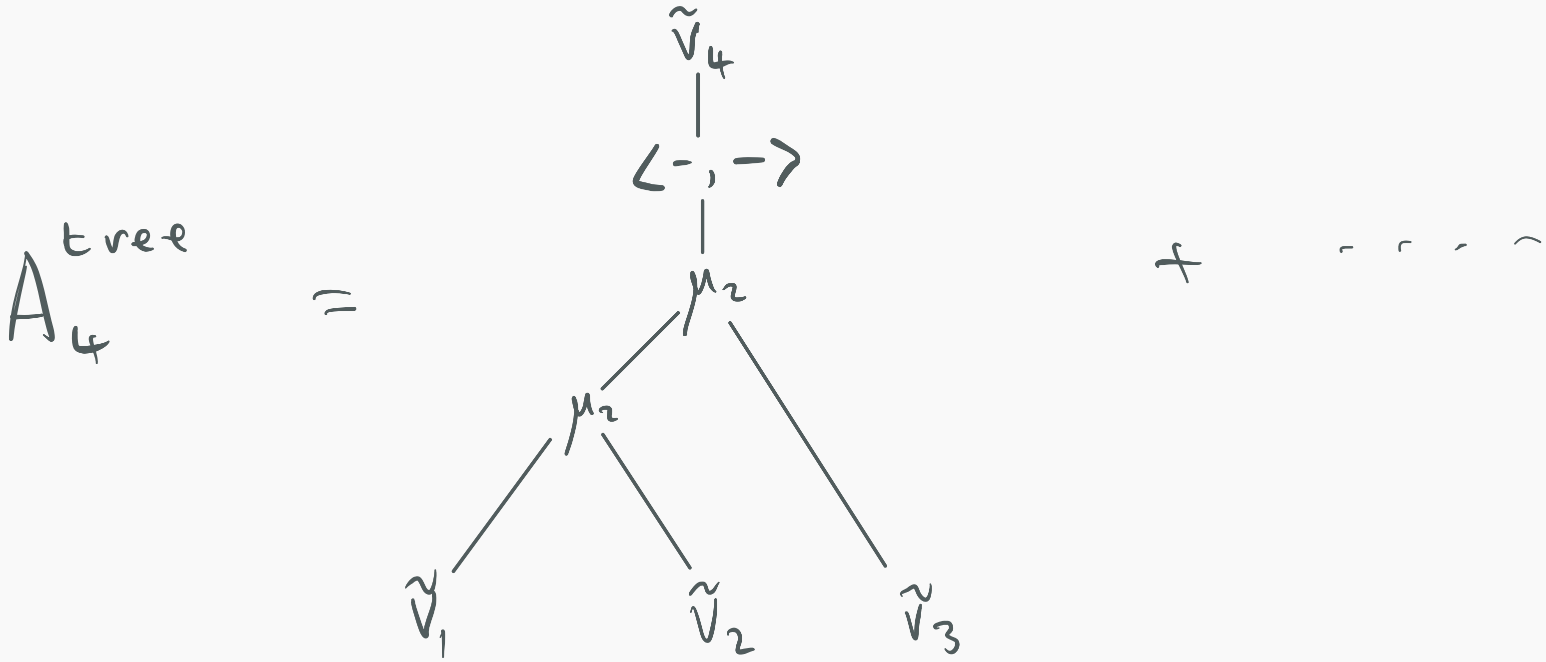




$$\tilde{\mu}_3(\tilde{v}_1, \tilde{v}_2, \tilde{v}_3) \sim \mathsf{p}\mu_3(\mathsf{e}(\tilde{v}_1), \mathsf{e}(\tilde{v}_2), \mathsf{e}(\tilde{v}_2)) + \mathsf{p}\mu_2(\mathsf{h}\mu_2(\mathsf{e}(\tilde{v}_1), \mathsf{e}(\tilde{v}_2)), \mathsf{e}(\tilde{v}_3)) + \cdots$$

Cyclic structure gives tree-level amplitudes

$$A_n^{\text{tree}}(\tilde{v}_1, \tilde{v}_2, \dots \tilde{v}_n) = \langle \tilde{v}_1, \tilde{\mu}_{n-1}(\tilde{v}_2, \dots, \tilde{v}_n) \rangle$$



Actions and amplitudes are unified as quasi-isomorphic L_∞-algebras

Colour-kinematics duality and the double copy

Colour-kinematics duality

Amplitude for gluons to scatter schematically:

Colour numerators $c_i \sim f_{ab}{}^{c} f_{cd}{}^{e}$

Kinematic numerators $n_i \sim \varepsilon_\mu p^\mu + \cdots$

$$A^{n,L}_{\mathsf{YM}} = \sum_{i \in \text{cubic diag}} \frac{1}{S_i} \int_L \frac{c_i n_i}{d_i}$$

Bern-Carrasco-Johansson colour-kinematics duality conjecture 2008:

$$c_i + c_j + c_k = 0 \Rightarrow n_i + n_j + n_k = 0$$

Proven at tree level [Stieberger '09; Bjerrum, Bohr, Damgaard, Vanhove '09; Du, Teng '16; Bridges, Mafra '19; Mizera '19; Reiterer '19. . .]

Conjectured at loop level with highly non-trivial examples [Bern, Carrasco, Johansson '08 '10; Carrasco, Johansson '11; Bern, Davies, Dennen, Huang, Nohle '13; Bern, Davies, Dennen '14. . .]

Colour-kinematics duality

Assuming colour-kinematics duality is realised, gravity comes for free:

$$A_{\text{glouns}} \sim \sum \int \frac{c_i n_i}{d_i}$$

$$c_i \longrightarrow n_i$$

double copy kinematics

$$A_{\text{gravitons}} \sim \sum \int \frac{n_i n_i}{d_i}$$

[Bern, Carrasco, Johansson '08, '10; Bern, Dennen, Huang, Kiermaier '10]

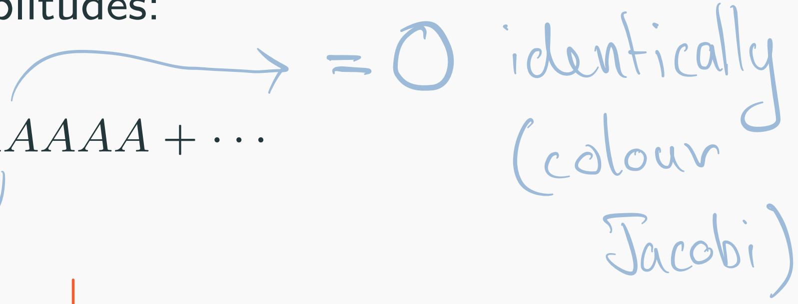
'Gluons for (almost) nothing, gravitons for free' JJ Carrasco

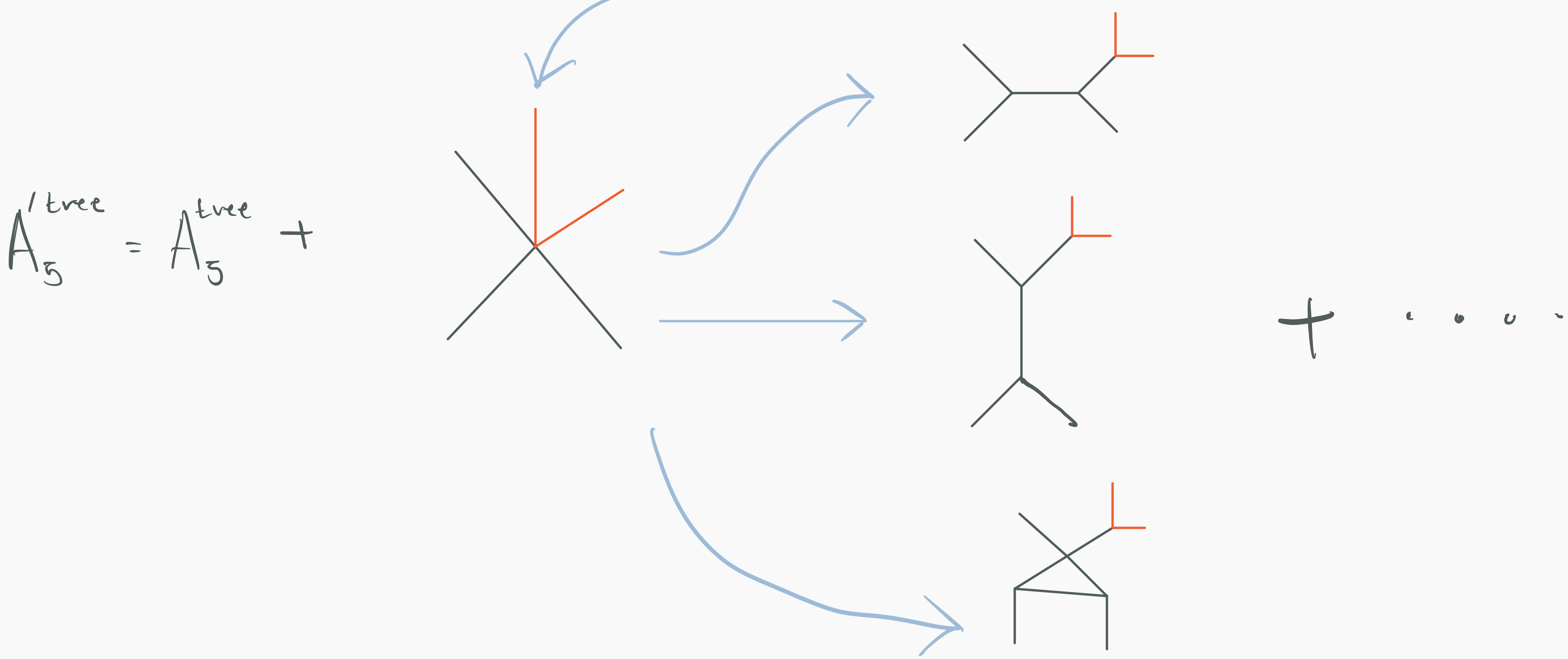
Manifesting colour-kinematics duality in the Batalin–Vilkovisky formalism

Manifest colour-kinematics duality of tree-level physical S-matrix

There is a Yang–Mills action such that the Feynman diagrams yield amplitudes manifesting colour-kinematics duality for tree-level amplitudes:

$$A\Box A + \partial AAA + \frac{\Box}{\Box}AAAA + \frac{\partial^3}{\Box^2}AAAAA + \cdots$$





[Bern, Dennen, Huang, Kiermaier '10; Tolotti, Weinzierl '13]

Manifest colour-kinematics duality of tree-level physical S-matrix

This can be strictified to have only cubic interactions through infinite tower of auxiliaries [Bern, Dennen, Huang, Kiermaier '10; Tolotti, Weinzierl '13; BJKMSW '21]

$$
\begin{aligned}
S^{\text{YM}}_{\text{on-shell CK}} = \operatorname{tr}\int & d^D x \tfrac{1}{2} A_\mu \Box A^\mu + \tfrac{1}{2} g \partial_\mu A_\nu [A^\mu, A^\nu] \\
& + \tfrac{1}{2} B^{\mu\nu\kappa} \Box B_{\mu\nu\kappa} - g(\partial_\mu A_\nu + \tfrac{1}{\sqrt{2}} \partial^\kappa B_{\kappa\mu\nu})[A^\mu, A^\nu] \\
& + C^{\mu\nu} \Box \bar{C}_{\mu\nu} + C^{\mu\nu\kappa} \Box \bar{C}_{\mu\nu\kappa} + C^{\mu\nu\kappa\lambda} \Box \bar{C}_{\mu\nu\kappa\lambda} + \\
& + g C^{\mu\nu} [A_\mu, A_\nu] + g \partial_\mu C^{\mu\nu\kappa} [A_\nu, A_\kappa] - \tfrac{g}{2} \partial_\mu C^{\mu\nu\kappa\lambda} [\partial_{[\nu} A_{\kappa]}, A_\lambda] \\
& + g \bar{C}^{\mu\nu} \big(\tfrac{1}{2} [\partial^\kappa \bar{C}_{\kappa\lambda\mu}, \partial^\lambda A_\nu] + [\partial^\kappa \bar{C}_{\kappa\lambda\nu\mu}, A^\lambda] \big) + \cdots
\end{aligned}
$$

Purely cubic colour-kinematics duality manifesting Feynman diagrams:

$$
A^{n,0}_{\text{YM}} = \sum_i \frac{c_i n_i}{d_i} \quad \text{s.t.} \quad c_i + c_j + c_k = 0 \Rightarrow n_i + n_j + n_k = 0
$$

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To lift to loop-level we should include off-shell unphysical/ghost modes in the external states so that we can glue trees into loops

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Include off-shell unphysical/ghost modes in the external states, the full BRST-extended state space

$$({A_\mu}^a, b^a, c^a, \bar{c}^a)$$

[Anastasiou, LB, Duff, Hughes, Nagy, Zoccali '14 '18; LB, Nagy '20; BJKMSW '20, '21, '22]

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$$
\begin{aligned}
S^{\text{YM}}_{\text{BRST-extended CK}} &= S^{\text{YM}}_{\text{on-shell CK}} + \int d^D x \tfrac{1}{2} b_a \Box b^a - \bar{c}_a \Box c^a \\
&- K^{\mu}_{1a} \Box \bar{K}^{1a}_{\mu} - K^{\mu}_{2a} \Box \bar{K}^{2a}_{\mu} - g f_{abc} \bar{c}^a \partial^\mu (A^b_\mu c^c) \\
&- \tfrac{1}{2} B^{\mu\nu\kappa}_a \Box B^a_{\mu\nu\kappa} + g f_{abc} \Big(\partial_\mu A^a_\nu + \tfrac{1}{\sqrt{2}} \partial^\kappa B^a_{\kappa\mu\nu} \Big) A^{\mu b} A^{\nu c} \\
&- g f_{abc} \Big\{ K^{a\mu}_1 (\partial^\nu A^b_\mu) A^c_\nu + [(\partial^\kappa A^a_\kappa) A^{b\mu} + \bar{c}^a \partial^\mu c^b] \bar{K}^{1c}_\mu \Big\} \\
&+ g f_{abc} \Big\{ K^{a\mu}_2 \Big[(\partial^\nu \partial_\mu c^b) A^c_\nu + (\partial^\nu A^b_\mu) \partial_\nu c^c \Big] + \bar{c}^a A^{b\mu} \bar{K}^{2c}_\mu \Big\} + \cdots
\end{aligned}
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- Strictifying yields cubic actions with tower of auxiliary fields
- Note, arguments apply to any theory with tree-level physical S-matrix colour-kinematics duality $\Rightarrow$

  $\exists$ on-shell BRST-extended "amplitudes" obeying colour-kinematics duality

Manifest colour-kinematics duality of off-shell BRST action

Off-shell momenta $p^2 \neq 0$: resulting CK duality violations are compensated by vertices $f\Box\phi$ generated by generically non-local field redefinitions:

$$\phi \mapsto \phi + f(\phi), \qquad \phi\Box\phi \mapsto \phi\Box\phi + f\Box\phi + \cdots$$

Cubic Feynman rules yield colour-kinematics duality manifesting loop amplitude integrands automatically! [BJKMSW '21]

We're finished, aren't we...

Manifest colour-kinematics duality of off-shell BRST action

Off-shell momenta $p^2 \neq 0$: resulting CK duality violations are compensated by vertices $f\Box\phi$ generated by generically non-local field redefinitions:

$$\phi \mapsto \phi + f(\phi), \qquad \phi\Box\phi \mapsto \phi\Box\phi + f\Box\phi + \cdots$$

Cubic Feynman rules yield colour-kinematics duality manifesting loop amplitude integrands automatically! [BJKMSW '21]

We're finished, aren't we...

...Jacobian determinants → counterterms ensuring unitarity

$$\det\left(\mathbb{1} + \frac{\delta f(\phi)}{\delta\phi}\right) = \int \mathcal{D}\bar{\chi}\,\mathcal{D}\chi\,\mathrm{e}^{\frac{\mathrm{i}}{\hbar}\int\left(\bar{\chi}_I\chi^I + \bar{\chi}_I\frac{\delta f^I}{\delta\phi^J}\chi^J\right)}$$

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Pure Yang–Mills: two-loop colour-kinematics duality with local lorentz-covariant cubic Feynman rule compatible numerators is impossible [Bern, Davies, Nohle '15]

We now understand this failure as a colour-kinematics duality anomaly [BJKMSW '21]

Colour-kinematics duality as a conventional (possibly anomalous) symmetry

CK duality can be realised as an infinite dimensional anomalous symmetry of Yang–Mills BRST action [BJKMSW '20, '21, '22]

$$C_{ij} c_{ab} A^{ia} \Box A^{jb} + F_{ijk} f_{abc} A^{ia} A^{jb} A^{kc}$$

$$c_{ab} = c_{(ab)} \qquad f_{abc} = f_{[abc]} \qquad c_{a(b} f^{a}_{c)d} = 0 \qquad f_{[ab|d} f^{d}_{c]e} = 0$$

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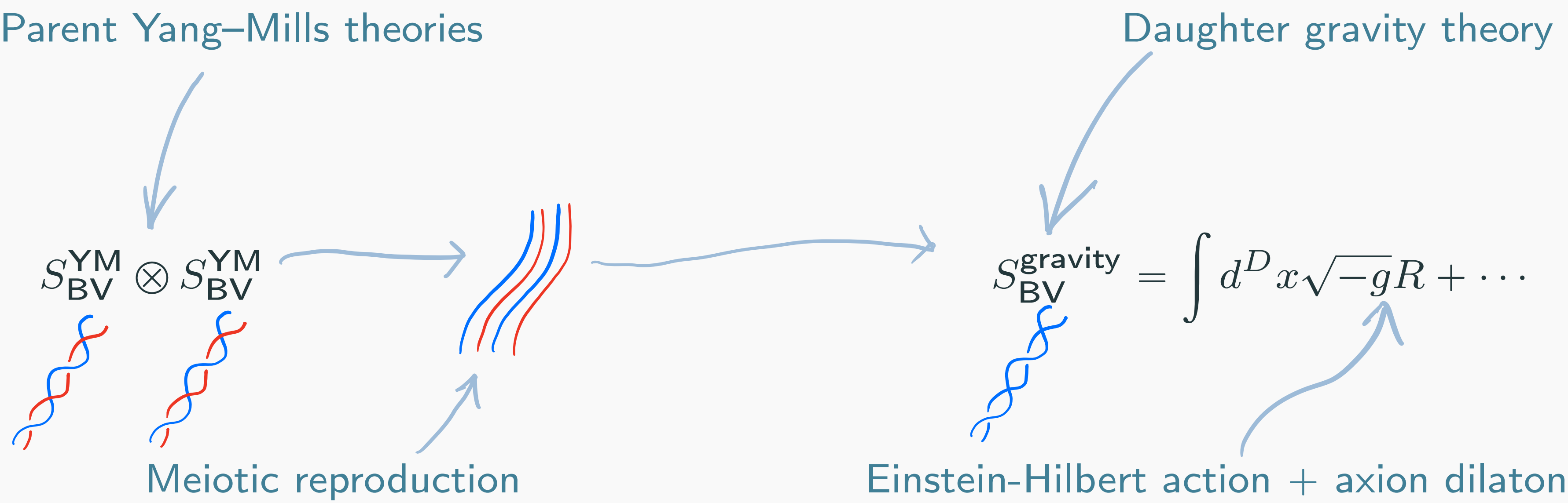
- Agrees with constraints from 2-loop Yang–Mills amplitudes [Bern, Davies, Nohle, '15]

BRST/BV action double copy **[LB, Nagy '20; BJKMSW '20; '21, '22, '23]**

$$C_{ij}c_{ab}A^{ia}\Box A^{ja} + F_{ijk}f_{abc}A^{ia}A^{jb}A^{kc} \rightarrow C_{ij}\tilde{C}_{\tilde{\imath}\tilde{\jmath}}A^{i\tilde{\imath}}\Box A^{j\tilde{\jmath}} + F_{ijk}\tilde{F}_{\tilde{\imath}\tilde{\jmath}\tilde{k}}A^{i\tilde{\imath}}A^{j\tilde{\jmath}}A^{k\tilde{k}}$$



Pertubative quantum gravity + axion-dilaton is the double copy of Yang–Mills (but counter terms for unitarity required)! **[BJKMSW '20]**

Questions

Okay, but. . .

- Proof is constructive and inductive: no theoretical understanding/control over higher vertices or the set of auxiliary fields with cubic interactions
- No closed form of colour-kinematics duality manifesting action
- No clue (generically) about the kinematic Lie algebra
- May need non-local field redefinitions ⇒ colour-kinematics duality anomaly
- Double copy is mathematically opaque

So we'd like. . .

- a clear mathematical characterisation of higher vertices
- a closed form colour-kinematics duality manifesting action
- to avoid the need for non-local field redefinitions ⇒ perfect all-loop colour-kinematics duality
- an understanding of kinematic Lie algebra
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Colour-kinematics duality, double copy and (homotopy) BV$^{\square}$ algebras

A clear mathematical characterisation of higher vertices: $BV^{■}_{\infty}$-algebras

Reiterer '18: Colour-kinematics duality of physical tree-level S-matrix is equivalent to a $BV^{■}_{\infty}$-algebra (deformation of BV_{∞}-algebras of [Galvez–Carrillo, Tonks, Vallette '09])

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Theory with kinematic Lie algebra $\Leftrightarrow \mathfrak{C} = \mathfrak{B}$ a $\mathrm{BV}^{\blacksquare}_{\infty}$-algebra [BJKMSW '21, '22]

$$\mathfrak{L} = \mathfrak{g} \otimes \mathfrak{C} = \mathfrak{g} \otimes \mathfrak{B} \quad (\mathfrak{C} \text{ is colour-stripped } C_\infty\text{-algebra})$$

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Higher products rough split into three types of vertices we introduced:

m^0_i	Colour-stripped vertices of gauge-fixed action for BRST colour-kinematics duality
$m^0_{i,j}$	Tolotti-Weinzerl corrections for tree on-shell colour-kinematics duality
$m^0_{i,j,k}$	Field red. vertices correcting for off-shell colour-kinematics duality

"homotopy Jacobi relations $\Leftrightarrow$ colour-kinematics duality "

See also [Bonezzi, Chiaffrino, Díaz–Jaramillo, and Hohm '23]

Perfect colour-kinematics duality

A strict BV$^{\square}$-algebra $\mathfrak{B}$ is dgca $(V, \mathsf{d}, \mathsf{m})$ with $\mathsf{b} : V \to V$ such that

$$\mathsf{b}^2 = 0, \qquad \square \; := \; [\mathsf{d}, \mathsf{b}] \; = \; \mathsf{d} \circ \mathsf{b} + \mathsf{b} \circ \mathsf{d}$$

and b is second order w.r.t $\mathsf{m}(-,-)$ so that

$$[x, y] = \mathsf{bm}(x, y) - \mathsf{m}(\mathsf{b}x, y) - (-1)^{x}\mathsf{m}(x, \mathsf{b}y)$$

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is a (shifted) Lie bracket: the kinematic Lie algebra

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$$\mathsf{h} \;=\; \mathsf{id}_{\mathfrak{g}} \otimes \frac{\mathsf{b}}{\blacksquare} \qquad \Rightarrow \qquad \mathrm{id}_{\mathfrak{L}} - \Pi \;=\; \mathsf{d} \circ \mathsf{h} + \mathsf{h} \circ \mathsf{d} = \mathsf{d} \circ \mathsf{b} + \mathsf{b} \circ \mathsf{d} = \blacksquare$$

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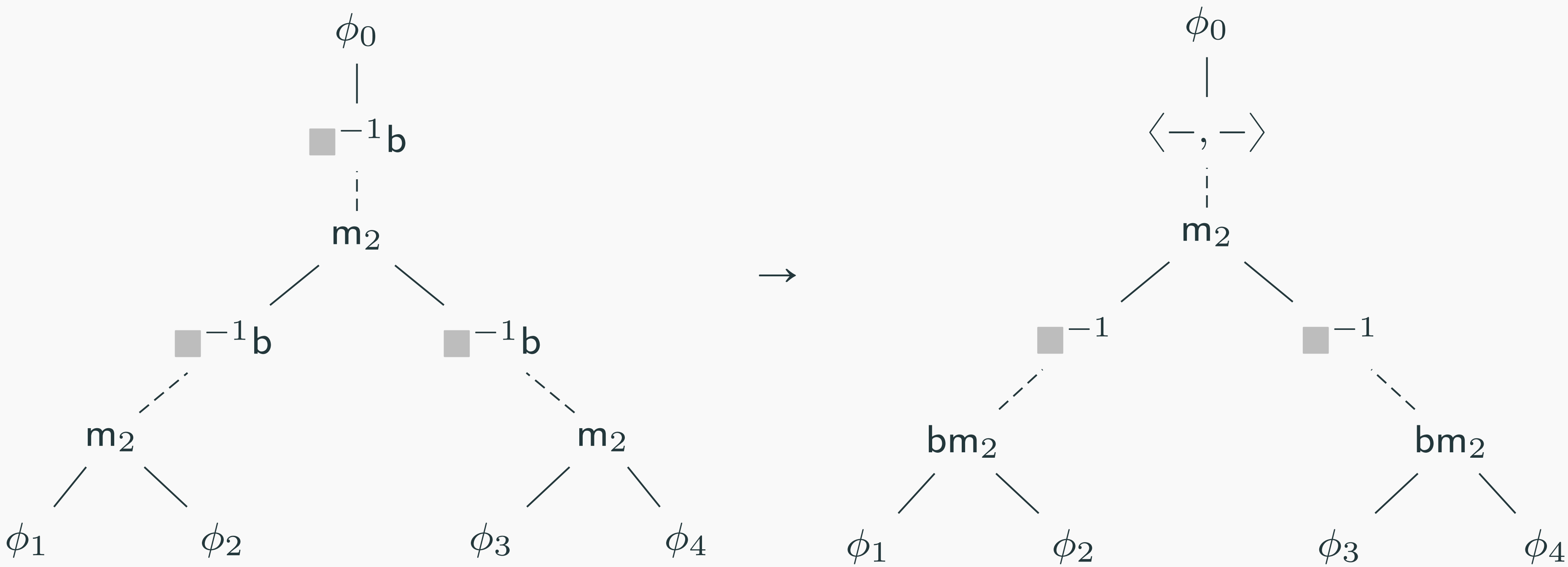

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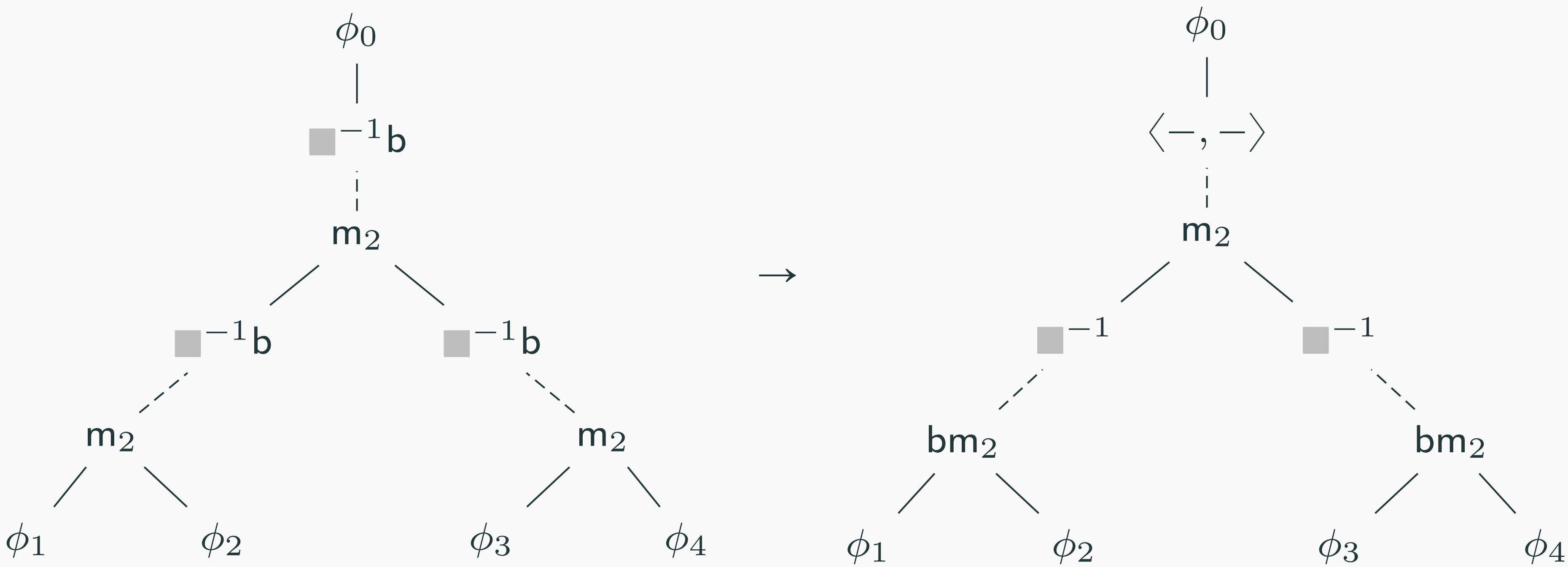

$$\langle \phi_0, \mathsf{m}(T_1, \mathsf{bm}(T_2, T_3)) \rangle + \langle \phi_0, \mathsf{m}(T_2, \mathsf{bm}_2(T_3, T_1)) \rangle + \langle \phi_0, \mathsf{m}(T_3, \mathsf{bm}(T_1, T_2)) \rangle = 0$$

$\mathsf{bm}(-,-) = [-,-]$ since "fields $= \mathrm{im}(\mathsf{b}) = \ker(\mathsf{b})$" post gauge-fixing

The double copy as restricted tensor product

$$C_{ij} c_{ab} A^{ia} \Box A^{ja} + F_{ijk} f_{abc} A^{ia} A^{jb} A^{kc} \to C_{ij} \tilde{C}_{\tilde{\imath}\tilde{\jmath}} A^{i\tilde{\imath}} \Box A^{j\tilde{\jmath}} + F_{ijk} \tilde{F}_{\tilde{\imath}\tilde{\jmath}\tilde{k}} A^{i\tilde{\imath}} A^{j\tilde{\jmath}} A^{k\tilde{k}}$$

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but the tensor product of differential graded commutative algebras is again a differential graded commutative algebra (i.e. not an L_∞-algebra as required)

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Let $\mathfrak{H}$ be a restrictedly tensorable cocommutative Hopf algebra. Furthermore, let $\mathfrak{B}_{\mathrm{L}} = (\mathfrak{B}_{\mathrm{L}}, \mathsf{d}_{\mathrm{L}}, \mathsf{m}_{\mathrm{L}}, \mathsf{b}_{\mathrm{L}})$ and $\mathfrak{B}_{\mathrm{R}} = (\mathfrak{B}_{\mathrm{R}}, \mathsf{d}_{\mathrm{R}}, \mathsf{m}_{\mathrm{R}}, \mathsf{b}_{\mathrm{R}})$ be two gauge-fixed $\mathrm{BV}^{\blacksquare}$-algebras over $\mathfrak{H}$ with $\blacksquare_{\mathrm{L}} = \blacksquare_{\mathrm{R}} = \blacksquare \in \mathfrak{H}$ and let $\hat{\mathfrak{B}} = (\hat{\mathfrak{B}}, \hat{\mathsf{d}}, \hat{\mathsf{m}}_2, \hat{\mathsf{b}}_-)$ be the restricted tensor product over $\mathfrak{H}$. The syngamy of $\mathfrak{B}_{\mathrm{L}}$ and $\mathfrak{B}_{\mathrm{R}}$ is the restricted kinematic dg Lie algebra $\mathfrak{Kin}^0(\hat{\mathfrak{B}})$

$$\mathsf{b} = \mathsf{b}_{\mathrm{L}} \otimes \mathsf{id} + \mathsf{id} \otimes \mathsf{b}_{\mathrm{R}} \quad \text{and} \quad \mu_2 = [-,-]_{\mathrm{L}} \otimes \mathsf{m}_{\mathrm{R}} + \mathsf{m}_{\mathrm{L}} \otimes [-,-]_{\mathrm{R}}$$

See also [Bonezzi, Chiaffrino, Díaz–Jaramillo, and Hohm '23]

Examples: Chern-Simons theory, (self-dual) super Yang–Mills theory and M2-brane models

The Chern-Simons paradigm

Chern–Simons theory has off–shell CK duality $\Rightarrow$ Chern–Simons has a BV$^{\square}$-algebra

[Ben–Shahar, Johansson '21; BJKMSW '22]

$$\mathfrak{L}^{\mathsf{CS}} = \Omega^0(M)\otimes\mathfrak{g} \xrightarrow{\mu_1=\mathrm{d}\otimes\mathrm{id}_{\mathfrak{g}}} \Omega^1(M)\otimes\mathfrak{g} \xrightarrow{\mathrm{d}\otimes\mathrm{id}_{\mathfrak{g}}} \Omega^2(M)\otimes\mathfrak{g} \xrightarrow{\mathrm{d}\otimes\mathrm{id}_{\mathfrak{g}}} \Omega^3(M)\otimes\mathfrak{g}$$

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$$S^{\mathsf{CS}}_{\mathsf{BV}} = \int \mathrm{tr}\Big(\tfrac{1}{2}A\wedge\mathrm{d}A + \tfrac{1}{3!}A\wedge[A,A] + A^+\wedge(\mathrm{d}c+[A,c]) + \tfrac{1}{2}c^+\wedge[c,c]\Big)$$

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$$\mathfrak{B}^{\mathsf{CS}} = \quad \Omega^0 \underset{\mathrm{d}^\dagger}{\overset{\mathrm{d}}{\rightleftarrows}} \Omega^1 \underset{\mathrm{d}^\dagger}{\overset{\mathrm{d}}{\rightleftarrows}} \Omega^2 \underset{\mathrm{d}^\dagger}{\overset{\mathrm{d}}{\rightleftarrows}} \Omega^3$$

$$\mathsf{d}A = \mathrm{d}A, \quad \mathsf{b}A = \mathrm{d}^\dagger A, \quad \mathsf{m}(A,B) = A\wedge B$$

$$\mathrm{d}\mathrm{d}^\dagger + \mathrm{d}^\dagger\mathrm{d} = \square$$

The Chern-Simons paradigm

Kinematic Lie algebra given by derived bracket

$$(-1)^p[\alpha,\beta] = -\mathrm{d}^\dagger(\alpha\wedge\beta) + \mathrm{d}^\dagger\alpha\wedge\beta + (-1)^p\alpha\wedge\mathrm{d}^\dagger\beta$$

is Schouten–Nijenhuis algebra of totally antisymmetric tensor fields, the natural Gerstenhaber algebra on three-dimensional Minkowski space [BJKMSW '22]

Restrciting to fields yields diffeomorphism algebra identified in [Ben–Shahar, Johansson '21]

The Chern-Simons paradigm

Kinematic Lie algebra given by derived bracket

$$(-1)^p[\alpha, \beta] = -\mathrm{d}^\dagger(\alpha \wedge \beta) + \mathrm{d}^\dagger\alpha \wedge \beta + (-1)^p \alpha \wedge \mathrm{d}^\dagger\beta$$

is Schouten–Nijenhuis algebra of totally antisymmetric tensor fields, the natural Gerstenhaber algebra on three-dimensional Minkowski space [BJKMSW '22]

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Look for Chern-Simons-type actions!

Holomorphic Chern-Simons theory on twistor space

Super self-dual Yang–Mills theory is equivalent to holomorphic Chern–Simons theory on super twistor space $Z \cong \mathbb{R}^{4|8} \times \mathbb{C}P^1$ with local coordinates $(x^\mu, \eta^i, \lambda^\alpha)$

$$S_{\mathsf{hCS}} = \int \Omega \wedge \operatorname{tr}\Big(\tfrac{1}{2} A \wedge \bar{\partial}_{\text{red}} A + \tfrac{1}{3!} A \wedge [A, A] + A^+ \wedge (\bar{\partial}_{\text{red}} c + [A, c]) + \tfrac{1}{2} c^+ \wedge [c, c]\Big),$$

$$\bar{\partial}_{\text{red}} = \hat{e}^\alpha \hat{E}_\alpha + \hat{e}^0 \hat{E}_0, \qquad \mathsf{b} = -\frac{4}{|\lambda|^2} \varepsilon^{\alpha\beta} \iota_{E_\alpha} \iota_{\hat{E}_\beta} \partial_{\text{red}} + 2\varepsilon^{\alpha\beta} \iota_{\hat{E}_\alpha} \iota_{\hat{E}_\beta} \hat{e}^0 \wedge$$

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- "Kaluza–Klein" expansion on $\mathbb{C}P^1$ gives infinite tower of auxiliary fields required for colour-kinematics duality

$$A^{a}(x, \eta, \lambda) \sim A(x, \eta)^{a} + A(x, \eta)^{\alpha a} \lambda_{\alpha} + A(x, \eta)^{\alpha\beta a} \lambda_{\alpha} \lambda_{\beta} + \cdots \leftrightarrow A^{ia}$$

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- Integrate out $\Rightarrow$ $\text{BV}^{\Box}_{\infty}$-algebra (cf. [Bonezzi, Diaz–Jaramillo, Nagy '23])
- Manifest kinematic Lie algebra and all-point all-loop order colour-kinematics duality for maximally supersymmetric case (but only 1-loop non-trivial!)
- Gauging away $\iota_{\hat{E}_0} A$ reproduces the kinematic Lie algebra of area-preserving diffeomorphisms on $\mathbb{C}^2$ identified in [Monteiro, O'Connell '11] (cf. [Bonezzi, Diaz–Jaramillo, Nagy '23])

Holomorphic Chern-Simons theory on ambitwistor space

Super Yang–Mills theory is equivalent to holomorphic Chern–Simons theory on the CR ambitwistor space [Movshev '04; Mason, Skinner '05]

$$\blacksquare = \bar{\partial}_{\mathsf{CR}}\mathsf{b} + \mathsf{b}\bar{\partial}_{\mathsf{CR}} = \Box_{\mathbb{R}^4} \;+\; 8\frac{\mu^{(\alpha}\hat{\mu}^{\beta)}\lambda^{(\dot{\alpha}}\hat{\lambda}^{\dot{\beta})}}{|\lambda|^2|\mu|^2}\frac{\partial}{\partial x^{\alpha\dot{\alpha}}}\frac{\partial}{\partial x^{\beta\dot{\beta}}}$$

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- Kinematic Lie algebra: yes
- Colour-kinematics duality: no (at least not obviously)

$$\frac{1}{\blacksquare} = \frac{1}{k^2}\left(\eta^{MN} - K^{MN}_{\mu\nu}\frac{k^\mu k^\nu}{k^2} + K^{MP}_{\mu\nu}K_P{}^N{}_{\rho\sigma}\frac{k^\mu k^\nu k^\rho k^\sigma}{k^4} - \cdots\right)$$

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- Comparing to standard R_ξ-gauge analysis of Yang–Mills, tempting to conjecture twistorial Ward identities kill spurious K contributions $\Rightarrow$ all-loop colour-kinematics duality (proving hard to establish or rule out)
- Can only work for supersymmetric case: twistorial anomaly for pure Yang-Mills
- Agrees with expectations from potential Jacobian counter term colour-kinematics duality anomalies

Pure spinors and super Yang-Mills

Super Yang–Mills theory as a pure spinor theory over $(x^\mu, \theta, \lambda, \bar{\lambda}, \mathrm{d}\bar{\lambda})$:

$$\int \Omega \mathrm{tr}\Big(\Psi Q \Psi + \frac{1}{3}\Psi\Psi\Psi\Big)$$

$\Rightarrow$ tree-level colour–kinematics duality, but b-ghost divergences obstruct loop-level

[Ben–Shahar, Guillum '21; BJKMSW '23]

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Cubic double copy action for supergravity

$$S := \int \Omega_{\mathsf{10d}\,\mathcal{N}=1} \wedge_{\mathbb{M}^{10|16}} \Omega_{\mathsf{10d}\,\mathcal{N}=1} \langle \Psi, (Q \otimes \mathsf{id} + \mathsf{id} \otimes Q)\Psi + \tfrac{1}{3}[\Psi, \Psi]\rangle_{\mathfrak{Kin}^0(\hat{\mathfrak{B}})}$$

[BJKMSW '23]

Pure spinors and M2-brane world-volume theories

Conjecture: Bagger–Lambert–Gustavsson (BLG) theory has 3-Lie algebra colour-kinematics duality **[Bargheer, He, McLoughlin '12; Huang, Johansson '12]**

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BV■-algebras can be extended to BV■-modules ("matter" fields)

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$\Rightarrow$ tree-level (standard Lie algebra) colour-kinematics duality **[BJKMSW '23]**

Future directions

A path to proving colour-kinematics duality for super Yang–Mills?

In progress [BJKSW]

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Curved backgrounds and classical double copy (beyond perturbation theory?)

Cf. [Lipstein, Nagy '23]

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String (field) theory

In progress [BJKSW]