Evaluating Feynman Integrals with the Help of the Landau Equations

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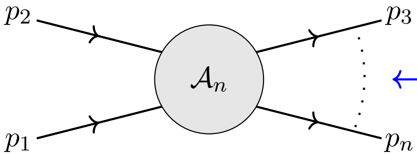
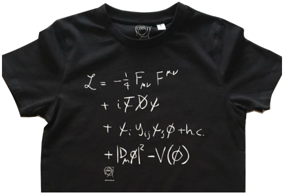
JHEP 10 (2023) 161 with Christoph Dlapa, Martin Helmer, Felix Tellander

Motivation: Scattering Amplitudes $\mathcal{A}_n$ in Quantum Field Theory





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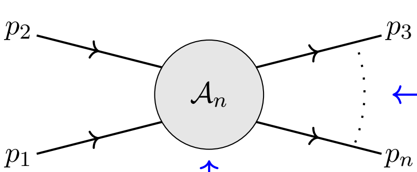
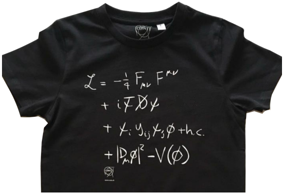
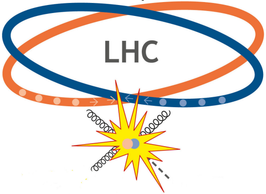




Collider Experiments



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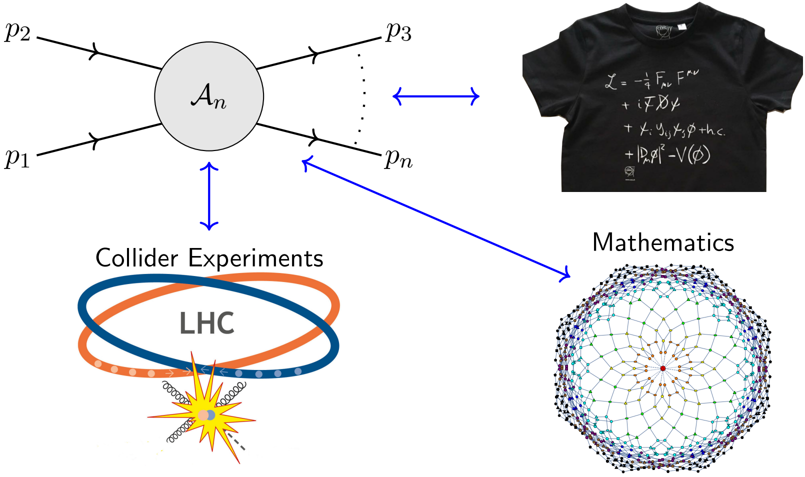

- Theoretical predictions for outcome of elementary particle collisions, central for experiments such as the LHC & High-Luminosity upgrade
- Exhibit remarkably deep mathematical structures

Amplitude calculation workflow

E.g. for $n=4$ gluons: $\mathcal{A}_4 = g_{YM}^2 \sum_{L=0,1\dots} g_{YM}^{2L} \mathcal{A}_4^{(L)}$, g_{YM} coupling const.

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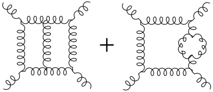
1. Draw contributing Feynman graphs $\mathcal{A}_4^{(2)} =$  $+ \ldots$

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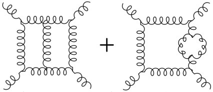
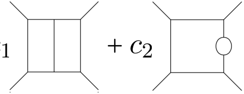
1. Draw contributing Feynman graphs $\quad \mathcal{A}_4^{(2)} =$  $+ \ldots$
2. Decompose to scalar Feynman integrals (FI) $\quad = c_1$  $+ \ldots$

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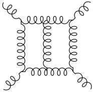
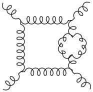
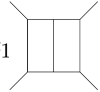
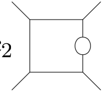
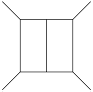
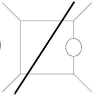
1. Draw contributing Feynman graphs $\quad \mathcal{A}_4^{(2)} =$  $+$  $+ \ldots$
2. Decompose to scalar Feynman integrals (FI) $\quad = c_1$  $+ c_2$  $+ \ldots$
3. Reduce to a basis $\vec{f}$ (master FI) $\quad = c_1'$  $+ 0$  $+ \ldots$

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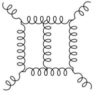
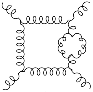
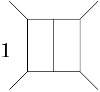
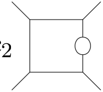
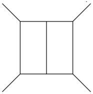
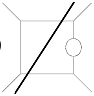
1. Draw contributing Feynman graphs $\quad \mathcal{A}_4^{(2)} =$  $+$  $+ \ldots$
2. Decompose to scalar Feynman integrals (FI) $\quad = c_1$  $+ c_2$  $+ \ldots$
3. Reduce to a basis $\vec{f}$ (master FI) $\quad = c_1'$  $+ 0$  $+ \ldots$
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Of the form

$$f_1 = \int \prod_{l=1}^{L} \frac{d^D k_l}{i\pi^{D/2}} \prod_{i=1}^{E} \frac{1}{D_i^{\nu_i}},$$

where $D_i = -q_i^2 + m_i^2$ and $D = D_0 - 2\epsilon$.

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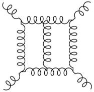
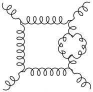
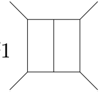
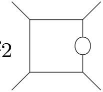
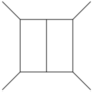
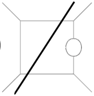
$\mathcal{A}_4^{(2)} =$  $+$  $+ \ldots$

$= c_1$  $+ c_2$  $+ \ldots$

$= c_1'$  $+ 0$  $+ \ldots$

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Evaluation of Feynman Integrals

Method of choice: Canonical differential equations

For polylogarithmic FI, find basis transformation $\vec{g} = T \cdot \vec{f}$ such that
[Gehrmann,Remiddi'99] [Henn'13]

$$d\vec{g} = \epsilon \, d\widetilde{M} \, \vec{g}, \qquad \widetilde{M} \equiv \sum_i \overbrace{\tilde{a}_i}^{\text{constant matrices}} \log \underbrace{W_i}_{\text{letters}},$$

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In line with strategy of state of the art precision calculations, e.g.
[Abreu,Ita,Moriello,Page,Tschernow,Zeng'20]

The Landau equations

Yield specific values of (kinematic) parameters of any (Feynman) integral, for which it may become singular. [Landau'59]

$$f_1 = \int \prod_{l=1}^{L} \frac{d^D k_l}{i\pi^{D/2}} \int_0^\infty \prod_{i=1}^{E} dx_i \frac{\delta(\sum_j x_j - 1)}{\left(\sum_j x_j D_j\right)^{\sum_k \nu_k}}$$

where $D_i = -q_i^2 + m_i^2$.

Landau equations:

$$x_i D_i = 0 \ \forall i = 1, \dots E$$
$$\frac{\partial}{\partial k_l} \sum_{i=1}^{E} x_i D_i = 0, \ \forall l = 1, \dots, L.$$

Formulated as conditions for the contour of integration to become trapped between two poles of integrand.

Believed for long to only provide information on where $W_i = 0$.

This work

Evidence through two loops: Rational letters of polylogarithmic FI captured by Landau equations, when recast as polynomial of the kinematic variables of integral, known as the *principal A-determinant* E_A!

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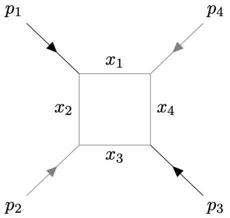
Example: 'Two-mass easy' box with $p_2^2 = p_4^2 = 0$, $p_1^2, p_3^2 \neq 0$:



E_A equipped with natural factorization, $(s = (p_1 + p_2)^2,\ t = (p_1 + p_4)^2)$

$$E_A = (p_1^2 p_3^2 - st) p_1^2 p_3^2 st (p_1^2 + p_3^2 - s - t)(p_3^2 - t)(p_3^2 - s)(p_1^2 - t)(p_1^2 - s).$$

where each factor is indeed a letter of the integral!

Outline

Feynman integrals in the Lee-Pomeransky representation:

$$f_1 = \frac{\Gamma(D/2)}{\Gamma((L+1)D/2 - \sum_i \nu_i)} \int_0^\infty \prod_{i=1}^{E} \left(\frac{x^{\nu_i - 1} dx_i}{\Gamma(\nu_i)} \right) \frac{1}{\mathcal{G}^{D/2}}$$

where $\mathcal{G} = \mathcal{U} + \mathcal{F}$, and for graph G associated to integral f_1,

$$\mathcal{U} = \sum_{\substack{T \text{ a spanning} \\ \text{tree}^1 \text{ of } G}} \prod_{e \notin T} x_e,$$

$$\mathcal{F} = \mathcal{U} \sum_{e \in E} m_e^2 x_e - \sum_{\substack{F \text{ a spanning} \\ 2\text{-forest}^2 \text{ of } G}} p(F)^2 \prod_{e \notin F} x_e,$$

are the 1^{st} and 2^{nd} Symanzik polynomials, of degree $L, L+1$ in the x_i.

In this form, f_1 is special case[3] of $\mathcal{A}$-hypergeometric function as defined by Gelfand, Graev, Kapranov & Zelevinsky (GKZ). [de la Cruz'19][Klausen'19]

[1] Connected subgraph of G containing all vertices but no loops.

[2] Defined similarly, but with 2 connected components.

[3] Generic case: All $\mathcal{G}$ polynomial coefficients are variables, different from each other.

Singularities of GKZ-systems

Let $\mathcal{G} = \sum_{j=1}^{m} c_j \prod_{i=1}^{E} x_i^{a_{ij}}$, c_j all independent variables.
Values of c_i for which GKZ-system becomes singular are solutions to

$$E_A(\mathcal{G}) = 0$$

where $E_A(\mathcal{G})$ is the *principal A-determinant of $\mathcal{G}$*: Polynomial in c_j with integer coefficients, that vanishes whenever the system of equations

$$\mathcal{G} = x_1 \frac{\partial \mathcal{G}}{\partial x_1} = \ldots = x_E \frac{\partial \mathcal{G}}{\partial x_E} = 0 \text{ has a solution for } \vec{x} \in (\mathbb{C}^*)^E .$$

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In practice, compute via theorem factorizing it into contributions from each face Γ of polytope with vertices $(a_{1j}, \ldots, a_{Ej})$, $j = 1, \ldots, m$

$$E_A(\mathcal{G}) = \prod_{\Gamma} \Delta_{\Gamma}(\mathcal{G})$$

where the *A-discriminant* $\Delta_{\Gamma}(\mathcal{G})$ also polynomial in c_i, that vanishes when

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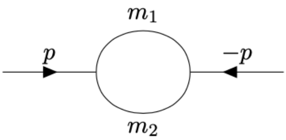
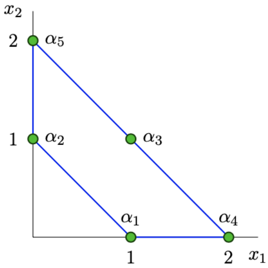
Example: Principal A-determinant of bubble





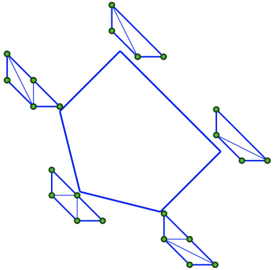
(Newton) polytope of $\mathcal{G}$ polynomial exponents, Newt($\mathcal{G}$)

$$\mathcal{G} = x_1 + x_2 + (m_1^2 + m_2^2 - p^2)x_1x_2 + m_1^2x_1^2 + m_2^2x_2^2\,,$$

$$\begin{aligned} E_A(\mathcal{G}) &= \Delta_{\alpha_4}\Delta_{\alpha_5}\Delta_{\alpha_4\alpha_5}\Delta_{\alpha_1\alpha_2\alpha_4\alpha_5} \\ &= m_1^2m_2^2(p^4 + m_1^4 + m_2^4 - 2p^2m_1^2 - 2p^2m_2^2 - 2m_1^2m_2^2)p^2\,, \end{aligned}$$

Interpretation of $E_A(\mathcal{G})$ polytope

Newt($E_A(\mathcal{G})$), built out of exponents of $E_A(\mathcal{G})$ polynomial: Keeps track of *triangulations* of Newt($\mathcal{G}$).



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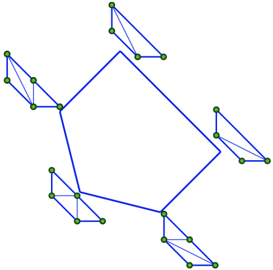
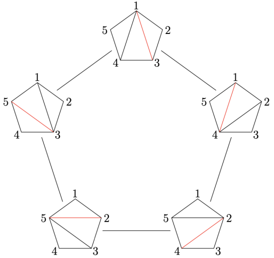


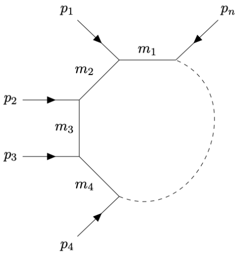


Cluster algebras also describe triangulations of geometric spaces [Fomin,Zelevinsky'01] [Felikson,Shapiro,Tumarkin'11]

First-principle derivation of observed cluster-algebraic structure of Feynman integrals? [Chicherin,Henn,Papathanasiou'20]. . . [He,Liu,Tang,Yang'22]

Generic n-point 1-loop integrals

All $m_i, p_i^2 \neq 0$ and different from each other



All Landau singularity information captured in *modified Cayley matrix* $\mathcal{Y}$,[Melrose'65]

$$\mathcal{Y} = \begin{pmatrix} 0 & 1 & 1 & \cdots & 1 \\ 1 & Y_{11} & Y_{12} & \cdots & Y_{1n} \\ 1 & Y_{12} & Y_{22} & \cdots & Y_{2n} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & Y_{1n} & Y_{2n} & \cdots & Y_{nn} \end{pmatrix} \quad \begin{aligned} Y_{ii} &= 2m_i^2 \\ Y_{ij} &= m_i^2 + m_j^2 - s_{ij-1} \\ s_{ij} &= (p_i + \ldots + p_j)^2 \end{aligned}$$

[1] Where all $x_i \neq 0$

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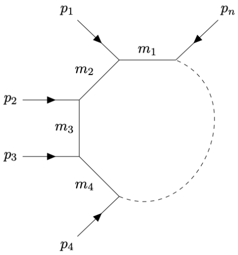

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because A-discriminants reduce to usual determinants:

- $\Delta_{\mathsf{Newt}(\mathcal{F})}(\mathcal{F}) = \det Y$: Leading[1] Landau singularity of type I (finite k)
- $\Delta_{\mathsf{Newt}(\mathcal{G})}(\mathcal{G}) = \det \mathcal{Y}$: Leading[1] Landau singularity of type II ($k \to \infty$)
- Subleading Landau singularity where $x_{i_1}, \ldots, x_{i_m} = 0 \sim$ Leading singularity of subgraph where internal edges $i_1, \ldots, i_m$ removed [Klausen'21]

[1] Where all $x_i \neq 0$

Minors of modified Cayley matrix

For any matrix A with elements a_{mn}, let (j, k)-th minor of A be

$$A\begin{bmatrix} j \\ k \end{bmatrix} \equiv \begin{vmatrix} a_{1,1} & a_{1,2} & a_{1,3} & \cdots & a_{1,k-1} & k & a_{1,k+1} & \cdots & a_{1,N} \\ a_{2,1} & a_{2,2} & a_{2,3} & \cdots & a_{2,k-1} & & a_{2,k+1} & \cdots & a_{2,N} \\ \vdots & \vdots & \vdots & & \vdots & & \vdots & & \vdots \\ a_{j-1,1} & a_{j-1,2} & a_{j-1,3} & \cdots & a_{j-1,k-1} & & a_{j-1,k+1} & \cdots & a_{j-1,N} \\ j & & & & & & & & \\ a_{j+1,1} & a_{j+1,2} & a_{j+1,3} & \cdots & a_{j+1,k-1} & & a_{j+1,k+1} & \cdots & a_{j+1,N} \\ \vdots & \vdots & \vdots & & \vdots & & \vdots & & \vdots \\ a_{N,1} & a_{N,2} & a_{N,3} & \cdots & a_{N,k-1} & & a_{N,k+1} & \cdots & a_{N,N} \end{vmatrix},$$

where shading indicates removal of row and column. Similarly $A\begin{bmatrix} i_1 \ldots i_k \\ j_1 \ldots j_k \end{bmatrix}$,
$A\begin{bmatrix} \cdot \\ \cdot \end{bmatrix} = \det A$.

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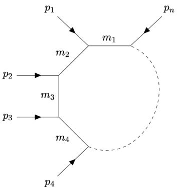
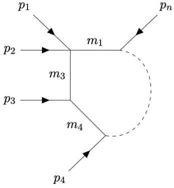
$$\mathcal{Y}\begin{bmatrix} 3 \\ 3 \end{bmatrix} \left(\right. $$



$$\left. \right) = \mathcal{Y}\begin{bmatrix} \cdot \\ \cdot \end{bmatrix} \left(\right.$$



$$\left. \right)$$

Principal A-determinant of generic 1-loop graphs

Gathering previous bits of information, arrive at

$$E_A(\mathcal{G}) = \mathcal{Y}\begin{bmatrix} \cdot \\ \cdot \end{bmatrix} \prod_{i=1}^{n+1} \mathcal{Y}\begin{bmatrix} i \\ i \end{bmatrix} \ldots \prod_{i_{n-1} > \ldots > i_1 = 1}^{n+1} \mathcal{Y}\begin{bmatrix} i_1 \ldots i_{n-1} \\ i_1 \ldots i_{n-1} \end{bmatrix} \prod_{i=2}^{n+1} \mathcal{Y}_{ii} \,.$$

Product of all diagonal k-dimensional minors of $\mathcal{Y}$ with $k = 1, \ldots, n+1$, except $\mathcal{Y}_{11} = 0$.

$2^{n+1} - n - 2$ factors, e.g. $1, 4, 11, 26, 57, 120$ factors for $n = 1, \ldots, 6$.

From 1-loop rational to square-root letters

Working assumption: Square-root letters produced by re-factorizing E_A using Jacobi determinant identities of the form

$$p \cdot q = f^2 - g = (f - \sqrt{g})(f + \sqrt{g})\,,$$

where

1. p, q factors of E_A, i.e. rational letters.
2. Square-root letters $f \pm \sqrt{g}$ obtained contain leading singularity of the Feynman integral considered in second term. [Cachazo'08]

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Motivated by interpretation of 1-loop integrals as volumes of spherical simplices. [Davydychev,Delbourgo'99] Jacobi identities,

$$A\begin{bmatrix} \cdot \\ \cdot \end{bmatrix} A\begin{bmatrix} i & j \\ i & j \end{bmatrix} = A\begin{bmatrix} i \\ i \end{bmatrix} A\begin{bmatrix} j \\ j \end{bmatrix} - A\begin{bmatrix} i \\ j \end{bmatrix} A\begin{bmatrix} j \\ i \end{bmatrix} \overset{A=A^T}{=} A\begin{bmatrix} i \\ i \end{bmatrix} A\begin{bmatrix} j \\ j \end{bmatrix} - A\begin{bmatrix} i \\ j \end{bmatrix}^2$$

crucial for their computation. Point 2 adopts widely observed pattern in 1- and 2-loop computations.

All 1-loop letters I

Need only ratio $\frac{f-\sqrt{g}}{f+\sqrt{g}}$, as product already contained in rational alphabet.
Letting $D = D_0 - 2\epsilon$, obtain N letters of type,

$$W_{1,\ldots,(i-1),\ldots,n} = \begin{cases} \dfrac{\mathcal{Y}\begin{bmatrix} i \\ 1 \end{bmatrix} - \sqrt{-\mathcal{Y}\begin{bmatrix} \cdot \\ \cdot \end{bmatrix}\mathcal{Y}\begin{bmatrix} 1 & i \\ 1 & i \end{bmatrix}}}{\mathcal{Y}\begin{bmatrix} i \\ 1 \end{bmatrix} + \sqrt{-\mathcal{Y}\begin{bmatrix} \cdot \\ \cdot \end{bmatrix}\mathcal{Y}\begin{bmatrix} 1 & i \\ 1 & i \end{bmatrix}}}, & D_0 + n \text{ odd}, \\ \\ \dfrac{\mathcal{Y}\begin{bmatrix} i \\ 1 \end{bmatrix} - \sqrt{\mathcal{Y}\begin{bmatrix} i \\ i \end{bmatrix}\mathcal{Y}\begin{bmatrix} 1 \\ 1 \end{bmatrix}}}{\mathcal{Y}\begin{bmatrix} i \\ 1 \end{bmatrix} + \sqrt{\mathcal{Y}\begin{bmatrix} i \\ i \end{bmatrix}\mathcal{Y}\begin{bmatrix} 1 \\ 1 \end{bmatrix}}}, & D_0 + n \text{ even}. \end{cases}$$

All 1-loop letters II

In addition, $n(n-1)/2$ letters of type,

$$W_{1,\dots,(i-1),\dots,(j-1),\dots,n} = \begin{cases} \dfrac{\mathcal{Y}\begin{bmatrix} i \\ j \end{bmatrix} - \sqrt{-\mathcal{Y}\begin{bmatrix} \cdot \\ \cdot \end{bmatrix}\mathcal{Y}\begin{bmatrix} i & j \\ i & j \end{bmatrix}}}{\mathcal{Y}\begin{bmatrix} i \\ j \end{bmatrix} + \sqrt{-\mathcal{Y}\begin{bmatrix} \cdot \\ \cdot \end{bmatrix}\mathcal{Y}\begin{bmatrix} i & j \\ i & j \end{bmatrix}}}, & D_0 + n \text{ odd}, \\ \\ \dfrac{\mathcal{Y}\begin{bmatrix} 1 & j \\ 1 & i \end{bmatrix} - \sqrt{-\mathcal{Y}\begin{bmatrix} 1 \\ 1 \end{bmatrix}\mathcal{Y}\begin{bmatrix} 1 & i & j \\ 1 & i & j \end{bmatrix}}}{\mathcal{Y}\begin{bmatrix} 1 & j \\ 1 & i \end{bmatrix} + \sqrt{-\mathcal{Y}\begin{bmatrix} 1 \\ 1 \end{bmatrix}\mathcal{Y}\begin{bmatrix} 1 & i & j \\ 1 & i & j \end{bmatrix}}}, & D_0 + n \text{ even}, \end{cases}$$

All 1-loop letters III

Our procedure also predicts $\mathcal{Y}\left[\begin{smallmatrix}\cdot\\\cdot\end{smallmatrix}\right]$ and $\mathcal{Y}\left[\begin{smallmatrix}1\\1\end{smallmatrix}\right]$ as individual rational letters, but in fact only the ratio

$$W_{1,2,\ldots,n} = \frac{\mathcal{Y}\begin{bmatrix}\cdot\\\cdot\end{bmatrix}}{\mathcal{Y}\begin{bmatrix}1\\1\end{bmatrix}},$$

appears, as we'll get back to in next slide.

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Total letter count: Assuming $n \le d+1$ for external kinematics dimension d,

$$|W| = 2^{n-3}\left(n^2+3n+8\right) - \frac{1}{6}\left(n^3+5n+6\right),$$

e.g. $|W| = 1, 5, 18, 57, 166$ for $n = 1, \dots, 5$ and D_0 even.

Verification through differential equations & comparison with literature

From letter prediction, derived canonical differential equations through numeric IBP identities ⇒ confirmation.

By explicit computation up to $n = 10$, infer general form, e.g. $n + D_0$ even:

$$
\begin{aligned}
d\mathcal{J}_{1\dots n} =& \epsilon \; d\log W_{1\dots n} \; \mathcal{J}_{1\dots n} \\
&+ \epsilon \sum_{1\leq i\leq n} (-1)^{i+\left\lfloor \frac{n}{2}\right\rfloor} d\log W_{1\dots(i)\dots n} \; \mathcal{J}_{1\dots\widehat{i}\dots n} \\
&+ \epsilon \sum_{1\leq i<j\leq n} (-1)^{i+j+\left\lfloor \frac{n}{2}\right\rfloor} d\log W_{1\dots(i)\dots(j)\dots n} \; \mathcal{J}_{1\dots\widehat{i}\dots\widehat{j}\dots n}.
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\end{aligned}
$$

Furthermore, compared to previous results for D_0 even based on

1. the diagrammatic coaction [Abreu,Britto,Duhr,Gardi'17]
2. the Baikov representation [Chen,Ma,Yang'22]

Agreement in form of CDE, as well as in letters for orientations presented in 2, see also. [Jiang,Yang'23]

Limits of generic to non-generic graphs

Proved that E_A has well-defined limit when any $m_i^2, p_j^2 \to 0$, namely it is unique regardless of the order with which we send them to zero.

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Limit of E_A when single parameter x takes value a may be defined as

$$\lim_{x\to a} E_A = \left.\frac{\partial^l \widetilde{E_A}}{\partial x^l}\right|_{x=a} \neq 0\,, \text{ with } \left.\frac{\partial^{l'} E_A}{\partial x^{l'}}\right|_{x=a} = 0 \text{ for } l' = 0, \ldots, l-1\,,$$

While multivariate generalization straightforward, highly nontrivial that limit does not depend on order. E.g. triangle Cayley in limit $p_i^2 \to 0$:

$$\det Y = 0 + 2\sum_{i=1}^{3} p_i^2 (m_i^2 - m_{i-1}^2)(m_{i+1}^2 - m_{i-1}^2) + \mathcal{O}(p_j^2 p_k^2)\,,$$

While limits of individual factors in E_A depend on limit order, E_A as a whole does not, since different orders produce factors it already contains.

Strong evidence that alphabet of non-generic FI correctly obtained as limit of generic one, in line with previous observations.

Mathematica Notebook

LandauAlphabetDE.nb

```
(* Symbol alphabets*)
(* Generic Box in even dimension *)
D0 = 4;
EvaluateLetter[AllLettersList[4]] // Short
(* Two-mass easy box limit *)
Factor[PLimit[%, ms[1] → 0, ms[2] → 0, ms[3] → 0, ms[4] → 0, s[1, 3] → 0, ps[2] → 0]];
(* In the limit the letters become multiplicatively dependent. Since all of them are rational, a basis may be found as follows *)
dlExpand[dl /@ %];
dlCollect /@ DeleteCases[RowReduce[CoefficientArrays[%, Variables[%]][[2]]].Variables[%], 0] /. dl[x_] → x
Length[%]
(* Product indeed yields corresponding limit of the principal A-determinant *)
LS2meBox - Times @@ %%

(* Differential equations *)
(*Box basis*)
basis @@ Range[4]
(*Box canonical differential equations*)
CDEs[%] // MatrixForm
```

```
Short=
ms[1] <<24>> (ms[3]^2 ps[1]^2 - 2 ms[3] × ms[4] ps[1]^2 + ms[4]^2 ps[1]^2 - 2 ms[1] × ms[3] × ps[1] × ps[2] +
   2 ms[1] × ms[4] × ps[1] × ps[2] + <<172>> + ms[3]^2 s[2, 3]^2 - 2 ms[1] × s[1, 2] s[2, 3]^2 - 2 ms[3] × s[1, 2] s[2, 3]^2 + s[1, 2]^2 s[2, 3]^2)

Out[180]= ps[1] × ps[3] (ps[1] - s[1, 2]) (ps[3] - s[1, 2]) s[1, 2] (ps[1] - s[2, 3]) (ps[3] - s[2, 3]) (ps[1] + ps[3] - s[1, 2] - s[2, 3]) s[2, 3] (ps[1] × ps[3] - s[1, 2] × s[2, 3])
```

Short=

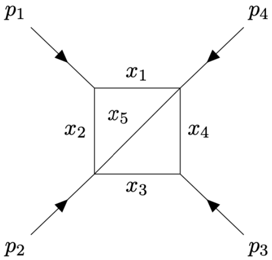
$$\left\{-\frac{1}{2\,\mathrm{ms}[1]},\ -\frac{1}{2\,\mathrm{ms}[2]},\ \ll 53 \gg,\ \frac{-\mathrm{ms}[1]\times\mathrm{ms}[3]\times\mathrm{ps}[1]+\ll 73\gg}{-\mathrm{ms}[1]\times\mathrm{ms}[3]\times\mathrm{ps}[1]+\ll 71\gg+\sqrt{\ll 1\gg}},\ \frac{-2\,\mathrm{ms}[1]\times\mathrm{ms}[2]\times\mathrm{ps}[1]+\ll 65\gg}{-2\,\mathrm{ms}[1]\times\mathrm{ms}[2]\times\mathrm{ps}[1]+\ll 63\gg+\sqrt{\left(-\mathrm{ms}[1]^2+2\,\mathrm{ms}[1]\times\mathrm{ms}[2]-\ll 1\gg^2+\ll 1\gg+2\,\mathrm{ms}[2]\times\mathrm{ps}[1]-\mathrm{ps}[1]^2\right)\ (\ll 1\gg)}}\right\}$$

```
Out[185]= {ps[1], ps[3], ps[1] - s[1, 2], ps[3] - s[1, 2], s[1, 2], ps[1] - s[2, 3], ps[3] - s[2, 3], ps[1] + ps[3] - s[1, 2] - s[2, 3], s[2, 3], ps[1] × ps[3] - s[1, 2] × s[2, 3]}

Out[186]= 10

Out[187]= 0

Out[188]= {IG[2][1], IG[2][2], IG[2][3], IG[2][4], IG[2][1, 2], IG[2][1, 3], IG[2][1, 4], IG[2][2, 3], IG[2][2, 4], IG[2][3, 4], IG[4][1, 2, 3], IG[4][1, 2, 4], IG[4][1, 3, 4], IG[4][2, 3, 4], IG[4][1, 2, 3, 4]}
```

MatrixForm=

w[1]	0	0	0	0	0	0	0	0	0
0	w[2]	0	0	0	0	0	0	0	0
0	0	w[3]	0	0	0	0	0	0	0
0	0	0	w[4]	0	0	0	0	0	0
-w[1, {2}]	w[{1}, 2]	0	0	w[1, 2]	0	0	0	0	0
-w[1, {3}]	0	w[{1}, 3]	0	0	w[1, 3]	0	0	0	0
-w[1, {4}]	0	0	w[{1}, 4]	0	0	w[1, 4]	0	0	0
0	-w[2, {3}]	w[{2}, 3]	0	0	0	0	w[2, 3]	0	0
0	-w[2, {4}]	0	w[{2}, 4]	0	0	0	0	w[2, 4]	0
0	0	-w[3, {4}]	w[{3}, 4]	0	0	0	0	0	w[3, 4]
-w[1, {2}, {3}]	w[1, {2}, {3}] + w[{1}, {2}, 3]	-w[{1}, {2}, 3]	0	-w[1, 2, {3}]	w[1, {2}, 3]	0	-w[{1}, 2, 3]	0	0

Two-loop example of principal A-determinant-alphabet relation



1-mass slashed box,

$$p_1^2 \neq 0, p_2^2 = p_3^2 = p_4^2 = 0$$

$$E_A(\mathcal{G}) = (p_1^2 - t)(p_1^2 - s)(p_1^2 - s - t)(s + t)stp_1^2.$$

Agrees precisely with (2dHPL) alphabet known to describe 2-loop master integrals with these kinematics! [Gehrmann,Remiddi'00]

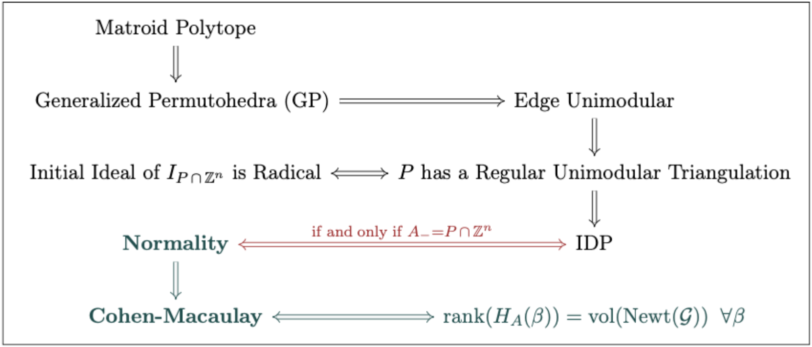
Further mathematical properties of Feynman integrals:Cohen-Macauley

Guarantees that

$$\# \text{ master integrals} = \text{volume of Newt}(\mathcal{G})$$

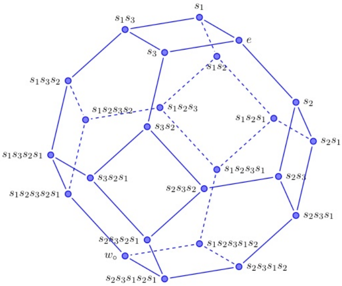
Proved it for currently largest known class of 1-loop integrals, including completely on-shell/massless. For earlier work, see [Tellander,Helmer'21] [Walther'22]

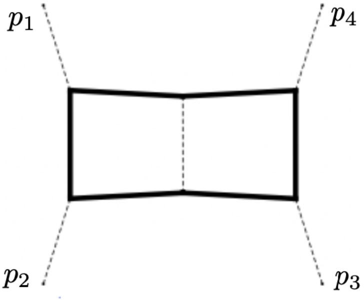
Relation to other properties:



Further mathematical properties of Feynman integrals
:Generalized permutohedron (GP) property

A polytope $P \subset \mathbb{R}^n$ is GP if and only if every edge is parallel to $\mathbf{e}_i - \mathbf{e}_j$, where $\mathbf{e}_i$ is unit vector on coordinate axis, for some $i, j \in \{1, \ldots, n\}$. E.g.





Practical utility: This property facilitates new methods for fast Monte Carlo evaluation of Feynman integrals. [Borinsky'20] [Borinsky,Munch,Tellander'23]

Previously proven for generic kinematics. [Schultka'18] Here: Generalized to any graph where all external vertices joined by massive path.

Conclusions and Outlook

Evidence that rational letters of polylogarithmic FI captured by polynomial form of Landau equations in terms of *principal A-determinant* E_A!

- Through 2 loops
- 1 loop: Also obtain square-root letters from Jacobi identities + CDE
- Strong evidence for well-defined limits to non-generic kinematics
- Easy-to-use Mathematica file with our results

Next Stage

1. More efficient evaluation of E_A + more 2-loop checks
2. New predictions for pheno, e.g. letters for $2 \to 3$ with 2 massive legs
   [Les Houches Standard Model Precision Wishlist'21]
3. Explore implications for beyond-polylogarithmic case