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Infinitesimal Higher Symmetries

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Outline

- Principal bundles, parallel transport and connections
- Higher principal bundles
- Higher connections as infinitesimal symmetries
- Case study: connections on n-gerbes

The classical story

Principal bundles, parallel transport and connections

Principal bundles

Fix a finite-dimensional smooth manifold M.

Definition (Principal G-bundle; global)

Let G be a Lie group. A principal G-bundle over M is a fibre bundle $\pi\colon P \to M$, together with a smooth G-action on P which preserves the fibres and satisfies that following map is a diffeomorphism:

$$P \times G \longrightarrow P \times_M P\,, \qquad (p, g) \longmapsto (p,\, pg)\,.$$

Definition (Principal G-bundle; local)

Let $\mathcal{U} = \{U_a\}_{a \in \Lambda}$ be a good open covering of M. A principal G-bundle is a family of smooth maps $g_{ab}\colon U_{ab} \to G$ satisfying the Čech 1-cocycle condition,

$$g_{ab}\, g_{bc} = g_{ac} \qquad \text{(restricted to } U_{abc}\text{)} \quad \forall\, a, b, c \in \Lambda\,, \qquad g_{aa} = 1\,.$$

Morphisms of principal bundles

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Definition (Morphisms of principal G-bundles; global)

Let P, Q be principal G-bundles over M. A morphism $P \to Q$ is a smooth map $f\colon P \to Q$ which preserves fibres and the G-action.

Definition (Morphisms of principal G-bundles; local)

Let (g_{ab}) and (g'_{ab}) define two principal G-bundles (over the same cover, for simplicity). A morphism $(g_{ab}) \to (g'_{ab})$ is a collection of smooth maps $h_a\colon U_a \to G$ which are a Čech coboundary:

$$h_b\, g_{ab} = g'_{ab}\, h_a\,, \qquad \forall\, a, b \in \Lambda\,.$$

Remark: Any morphism of principal G-bundles is an isomorphism; we obtain a **groupoid** $\mathcal{B}\mathrm{un}(M; G)$.

Parallel transport

Definition (Parallel transport)

Let $P \to M$ be a principal G-bundle. A parallel transport on P is assignment as follows: to each smooth path $\gamma\colon [0,1] \to M$, we assign a diffeomorphism $\mathtt{PT}_\gamma : P_{|\gamma(0)} \longrightarrow P_{|\gamma(1)}$, such that

(1) $\mathtt{PT}_\gamma$ preserves the G-action,

(2) $\mathtt{PT}_{\gamma'} \circ \mathtt{PT}_\gamma = \mathtt{PT}_{\gamma' \circ \gamma}$,

(3) $\mathtt{PT}_\gamma = \mathrm{id}$ whenever γ is a constant path,

(4) $\mathtt{PT}_\gamma$ depends smoothly on γ,

(5) $\mathtt{PT}_\gamma = \mathtt{PT}_{\gamma'}$ whenever γ and γ' are thin-homotopic.

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Proposition

A parallel transport is **flat** if and only if it is invariant under **all** homotopies $h\colon \gamma_0 \to \gamma_1$.

Relevance: Gauge theory; general relativity (geodesics); Aharonov-Bohm effect; machine learning; ...

Infinitesimal deformations of bundles—the Atiyah algebroid

A parallel transport is a rule for comparing fibres of $P \to M$ over different points.

Goal now: **Infinitesimal** parallel transport.

- Locally, we can trivialise P to $P_{|U} \cong U \times G \xrightarrow{\mathrm{pr}} U$.
- PT within U then assigns to a path γ an element $g \in G$ (fibre translation).
- Passing to infinitesimals, this is a pair $(X, \xi) \in T_x M \times \mathfrak{g}$.

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Definition (Atiyah Lie-algebroid, local version)

Let $g = (g_{ab})$ be the cocycle defining the bundle $P \to M$. The **Atiyah (Lie-)algebroid** of g is the $C^\infty(M)$-module

$$\mathrm{At}(g) = \left\{(X, \xi) \,\middle|\, X \in \mathfrak{X}(M),\ \xi = (\xi_a \colon U_a \to \mathfrak{g}),\ \xi_b = \mathrm{Ad}_{g_{ab}} \xi_a + g_{ab}^* \mu_G(X)\right\}$$

with anchor map $\rho\colon \mathrm{At}(g) \longrightarrow \mathfrak{X}(M)$, $(X, \xi) \longmapsto X$ and bracket

$$\big[(X, \xi), (Y, \eta)\big]_{\mathrm{At}(g)} = \left([X, Y]_{\mathfrak{X}(M)}, \mathcal{L}_X \eta - \mathcal{L}_Y \xi - [\xi, \eta]_{\mathfrak{g}}\right).$$

Infinitesimal parallel transport: connections

- An infinitesimal parallel transport, also called a **connection** on P, assigns to an infinitesimal path an infinitesimal fibre translation.
- Globalising, it is a smooth map

$$(\mathrm{id}_{\mathfrak{X}(M)}, A) \colon \mathfrak{X}(M) \longrightarrow \mathrm{At}(g)\,, \qquad X \longmapsto \big(X,\, A(X)\big)\,.$$

- This does **not** respect the Lie structures: the failure is called the **curvature of A**,

$$\big[\big(X,\, A(X)\big),\, \big(Y,\, A(Y)\big)\big]_{\mathrm{At}(g)} - (\mathrm{id}_{\mathfrak{X}(M)}, A)\big([X,Y]_{\mathfrak{g}}\big) = \big(0,\, F_A(X,Y)\big)\,.$$

Proposition

A connection on a principal bundle defined by g corresponds to a collection of 1-forms $A = \big(A_a \in \Omega^1(U_a;\mathfrak{g})\big)$ such that

$$A_b = \mathrm{Ad}_{g_{ab}} A_a + g_{ab}^* \mu_G = g_{ab}\, A_a\, g_{ab}^{-1} + g_{ab}^{-1} \mathrm{d} g_{ab}\,.$$

Parallel transports and connections on P are in 1:1-correspondence.

Remarks on universal symmetries

The **universal symmetry group** $\mathrm{Sym}(P)$ of $P \to M$ has the following descriptions:

- The group of smooth maps $\hat{f}\colon P \to P$ which preserve the G-action and cover a diffeomorphism $f\colon M \to M$.
- The group of pairs (f, α), where $f \in \mathrm{Diff}(M)$ and $\alpha\colon P \to \alpha^* P$ is a morphism of G-bundles.

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Proposition

(1) Let H be a Lie group which acts smoothly on M by a map $\Phi\colon H \to \mathrm{Diff}(M)$. Then, H-equivariant structures on P are in bijections with lifts of Φ to $\hat{\Phi}\colon H \to \mathrm{Sym}(P)$.

(2) The Lie algebra of $\mathrm{Sym}(P)$ is

$$\mathfrak{sym}(P) = \mathrm{At}(P)\,.$$

Previously: $\mathrm{Sym}(P)$ for gerbes [SB, Müller, Szabo] and general smooth principal ∞-bundles [SB, Shahbazi]; applications to QFT anomalies and NSNS supergravity.

Higher structure

∞-groups and ∞-bundles

2-groups:

- Let G be a Lie group. Consider its fundamental groupoid $\pi_{\leqslant 1}G$:
  - objects = points $g \in G$, morphisms = $\{$paths γ in $G\}$/homotopies fixing endpoints.

  This inherits a monoidal structure from the group structure of G.
- A **2-group** is a monoidal groupoid in which every object has an inverse [Baez? Older?].
- **Example:** $\mathrm{BU}(1)$ is the groupoid with objects $= \{*\}$ and morphisms $= \mathrm{U}(1)$.
  We set $* \otimes * := *$, $z \circ z' := z\, z'$ and $z \otimes z' := z\, z'$.

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Even higher groups:

A group is (1) a set with a multiplication and (...), or (2) a groupoid with a single object.
An **∞-group** is, equivalently, [Stasheff; Lurie]

- a coherently monoidal ∞-groupoids where each object has an inverse (multiplication encoded as monoidal structure), or
- an ∞-groupoid with a single object (multiplication encoded as composition).

Example: The based loop group $\Omega_x X$ of a topological space X [Stasheff].

Smooth spaces and smooth groups

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Let $\mathcal{C}\mathrm{art}$ denote the category with objects $\{\mathbb{R}^n \,|\, n \in \mathbb{N}_0\}$ and morphisms all smooth maps $\mathbb{R}^n \to \mathbb{R}^m$.
Let $\mathcal{S}$ denote the ∞-category of spaces (‘spaces’ = ‘∞-groupoids’).

Definition (Smooth space) [Schreiber]

A **smooth space** is a functor $X \colon \mathcal{C}\mathrm{art}^{\mathrm{op}} \longrightarrow \mathcal{S}$. We write $\mathbb{H}$ for the ∞-category of smooth spaces. A **smooth ∞-group** is a functor $G \colon \mathcal{C}\mathrm{art}^{\mathrm{op}} \longrightarrow \mathcal{G}\mathrm{rp}$.

- Interpretation: X assigns to each $\mathbb{R}^n$, $n \in \mathbb{N}_0$, the space of smooth maps $\mathbb{R}^n \to X$.

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- Interpretation: X assigns to each $\mathbb{R}^n$, $n \in \mathbb{N}_0$, the space of smooth maps $\mathbb{R}^n \to X$.
- **Example:** If M is a manifold, then $\mathbb{R}^n \mapsto \mathcal{M}\mathrm{fd}(\mathbb{R}^n, M)$ defines an object $M \in \mathbb{H}$.
  This furnishes an embedding $\mathcal{M}\mathrm{fd} \hookrightarrow \mathbb{H}$.
- $\mathrm{BU}(1)$ is a smooth ∞-group with $\mathrm{BU}(1)(\mathbb{R}^n) = N\big(\mathcal{M}\mathrm{fd}(\mathbb{R}^n, \mathrm{U}(1)) \rightrightarrows *\big) \simeq \mathcal{B}\mathrm{un}\big(\mathbb{R}^n; \mathrm{U}(1)\big)$.
- There is a smooth ∞-group $\mathcal{B}\mathrm{un}_\nabla\big(-; \mathrm{U}(1)\big)$.

Principal ∞-bundles

With a notion of (smooth) higher groups at hand, we can build higher principal bundles.

- Works in a particular type of ∞-categories, the **∞-topoi** [Giraud; Rezk; Lurie].
- **Example:** Both $\mathcal{S}$ and $\mathbb{H}$ are ∞-topoi. The EEpis in $\mathcal{S}$ are those maps which are surjective on π_0.

Definition (Principal ∞-bundle) [Nikolaus, Schreiber, Stevenson; SB]

Let $\mathcal{X}$ be an ∞-topos and G a group object in $\mathcal{X}$. A **G-principal ∞-bundle** consists of an effective epimorphism $P \to X$ in $\mathcal{X}$ and a fibre-preserving G-action on P such that the canonical morphism $P \times G \longrightarrow P \times_X P$ is an equivalence.

Examples:

- For G a Lie group, the canonical map $* \to \mathrm{B}G$ is a G-principal ∞-bundle (G acts trivially).
- A **(bundle) gerbe** is equivalently a $\mathrm{BU}(1)$-principal ∞-bundle.

Higher connections—(some of) what happened so far

- Parallel transport for a **strict** type of **2-bundles** was introduced by [Baez, Schreiber '04], later linked to connections [Schreiber, Waldorf '07; Faria Martins, Picken '10; Waldorf '17; Saemann, Schmidt, Kim '19;...].
- For higher bundles whose structure ∞-group arises as an **integration of an L_∞-algebra**, a general formalism was provided by [Sati, Schreiber, Stasheff '08].
- For ∞-groups with another **strictness condition**, connections and parallel transport on **trivial** bundles was given by [Kapranov '07, '15] using the free Lie algebroid on the tangent bundle TM.
- A theory of holonomy for **flat** connections and its relation to ∞-local systems was developed by [Abad, Schätz '14].
- An approach using rational homotopy for bundles controlled/classified by a **discrete** space by [Fiorenza, Sati, Schreiber '20].
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Higher connections as infinitesimal symmetries

Derived geometry and deformation theory

Classifying stacks

For $G \in \mathcal{G}\mathrm{rp}(\mathcal{X})$, let $\mathrm{B}G \in \mathcal{X}$ denote the quotient of the trivial action of G on the point $*$.

Definition (Classifying object) [Nikolaus, Schreiber, Stevenson]

The object $\mathrm{B}G$ is called the **classifying object of G**.

Theorem [Nikolaus, Schreiber, Stevenson]

Let $X \in \mathcal{X}$. There is an equivalence of ∞-groupoids

$$\mathcal{B}\mathrm{un}(X; G) \simeq \mathcal{X}(X, \mathrm{B}G)\,.$$

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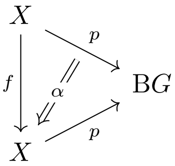
If $p\colon X \to \mathrm{B}G$ classifies a G-principal ∞-bundle $P \to X$, the symmetries of P are the **'deformations'**



Incorporating deformations: formal smooth spaces

For **infinitesimal symmetries** of P, study **infinitesimal deformations** of its classifying map

$$p\colon X \to \mathrm{B}G\,.$$

Incorporate infinitesimals into the formalism of smooth spaces: **derived differential geometry** (DDG) [Lawvere; Dubuc; Moerdijk, Reyes; Kock; Spivak; Carchedi, Steffens; Nuiten; ...].

- This works by incorporating **algebra**: the functions $C^\infty(M;\mathbb{R})$ on each manifold form a C^∞**-ring**.
- Roughly speaking, DDG is algebraic geometry over dg- or simplicial C^∞-rings; it behaves differently form (derived) algebraic geometry, e.g. due to existence of partitions of unity.
- Strongly related to dg/higher Lie geometry [Xu, Zhu, Behrend, Weinstein, Gualtieri, Ševera, ...].

Incorporating deformations: formal smooth spaces

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We replace $\mathcal{C}\mathrm{art}$ by $\mathcal{C}\mathrm{art}_{th}$, whose function algebras are of the form $C^\infty(\mathbb{R}^n;\mathbb{R})\otimes W$, where W is a local algebra with nilpotent ideal. These are the infinitesimal thickenings of the $\mathbb{R}^n$s.

Definition (Formal smooth space)

A **formal smooth space** is a ∞-functor $X\colon \mathcal{C}\mathrm{art}_{th}^{\mathrm{op}} \longrightarrow \mathcal{S}$. These assemble into an ∞-topos denoted $\mathbb{H}_{th}$.

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Example: We now indeed capture infinitesimal deformations of smooth geometric data intrinsically: Consider the space $\mathbb{R}_\epsilon$ with function C^∞-ring $\mathbb{R}\otimes\mathbb{R}[\epsilon]/\epsilon^2$. Then,

$$\mathbb{H}_{th}(\mathbb{R}_\epsilon, M) \cong TM \qquad \text{(as a set)}\,.$$

Remark: There is a fully faithful embedding $\mathbb{H}\hookrightarrow\mathbb{H}_{th}$.

L_∞-algebroids and dg Lie algebroids

Let k be a field of characteristic zero and A a connective commutative dg algebra over k.

Definition (L_∞-algebroid)

A **L_∞-algebroid** over A is a dg module E over A together with an **anchor map** $\rho\colon E \to T_A$ and a family of brackets $[-]_{n,E}\colon E^{\otimes n} \to E$ of degree $2-n$ such that

(1) the brackets turn E into an L_∞-algebra (antisymmetry, coherent Jacobi),

(2) ρ is a morphism of L_∞-algebras, and

(3) $[-]_{n,E}$ satisfies the Leibniz rule

$$[\xi, f\cdot\eta]_E = (-1)^{|\xi|\,|f|} f\cdot[\xi,\eta]_E + \rho(\xi)(f)\cdot\eta\,.$$

for $n=2$ and is graded A-linear for $n>2$.

If $[-]_{n,E}=0$ for all $n>2$, then $(E, [-,-]_E, \rho)$ is called a **dg Lie algebroid** over A.

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Morphisms of L_∞-algebroids: tower of $\phi_1\colon \mathfrak{g}\to\mathfrak{h}$ and $\phi_n\colon \mathfrak{g}^{\otimes n}\to\mathfrak{h}[n]$ with coherences. Conveniently encoded using Chevalley-Eilenberg CDGCs.

Our situation: $A = C^\infty(M)$; we then speak of L_∞-algebroids on M.

Nuiten's Family Lurie-Pridham Theorem

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Definition (Formal moduli problem over A) [Nuiten '17]

Let k be a field of characteristic zero and A be a connective commutative k-algebra. A **formal moduli problem (FMP) over A** is a functor $F\colon (\mathcal{C}\mathcal{A}\mathrm{lg}_k^{\mathrm{Art}})_{/A} \longrightarrow \mathcal{S}$ such that

(1) $F(A) \simeq *$, and

(2) F maps square-zero extensions to pullbacks.

We use the following extension of the famous **Lurie-Pridham Theorem** [Pridham '07; Lurie '10]:

Theorem [Nuiten '17]

There is an equivalence of ∞-categories

$$\mathrm{MC}\colon L_\infty\mathcal{A}\mathrm{gd}_A \underset{\dashleftarrow}{\overset{\simeq}{\longrightarrow}} \mathrm{FMP}(A)\,.$$

Defining higher connections, circumventing flatness

We are interested in the FMP describing deformations of the classifying map $p\colon M \to \mathrm{B}G$.

Definition (Atiyah L_∞-algebroid) [SB, Müller, Nuiten, Szabo]

Let G be a smooth ∞-group and $P \to M$ a G-principal ∞-bundle classified by a morphism $p\colon M \to \mathrm{B}G$ in $\mathbb{H}_{(th)}$. The **Atiyah L_∞-algebroid $\mathrm{At}(P)$ of P** is the L_∞-algebroid classifying the above FMP under Nuiten's theorem.

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Goal: Define (not necessarily flat) connections on generic ∞-bundles P.

For $l \in \mathbb{N}$, there is an ∞-functor $Q^{(l)}\colon L_\infty\mathcal{A}\mathrm{gd}_A \to L_\infty\mathcal{A}\mathrm{gd}_A$ which truncates away all terms in $\mathrm{CE}_*(\mathfrak{g})$ containing more than l tensor factors, i.e. $\mathrm{CE}_*(Q^{(l)}\mathfrak{g}) = \mathrm{Sym}^{1\leqslant\bullet\leqslant l}_{C^\infty(M)}(\mathfrak{g})$ [Nuiten].

Definition (Space of l-connections) [SB, Müller, Nuiten, Szabo]

The **∞-groupoid of l-connections on P** is the **mapping space**

$$\mathrm{Con}_l(P) := L_\infty\mathcal{A}\mathrm{gd}^{\mathrm{dg}}_{C^\infty(M)}\big(Q^{(l)}\mathfrak{X}(M),\, \mathrm{At}(P)\big) \quad \in \mathcal{S}\,.$$

Case studies

Testing the new model

First check: If $P \to M$ is an ordinary principal bundle (G a Lie group), then

$$\mathrm{Con}_1(P) = \{\text{classical connections on } P\}\,, \quad \mathrm{Con}_l(P) = \{\text{flat conns. on } P\}\,,\ \forall\, l > 1\,. \quad \checkmark$$

Higher $\mathrm{U}(1)$-bundles/n-gerbes

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Particularly well-known higher case: connections on n-gerbes/higher $\mathrm{U}(1)$-bundles.

Definition (n-gerbe with l-connection) [Deligne; Gajer; SB, Shahbazi]

Let $\mathcal{U} = \{U_a\}_{a\in\Lambda}$ be a good open covering of M.

(1) An **$(n{-}1)$-gerbe/$\mathrm{B}^n\mathrm{U}(1)$-bundle on M** is a collection $g = \big(g_{a_0\cdots a_n} : U_{a_0\cdots a_n} \to \mathrm{U}(1)\big)$ satisfying the Čech cocycle condition, $\delta g := \prod_{i=0}^{n} (-1)^i g_{a_0\cdots\widehat{a_i}\cdots a_n} = 1$.

Higher U(1)-bundles/n-gerbes

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(2) An **l-connection on an n-gerbe g** is a tuple $(A^{(1)}, \ldots, A^{(l)})$, where

$$A^{(p)} = \big(A^{(p)}_{a_0\cdots a_{n-p}} \in \Omega^p(U_{a_0\cdots a_{n-p}})\big)$$

and such that

$$\mathrm{d}\log(g) = \delta A^{(1)}\,, \qquad \mathrm{d}A^{(p)} = (-1)^p\,\delta A^{(p+1)} \quad \forall\, p = 1, \ldots, l-1\,.$$

For each $0 \leqslant l \leqslant n+1$, there is an ∞-groupoid $\mathrm{Grb}^n_{\nabla|l}(M)$ of n-gerbes with l-connections.

Example: 0-gerbes are the same as $\mathrm{U}(1)$-bundles.

1-gerbes with connections model the B-field in string theory/SuGra [Kapustin; Witten]

n-gerbes with $(n+1)$-connection model differential cohomology [Deligne; Brylinski; Gajer; Schreiber]

Example: 0-gerbes are the same as $\mathrm{U}(1)$-bundles.

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The **space** of l-connections on an n-gerbe g is the (homotopy) fibre [SB, Shahbazi]

$$\begin{array}{ccc} \mathrm{Con}_{geo,l}(g) & \longrightarrow & \mathrm{Grb}^n_{\nabla|l}(M) \\ \downarrow & & \downarrow \\ * & \xrightarrow[\{g\}]{} & \mathrm{Grb}^n(M) \end{array}$$

Question: Is this space equivalent to the one obtained from our L_∞-algebroid picture?

Theorem [Nuiten; SB, Müller, Nuiten, Szabo]

Let g describe an n-gerbe on M. Its Atiyah L_∞-algebroid is the dg Lie algebroid

$$C^\infty(\mathcal{U}^{[0]}) \xrightarrow{\delta} \cdots \xrightarrow{\delta} C^\infty(\mathcal{U}^{[n-1]}) \xrightarrow{(0,\delta)} E_n(g) \quad ,$$

where $$E_n(g) = \big\{(X, f) \in \mathfrak{X}(M) \times C^\infty(\mathcal{U}^{[n]}) \,|\, \delta f = (-1)^{n+1} \mathrm{d}\log(g)(X)\big\}\,.$$

The anchor map is the projection onto $\mathfrak{X}(M)$, and the bracket is the Lie derivative of functions and vector fields.

The Atiyah L_∞-algebroid of an n-gerbe

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The anchor map is the projection onto $\mathfrak{X}(M)$, and the bracket is the Lie derivative of functions and vector fields.

Theorem [SB, Müller, Nuiten, Szabo]

For any n-gerbe g on M, there is an equivalence of $(l-1)$-groupoids

$$\mathrm{Con}_{geo,l}(g) \simeq \mathrm{Con}_l(g)\,.$$

This is an algebraic description of differential cohomology.

Proof (sketch)

- Goal: compute explicitly the mapping space $\mathrm{Map}_{L_\infty\mathcal{A}\mathrm{gd}^{\mathrm{dg}}_{C^\infty(M)}}\big(Q^{(l)}\mathfrak{X}(M),\ \mathrm{At}(g)\big)$.
- Use model structure on $L_\infty\mathcal{A}\mathrm{gd}^{\mathrm{dg}}_{C^\infty(M)}$: $Q^{(l)}\mathfrak{X}(M)$ is $C^\infty(M)$-cofibrant.

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- Find manageable simplicial resolution of $\mathrm{At}(g)$: we give a general, explicit construction for 'semi-abelian extensions' of a dg Lie algebroid structure on

$$[n] \longmapsto \mathrm{Hom}_k\left(C_*(\Delta^n; k),\, \mathrm{ch}_k(\mathfrak{g})\right),$$

  which in this case allows us to simplify formal constructions of [Getzler; Robert-Nicoud, Vallette].
- Lemma: if $\mathfrak{g}$ is fibrant (surjective anchor map) this produces a simplicial resolution $\widehat{\mathfrak{g}}$ of $\mathfrak{g}$.

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- Lemma: if $\mathfrak{g}$ is fibrant (surjective anchor map) this produces a simplicial resolution $\widehat{\mathfrak{g}}$ of $\mathfrak{g}$.
- The mapping space is thus modelled by the simplicial set

$$[n] \longmapsto L_\infty\mathcal{A}\mathrm{gd}^{\mathrm{dg}}_{C^\infty(M)}\big(Q^{(l)}\mathfrak{X}(M),\, \widehat{\mathfrak{g}}_n\big)\,.$$

- Computation: we have an isomorphism of simplicial sets

$$L_\infty\mathcal{A}\mathrm{gd}^{\mathrm{dg}}_{C^\infty(M)}\big(Q^{(l)}\mathfrak{X}(M),\, \widehat{\mathfrak{g}}_n\big) \cong \mathrm{Con}_{geo,l}(g)\,.$$

□

Thank you for your attention!