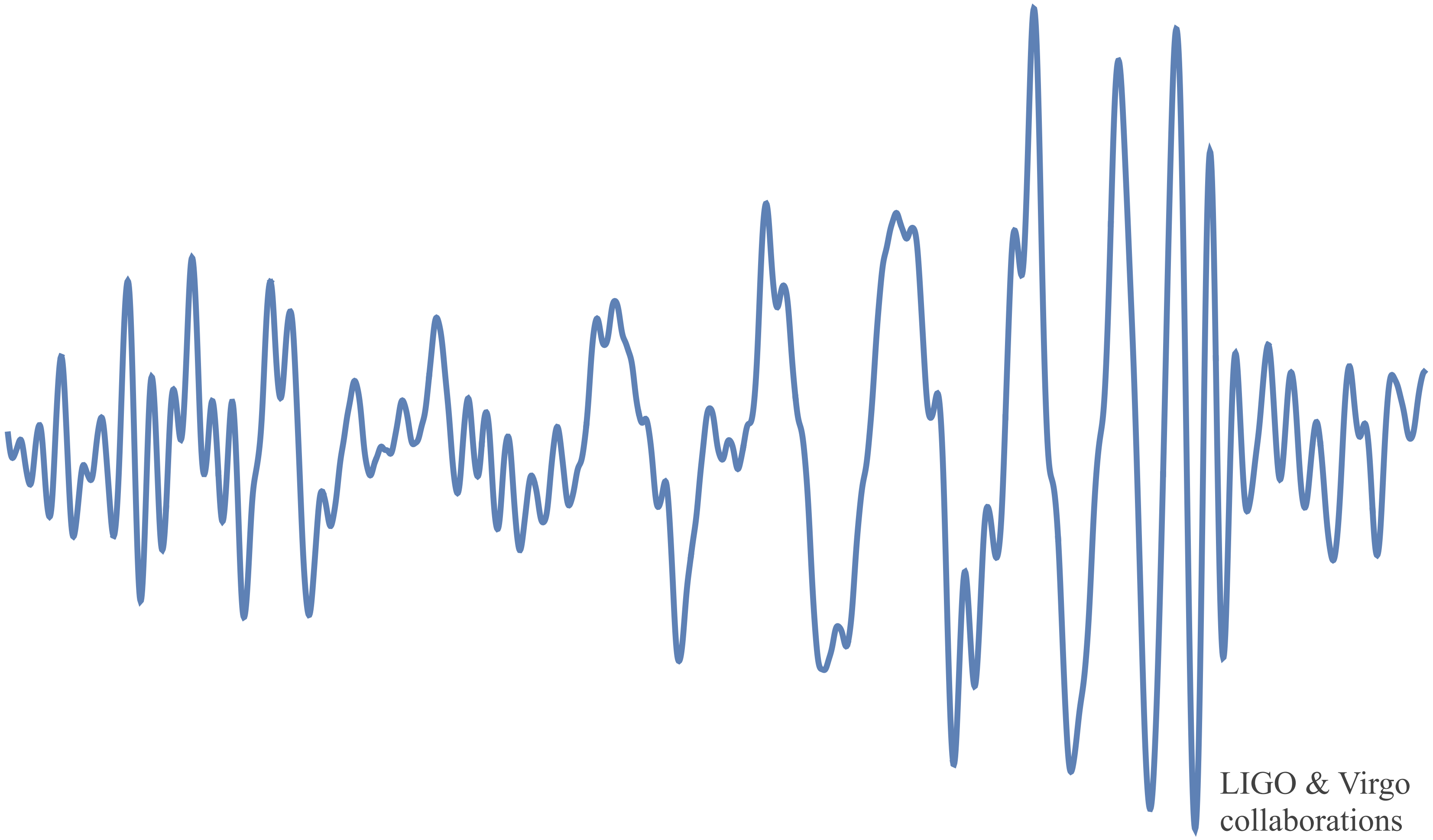
University of Hertfordshire, March 2024

Amplitudes and Waveforms

Donal O'Connell
Edinburgh

Motivation

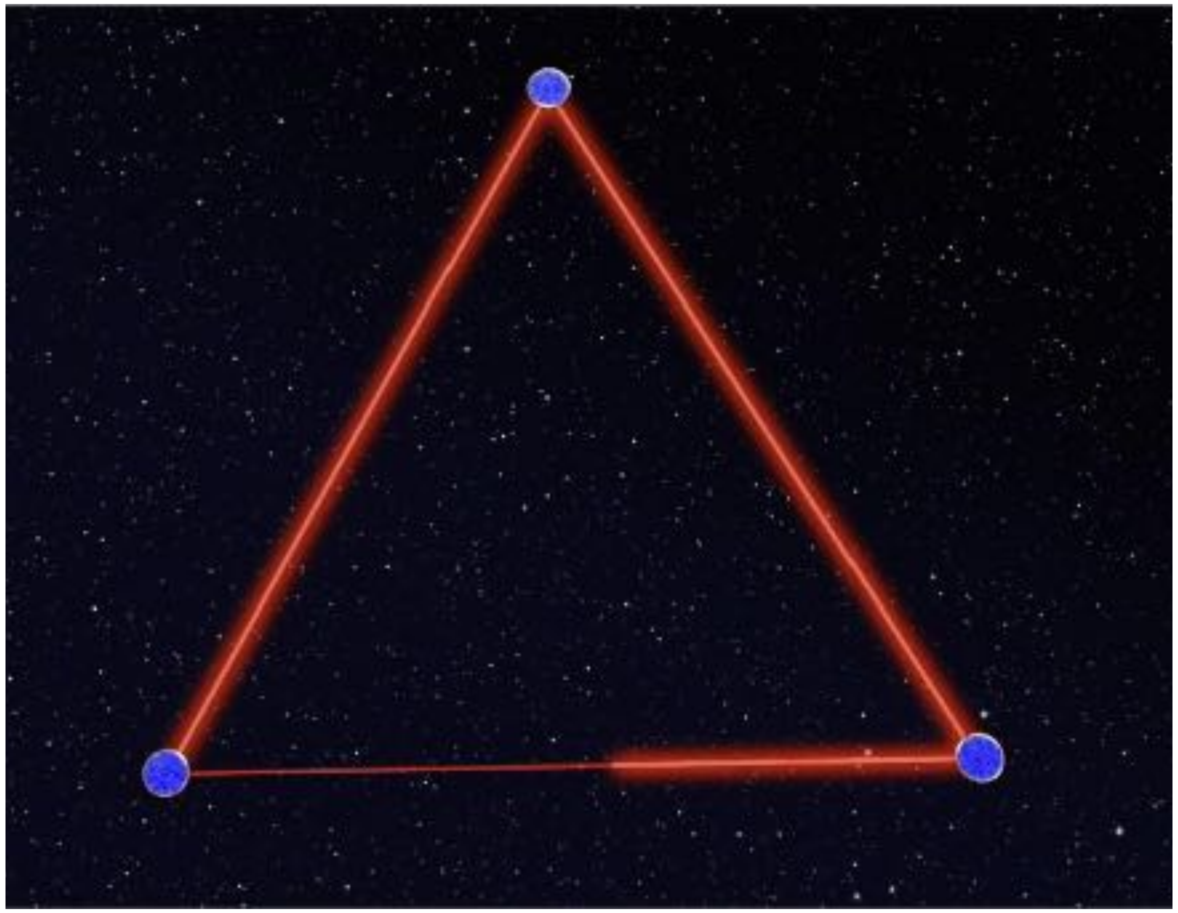


LIGO & Virgo collaborations

Gravity: data rich

Motivation

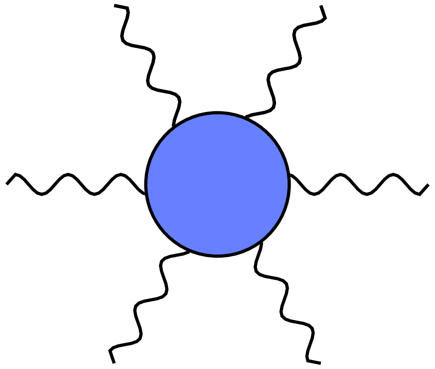
Eg LISA



High-precision gravitational wave theory required

Motivation

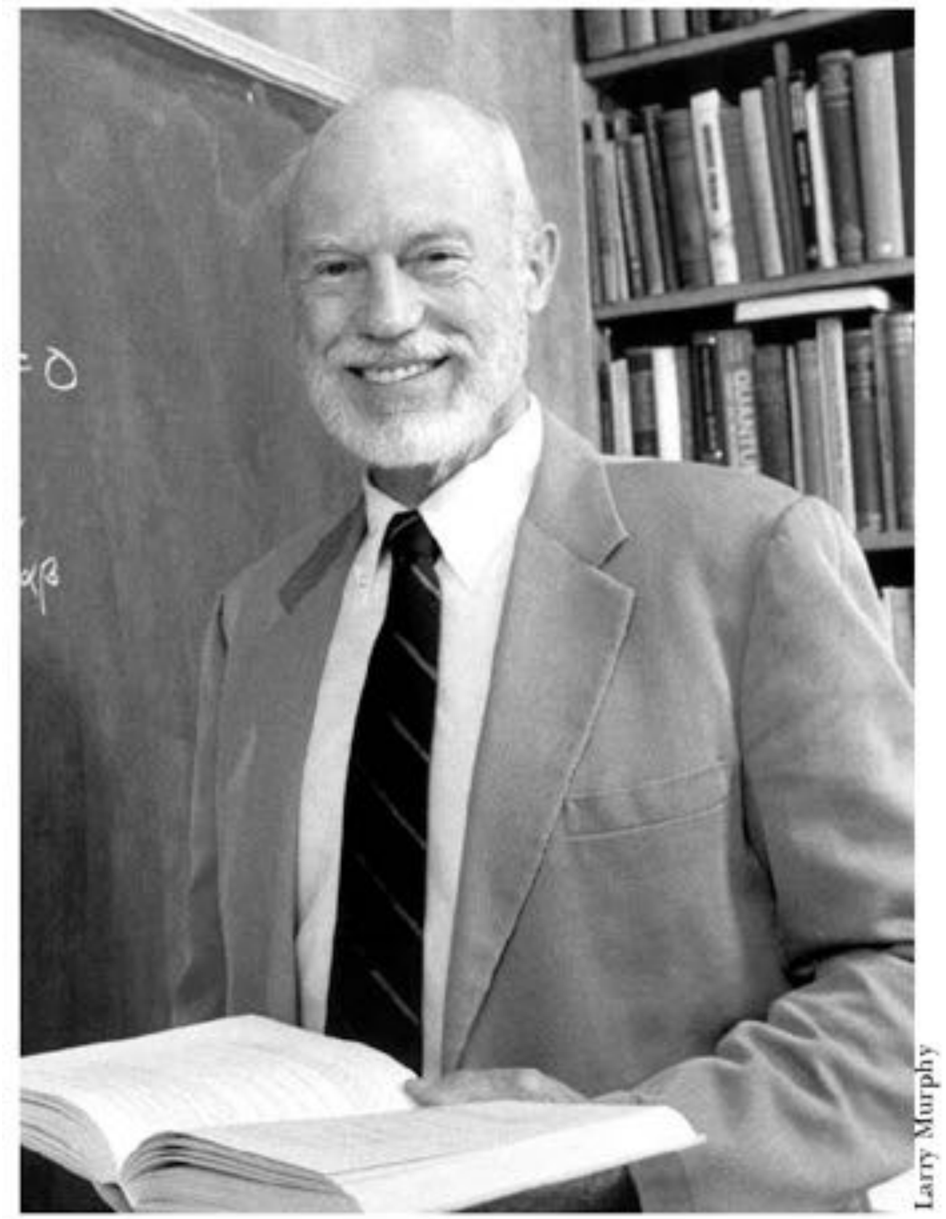
Early inspiral is perturbative



Motivation

Describe method for computing classical observables from amplitudes

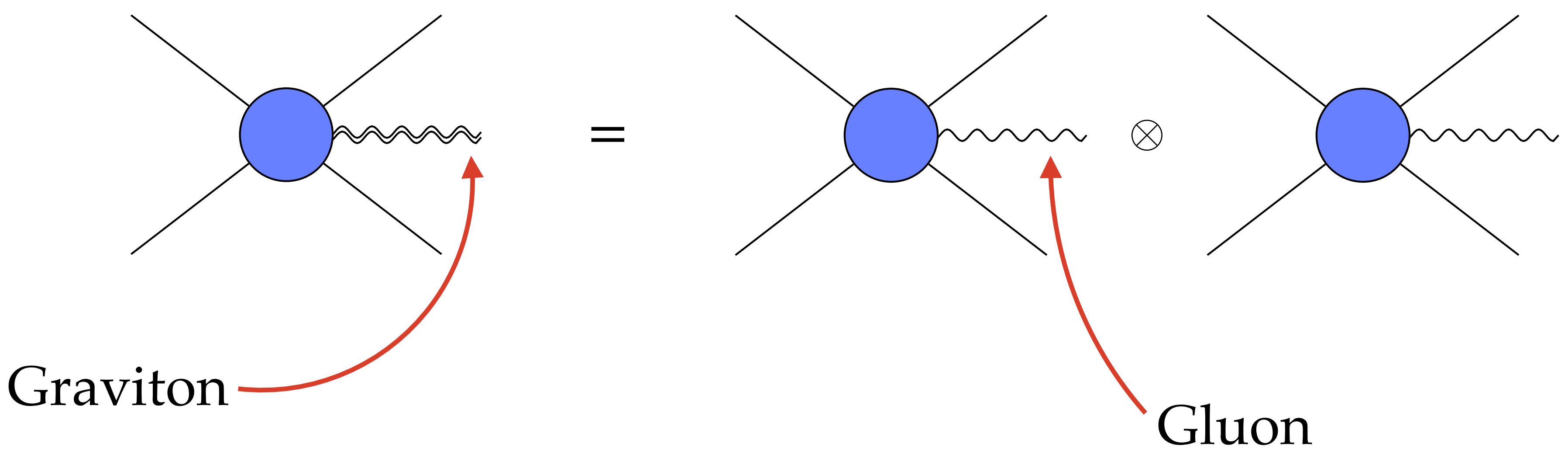
"Only observable in (quantum) gravity in asymptotically flat space is the S-matrix"



Motivation

Amplitudes: different perspective on gravity

The "double copy":



Kawai, Lewellen, Tye
Bern, Carrasco, Johansson
Cachazo, He, Yuan

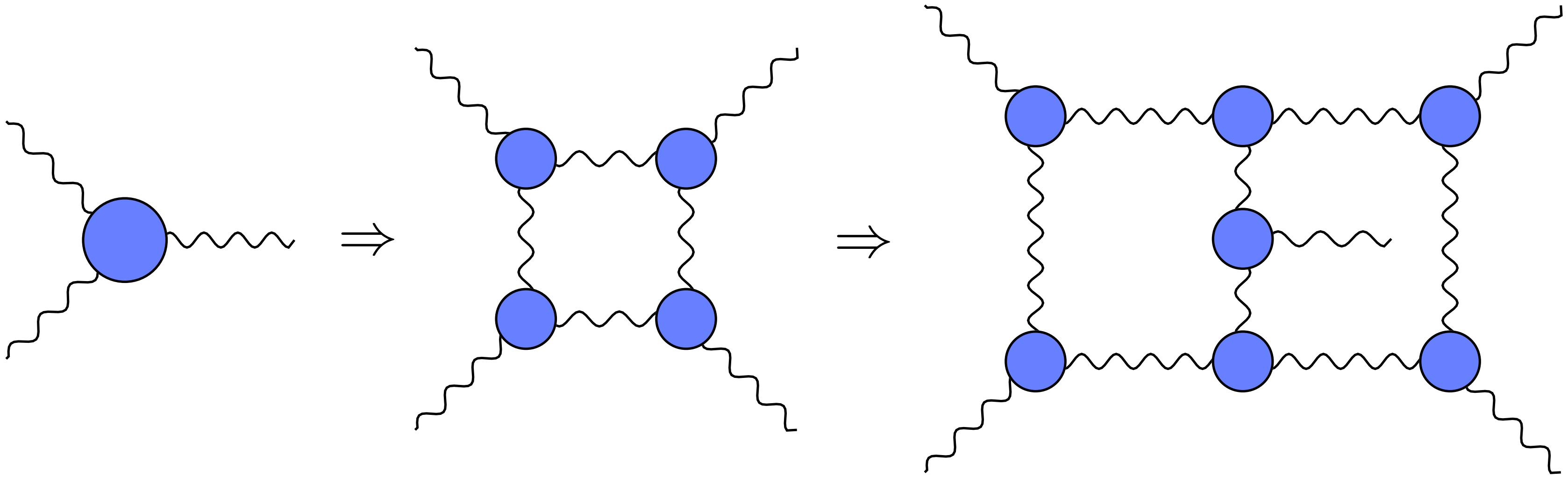
Gravity = Yang-Mills^2

Very bizarre from geometric point of view

Motivation

Amplitudes: different perspective on gravity

The unitarity method:



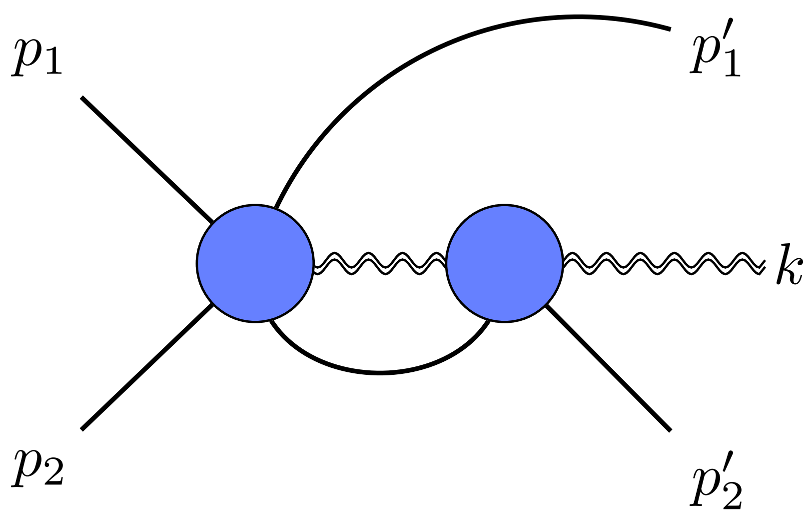
Bern, Dixon, Dunbar, Kosower
Neill, Rothstein

Gravitational processes determined by locality / quantum unitarity

At least novel from classical point of view

Motivation

My motivation: *learn about amplitudes*



Waveform $\neq$ cross-section

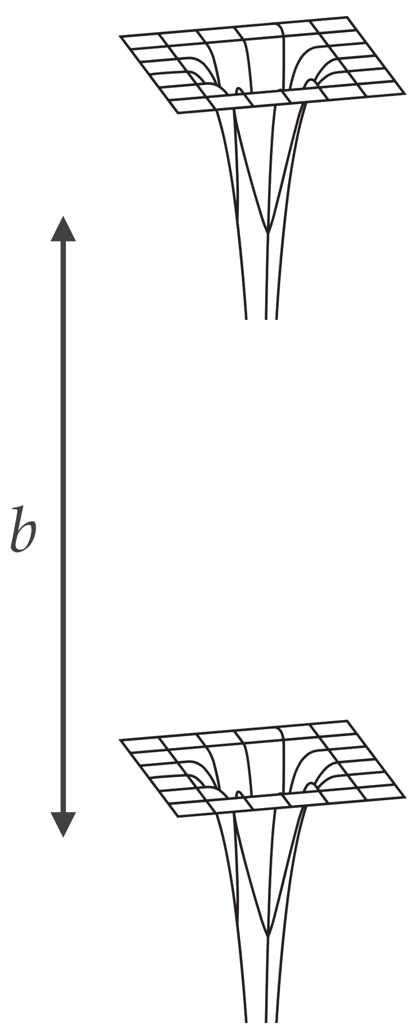
New questions, new challenges

This talk

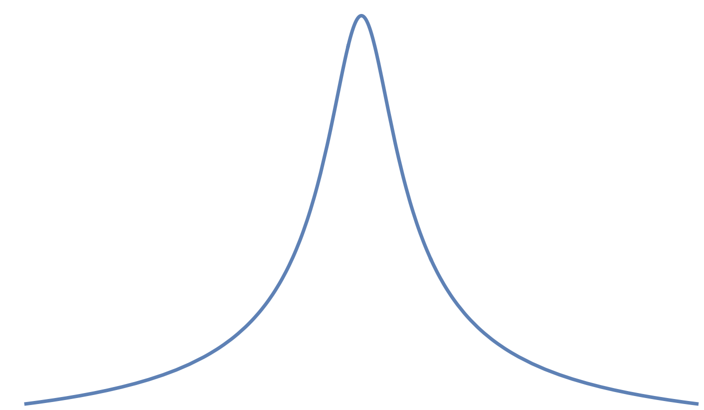
1. Waveforms from amplitudes
2. Waveform @ NLO
    1. In-in expectation
    2. Radiation reaction
    3. Integrals
    4. Surprises
3. Conclusions

Waveforms from Amplitudes

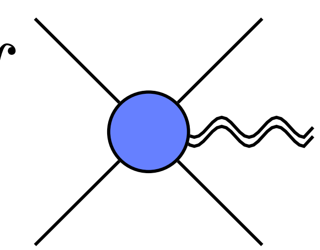
Waves from Amplitudes



$$\text{waveform}(t) =$$



$$= \frac{1}{\text{distance}} \int$$



$$+ \cdots$$

Classical point particle approximation: finite size under control

Kosower, Maybee & DOC

Cristofoli, Gonzo, Kosower & DOC

Waveforms & Amplitudes

Amplitudes arise from time evolution

$$|\psi\rangle \xrightarrow{\text{time}\to\infty} U(\infty,-\infty)|\psi\rangle$$

Initial state
Far past

Final state
Far future

Waveforms & Amplitudes

Amplitudes arise from time evolution

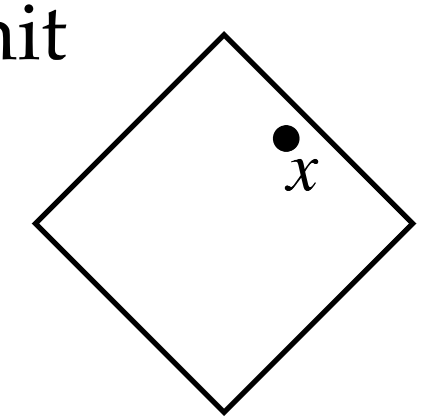
$$|\psi\rangle \xrightarrow{\text{time}\to\infty} S|\psi\rangle \qquad U(-\infty,\infty) = S = 1 + iT$$

Matrix elements of T are the amplitudes

$$\langle q_1 \cdots q_m | T | p_1 \cdots p_n \rangle = \mathcal{A}(p_1 \ldots p_n \to q_1 \ldots q_m)\, \delta^4(\text{total momentum})$$

Waveforms & Amplitudes

Measure *expectation* of curvature component in classical limit



Riemann curvature *operator* (linear)

$$\text{waveform} \equiv \langle\psi|S^\dagger \mathbb{R}_{....}(x) S|\psi\rangle$$

Final state Amplitudes!

Newman-Penrose scalar Ψ_4
Dominant curvature component at large distances

$$|\psi\rangle \sim \int \bigwedge \bigwedge e^{ip_1\cdot b}|p_1\, p_2\rangle$$

Classical initial conditions

E&M analogue useful

$$\text{EM waveform} \equiv \langle\psi|S^\dagger \mathbb{F}_{....}(x) S|\psi\rangle$$

Waveforms & Amplitudes

Measure *expectation* of curvature component in classical limit

$$\mathbb{R}....(x) = \partial.\partial.\mathbb{h}..(x)$$

Graviton polarisation

$$\text{waveform} = \int \widetilde{\mathrm{d}k}\,\left[k k\, \varepsilon\varepsilon\, e^{-ik\cdot x} \langle\psi|S^\dagger a(k) S|\psi\rangle + \text{c.c.}\right]$$

$$= i \int \widetilde{\mathrm{d}k}\, k k\, \varepsilon\varepsilon\, e^{-ik\cdot x} \langle\psi|a(k) T|\psi\rangle + \cdots + \text{c.c.}$$

LO!

Waveforms & Amplitudes

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Graviton polarisation

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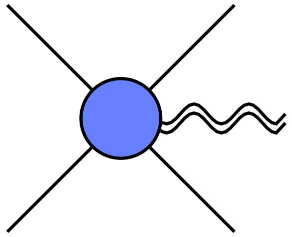
LO!

5 point amplitude

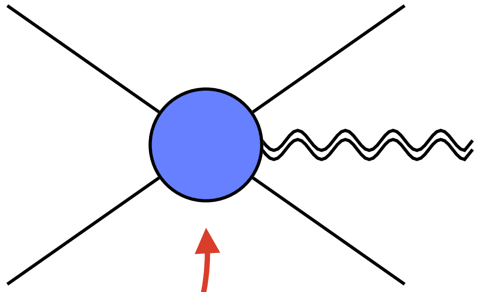


Waveforms & Amplitudes

Leading order:

Fourier integral: one bottleneck

$$\Psi_4(\omega) = \frac{1}{\text{distance}} \int \mathrm{d}^4 q_1 \mathrm{d}^4 q_2 \, \delta(p_1 \cdot q_1)\delta(p_2 \cdot q_2)\delta^4(k - q_1 - q_2) \, e^{ib\cdot(q_1 - q_2)}$$

$\times$ 

Waveform as a function of frequency
$k^\mu = (\omega, \omega\hat{\mathbf{n}})$

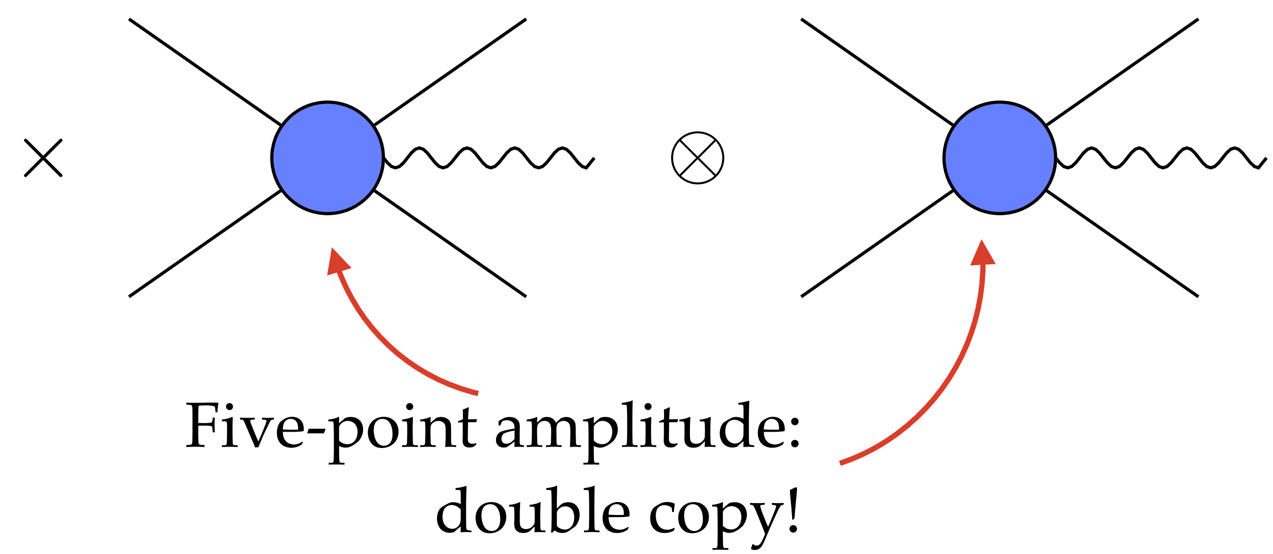
Five-point amplitude: double copy!

Cristofoli, Gonzo, Kosower & DOC

Waveforms & Amplitudes

Leading order:

$$\Psi_4(\omega) = \frac{1}{\text{distance}} \int \mathrm{d}^4 q_1 \mathrm{d}^4 q_2 \, \delta(p_1 \cdot q_1) \delta(p_2 \cdot q_2) \delta^4(k - q_1 - q_2) \, e^{ib\cdot(q_1 - q_2)}$$

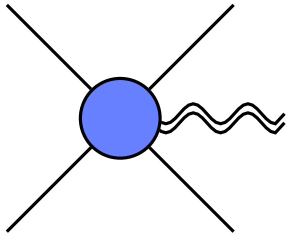


LO: Kovacs & Thorne, long ago

Eikonal connection

Seems contrary to intuition:

$$\text{waveform} = \frac{1}{\text{distance}} \int$$



$$+ \cdots$$

One graviton $\neq$ classical field

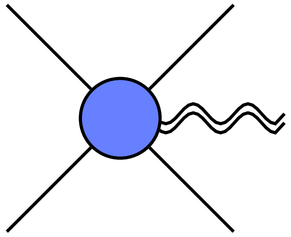
Classical field: expectation of a coherent state

$$\exp\left(\int \widetilde{\mathrm{d}k}\, \alpha(k)\, a^\dagger(k)\right) |0\rangle$$

Waveshape:
Fourier modes of classical field

Eikonal connection

Seems contrary to intuition:

$$\text{waveform} = \frac{1}{\text{distance}} \int$$  $$+ \cdots$$

Problem resolved if amplitude exponentiates in classical region

$$S|\psi\rangle \sim \int \exp\left(i\tilde{\mathcal{M}}_4(x,q)\right) \exp\left(\int \widetilde{\mathrm{d}k}\ (\mathcal{M}_5(x_1,x_2,k)+\cdots)\, a^\dagger(k)\right)|p_1',p_2'\rangle$$

Generalisation of eikonal exponentiation

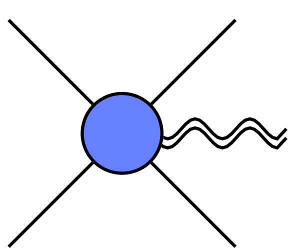
Ciafaloni, Colferai, Veneziano
Cristofoli, Gonzo, Moynihan, Ross, Sergola, White, DOC
Di Vecchia, Heissenberg, Russo, Veneziano

Bound binaries

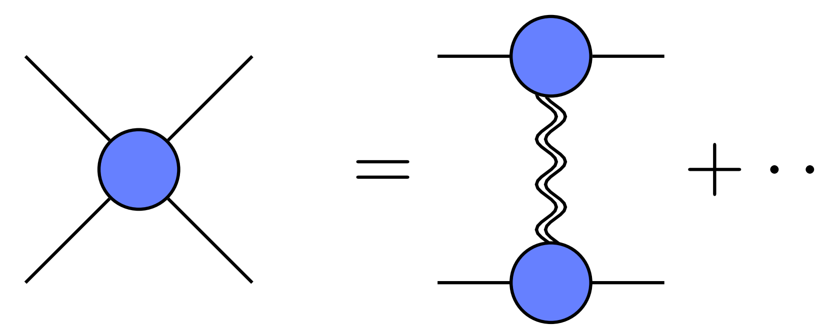
Bound case?

$$\text{Gravitational wave power} = -\frac{\mathrm{d}}{\mathrm{d}t}\text{Potential}$$

Einstein quadrupole power
+ corrections



Newton potential
+ corrections



Build EFT, valid in both bound & scattering cases

Match Wilson coefficients to scattering

Neill, Rothstein
Cheung, Rothstein, Solon
Bern, Cheung, Roiban, Shen, Solon, Zeng
Kälin, Porto

Waveform at NLO

Alessio, Bini, Brandhuber, Brown, Bohnenblust, Caron-Huot, Chen, Damour, De Angelis, Di Vecchia, Elkhidir, Giroux, Geralico, Georgoudis, Gowdy, Hannesdottir, Heissenberg, Herderschee, Ita, Krauss, Mizera, Roiban, Russo, Schlenk, Sergola, Teng, Travaglini, Vazquez-Holm, DOC, ..

1. In-in Expectation

$$\text{waveform} = \int \widetilde{\mathrm{d}k} \left[kk\, \varepsilon\varepsilon\, e^{-ik\cdot x} \langle \psi | S^\dagger a(k) S | \psi \rangle + \text{c.c.} \right]$$

Key object to understand

$$\langle \psi | S^\dagger a(k) S | \psi \rangle = \int_{p_i, p_i'} \psi^*(p_1', p_2') \psi(p_1, p_2) \boxed{\langle p_1', p_2' | S^\dagger a(k) S | p_1, p_2 \rangle}$$

In-in expectation $\mathcal{E}$

Elkhidir, Sergola, Vazquez-Holm, DOC

1. In-in Expectation

Expectation closely related to amplitudes

$$\mathcal{E}\,\delta^4(\text{momentum}) \equiv -i\langle p_1' p_2' | S^\dagger a(k) S | p_1 p_2 \rangle$$

$$= \langle p_1' p_2' k | T | p_1 p_2 \rangle - i \langle p_1' p_2' | T^\dagger a(k) T | p_1 p_2 \rangle$$

LO: tree amplitude

Higher orders: subtraction term

1. In-in Expectation

Define future creation / annihilation operators

$$b(k) \equiv S^\dagger a(k) S$$

$$i\mathcal{A}\delta^4(\text{momentum}) = \langle 0|b(p_1')b(p_2')b(k)a^\dagger(p_1)a^\dagger(p_2)|0\rangle$$

$$i\mathcal{E}\delta^4(\text{momentum}) = \langle 0|a(p_1')a(p_2')b(k)a^\dagger(p_1)a^\dagger(p_2)|0\rangle$$

Physics: different $i\epsilon$ prescription. In-out vs in-in, Schwinger-Keldysh

Expectation admits classical limit, amplitudes doesn’t

Caron-Huot, Giroux, Hannesdóttir, Mizera

1. In-in Expectation

Real parts equal

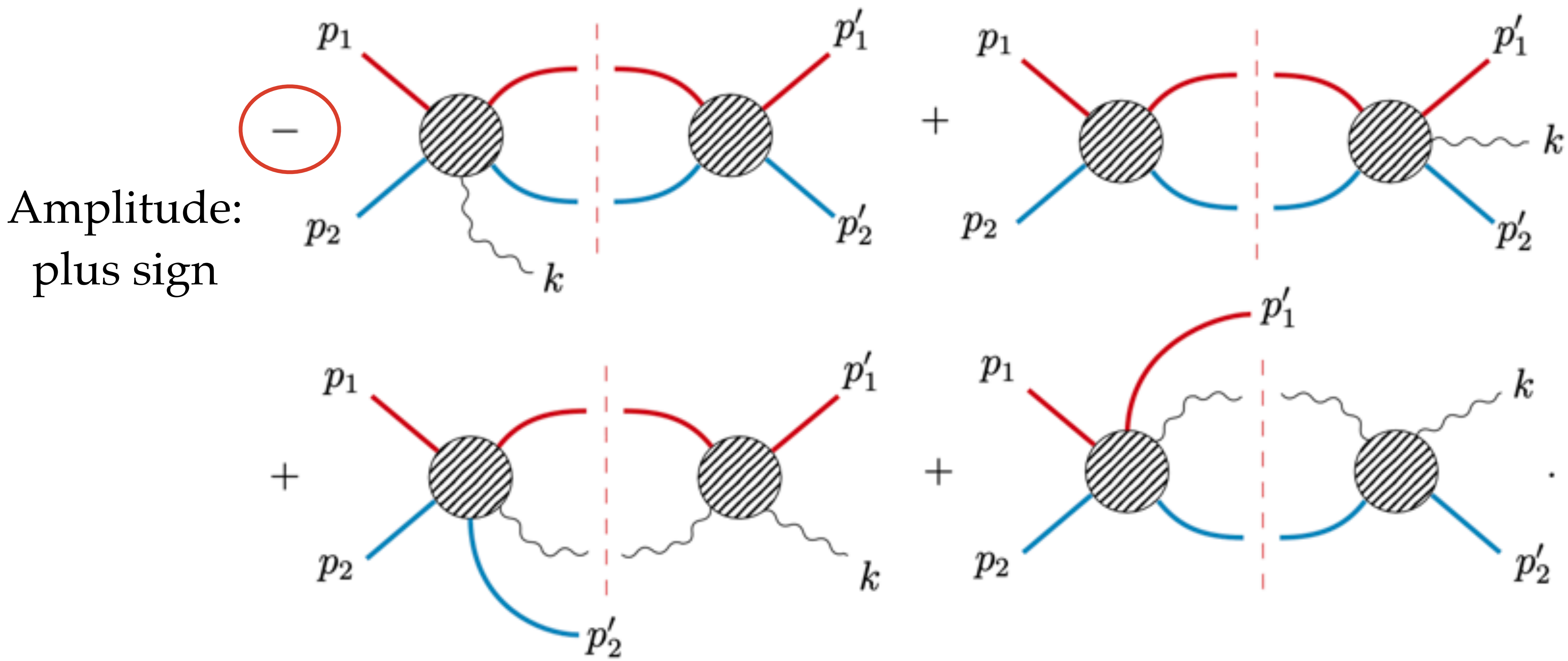
Treat polarisation vectors as real

$$\mathrm{Re}'\,\mathcal{E} = \mathrm{Re}'\,\mathcal{A}$$

1. In-in Expectation

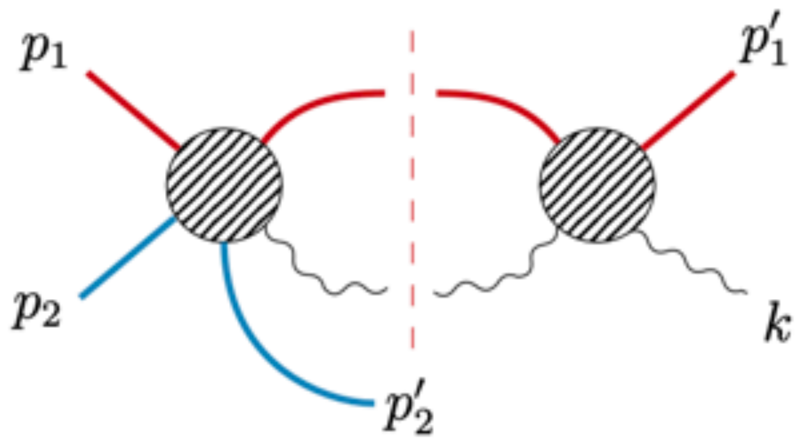
Im parts differ. At NLO:

$$2\,\mathrm{Im}'\,\mathcal{E}(p_1p_2 \to p_1'p_2'k_\eta)\hat{\delta}^D(p_1+p_2-p_1'-p_2'-k) =$$

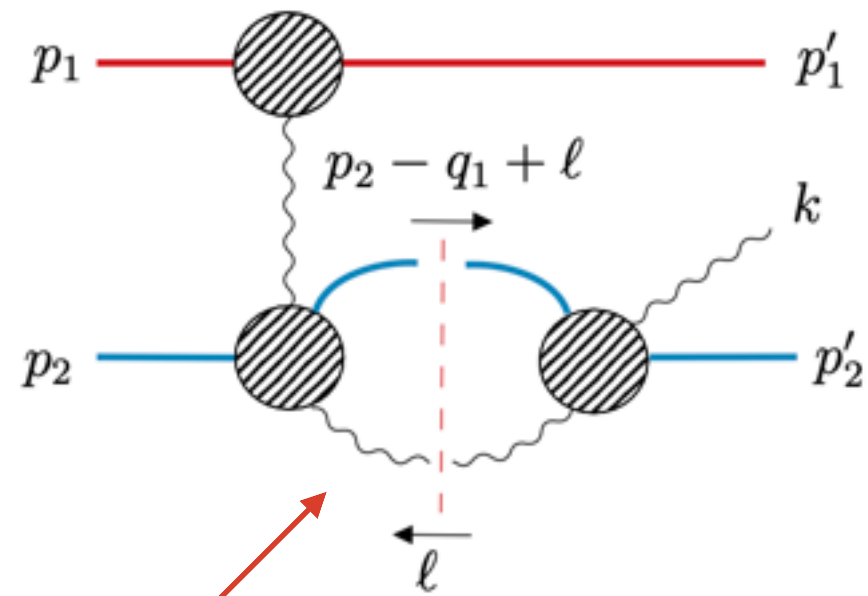


2. Radiation reaction

Focus on E&M



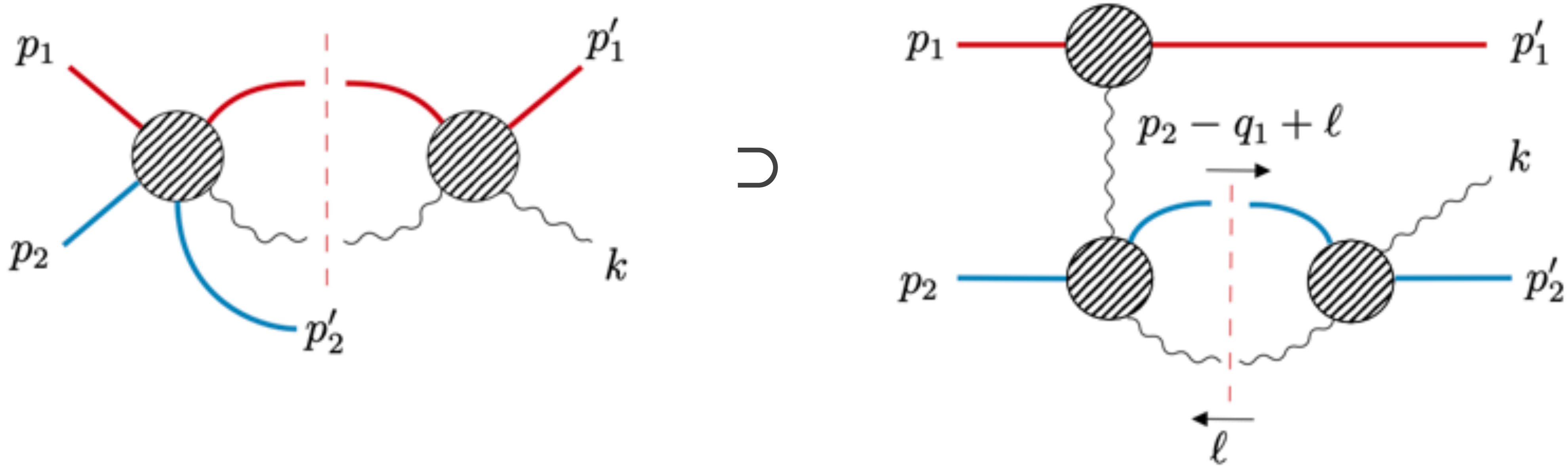
$\supset$



Particle 2 interacts with its own field

Elkhidir, Sergola, Vazquez-Holm, DOC

2. Radiation reaction



Omit self-field in undergrad EM — divergent, Abraham-Lorentz-Dirac

Amplitudes: textbook one-loop renormalization

Im part well-defined, physical, easily reproduces ALD

Would be interesting to understand in GR, YM

3. Integration

Loop and Fourier integrals

$$\int \mathrm{d}^D \ell \, v^\mu \frac{\partial}{\partial \ell^\mu} \frac{N(\ell)}{(\ell^2 + i\epsilon)(p \cdot \ell + i\epsilon) \cdots} = 0$$

$$\int \mathrm{d}^D q \, v^\mu \frac{\partial}{\partial q^\mu} e^{iq \cdot b} \frac{N(q)}{(q^2 + i\epsilon)(p \cdot q + i\epsilon) \cdots} = 0$$

One loop waveform ~ two loop complexity (in frequency domain)

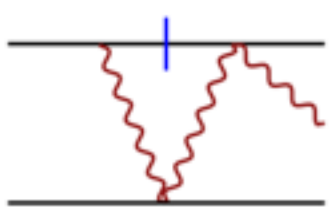
Matsumoto; Majima, Matsumoto, Takayama; Cacciatori, Mastrolia;
Brunello, Crisanti, Giroux, Mastrolia, Smith; Brunello, De Angelis

3. Integration

GR waveform currently only numerical

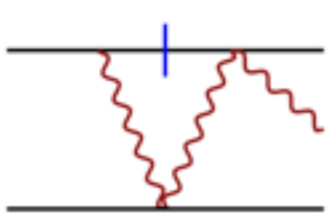
E&M: computed analytically (frequency space) last week!

Brunello & De Angelis



$$= \frac{\pi + 2i\log\frac{w_1+\sqrt{w_1^2-q_2^2}}{\sqrt{-q_2^2}}}{8\pi\sqrt{w_1^2-q_2^2}} + \mathcal{O}\left(\epsilon^1\right)$$

Struve function, index -1

$\mathcal{F}$ 

$$= \frac{i}{16\pi\sqrt{-b^2 p_\infty}}\left\{z\int_0^\infty \mathrm{d}x\left[e^{-z\cosh x}\,\mathbf{H}_{-1}\left(z\sqrt{p_\infty}\sinh x\right)\right] - i\,\frac{e^{-z\sqrt{1+p_\infty}}}{\sqrt{p_\infty}}\right\}$$

Bessel type

$$z = \frac{w_2|\mathbf{b}|}{\sqrt{\gamma^2-1}}$$

4. Surprises

- Waveform is IR divergent
  - But IR divergence is different to divergence of amplitudes

Caron-Huot, Giroux, Hannesdóttir, Mizera

4. Surprises

- Waveform is IR divergent
  - But IR divergence is different to divergence of amplitudes

Caron-Huot, Giroux, Hannesdóttir, Mizera

- Post-newtonian (small velocity) expansion compared to classical results (matched multipolar post-Minkowski approach)
  - Agreement found but choice of BMS frame important
  - "Intrinsic" BMS frame: include zero-energy 3-point amplitudes

Venziano, Vilkovisky, Bini, Damour, De Angelis, Geralico, Herderschee, Roiban, Teng

Conclusions

- Interesting dialogue between amplitudes and classical gravity
- Waveform constructed from in-in expectation
  - Close relationship to amplitude
  - Different pole prescription for some particles
- Technical challenges being overcome at one loop
- Conceptual challenges remain: IR divergence, BMS
- Need to understand bound states