The Heterotic-Ricci flow and its three-dimensional solitons

C. S. Shahbazi

UNED - Madrid and Hamburg University
In collaboration with Andrei Moroianu

Objective

The main goal of this talk is to introduce a novel curvature flow, the Heterotic-Ricci flow, as the two-loop renormalization group flow of the Heterotic string common sector and present a classification result of a particular class of its solitons.

Key characteristics of Heterotic-Ricci flow

- The Heterotic-Ricci flow is a coupled curvature evolution flow, depending on a non-negative real parameter κ, for a family of complete Riemannian metrics g_t and a family of three-forms H_t on a manifold M.
- The Heterotic-Ricci flow involves two terms quadratic in the curvature tensor of a metric connection with skew-torsion H.
- Solutions of Heterotic supergravity with trivial gauge bundle define a particular class of solitons of the Heterotic-Ricci flow.
- Similarly, solutions to the Hull-Strominger system with trivial gauge bundle and Bismut connection on the tangent bundle constitute a particular class of solitons of the Heterotic-Ricci flow.

Relations to existing flows

- When $\kappa = 0$ and $H_t = 0$ the Heterotic-Ricci flow the classical Ricci-flow.
- When $\kappa = 0$ the Heterotic-Ricci flow reduces to the generalized Ricci-flow [Oliynyk,Suneeta,Woolgar]: *higher order correction* of the latter.
- When $H = 0$ and $\kappa > 0$ the Heterotic-Ricci flow reduces to a constrained version of the RG-2 flow [Friedan] and hence it can alternatively be understood as a generalization of the latter via the three-form H.
- The Heterotic-Ricci flow on a complex manifold with trivial canonical bundle and $H_t = -\mathrm{d}^c\omega_t$ contains the Anomaly Flow [Phong,Picard,Zhang].
- The Heterotic-Ricci flow should be a particular case of generalized Ricci flow on a string Courant algebroid as introduced by García-Fernández.
- Being the renormalization group flow of a string theory, the Heterotic-Ricci flow can be expected to be related to other curvature evolution flows inspired by string theory: Type IIA/IIB/11d [Collins,Fei,Guo,Phong,Picard,Zhang].

Let X be a compact and oriented real two-dimensional manifold and let M be an oriented manifold. Given a Riemannian metric g on M, a two-form $b \in \Omega^2(M)$ and a function $\phi \in C^\infty(M)$, the bosonic string action determined by the triple (g, b, ϕ) on the pair (X, M) is the action functional [Becker,Becher,Schwarz]:

$$\mathcal{S}\colon \mathrm{Met}(X) \times C^\infty(X, M) \to \mathbb{R}\,,$$

defined on $\mathrm{Met}(X) \times C^\infty(X, M)$ by the following formula:

$$\mathcal{S}[h, \Psi] = -\frac{1}{\kappa}\int_X \left\{ |\mathrm{d}\Psi|^2_{h,g} + *_h(\Psi^* b) - \kappa\, \phi(\Psi)\, \mathrm{R}^h \right\} \nu_h\,.$$

These configuration space admits a large automorphism group of transformations preserving $\mathcal{S}$: among these, Weyl transformations, namely conformal rescalings of h by a positive real function, are particularly important.

The triple (g, b, ϕ) that determines the action functional $\mathcal{S}$ is not *dynamical* and is instead interpreted in this framework as the *couplings* of the bosonic string action. Every such choice (g, b, ϕ) gives rise to a well-defined theory at the *classical level*, that is, at the level of the classical equations of motion. In contrast, not every triple (g, b, ϕ) leads to a bosonic string action admitting a consistent quantization.

The quantization scheme of the bosonic string theory involves a regularization procedure that introduces an *ultra-violet cutoff* λ. Through this procedure, physical quantities, in particular, the couplings of the theory, normally acquire a dependence on the scale λ, in which case the theory is no longer conformally invariant. This implies that Weyl transformations are not guaranteed to be a symmetry of the bosonic string at the quantum level, something that cannot be allowed physically.

The dependence of the couplings of $\mathcal{S}$ on the renormalization scale λ is controlled through the *renormalization group flow equations*, which in the present case are given by:

$$\frac{\partial g_t}{\partial t} = -\beta_{g_t}\,, \qquad \frac{\partial b_t}{\partial t} = -\beta_{b_t}\,, \qquad \frac{\partial \phi_t}{\partial t} = -\beta_{\phi_t}\,,$$

where $t \in \mathbb{R}$ is the logarithm of the renormalization scale, (g_t, b_t, ϕ_t) denotes a one-parameter family of Riemannian metrics, two-forms and functions on M and:

$$\beta_{g_t} \in \Gamma(T^*M \odot T^*M)\,, \quad \beta_{b_t} \in \Omega^2(M)\,, \quad \beta_{\phi_t} \in C^\infty(M)\,,$$

denote the *beta functionals* of g_t, b_t, and ϕ_t.

Weyl invariance at the quantum level is controlled by $\beta_g = \beta_b = \beta_\phi = 0$ modulo *time dependent* diffeomorphisms.

Computing the beta functionals of the bosonic string is a complicated task that is usually performed perturbatively in the constant κ. To the lowest order in κ:

$$\frac{\partial g_t}{\partial t} = -2\kappa(\mathrm{Ric}^{g_t} - \tfrac{1}{4} H_t \circ H_t) + o(\kappa^2)\,,$$
$$\frac{\partial b_t}{\partial t} = -\kappa\,\delta^{g_t} H_t + o(\kappa^2)\,, \qquad \frac{\partial \phi_t}{\partial t} = c + \tfrac{\kappa}{2}\,(-\delta^{g_t}\varphi_t + |H_t|^2_{g_t}) + o(\kappa^2)\,,$$

where c is a constant that depends on the dimension of M and we have set $H_t := \mathrm{d}b_t$ and $\varphi_t := \mathrm{d}\phi_t$. Assuming $c = 0$ we obtain an evolution flow whose self-similar solutions solve the bosonic sector of NS-NS supergravity on M.

• By virtue of the evolution equation satisfied by b_t, we obtain the following evolution equation for H_t:

$$\frac{\partial H_t}{\partial t} = -\kappa\, \mathrm{d}\delta^{g_t} H_t + o(\kappa^2)\,.$$

This equation is sometimes considered in the renormalization group flow equations of the NS-NS worldsheet at first order in κ.

• The evolution equation for ϕ_t decouples and therefore can be considered separately.

• The *generalized Ricci flow* can therefore be introduced as the first-order renormalization group flow for (g_t, H_t), which after an appropriate *time* rescaling by κ is given by the following system of evolution equations:

$$\frac{\partial g_t}{\partial t} = -2\mathrm{Ric}^{g_t} + \frac{1}{2} H_t \circ H_t\,, \quad \frac{\partial H_t}{\partial t} = -\mathrm{d}\delta^{g_t} H_t\,,$$

for pairs (g_t, H_t).

• The generalized Ricci flow is being intensively studied in the literature: see [Streets, García-Fernández] and references therein. Various potential mathematical applications, including to the classification of compact complex surfaces.

• The generalized Ricci flow is the RG-flow of the bosonic string at first order in the parameter κ. Natural question: what about higher order corrections in κ?

• In order to consider higher order corrections in κ, we need to choose a particular string theory where to compute such corrections. This string theory can be the bosonic string itself or any of the five superstring theories, since all of them have the bosonic string as a common subsector.

• Warning: we do not compute the higher correction ourselves, we check and interpret the corresponding literature.

Considering the bosonic string as the NS-NS truncation of Heterotic string theory, a careful inspection of the computation of the beta functionals of the Heterotic worldsheet leads to the following RG flow equations for (g_t, H_t, ϕ_t) at second-order in κ [Metsaev and Tseytlin, Phys. Lett. B and Nucl. Phys. B, 1987]:

$$\frac{\partial g_t}{\partial t} = -2\kappa\,(\mathrm{Ric}^{g_t} - \tfrac{1}{4} H_t \circ_{g_t} H_t) - 2\kappa^2\,\mathcal{R}^{g_t,H_t} \circ_{g_t} \mathcal{R}^{g_t,H_t} + o(\kappa^3)\,,$$
$$\frac{\partial H_t}{\partial t} = -\kappa\,\mathrm{d}\delta^{g_t} H_t + o(\kappa^3)\,, \quad \frac{\partial \phi_t}{\partial t} = \frac{\kappa}{2}\,(|H_t|^2_{g_t} - \delta^{g_t}\mathrm{d}\phi_t - \kappa\,|\mathcal{R}^{g_t,H_t}|^2_{g_t}) + o(\kappa^3)\,.$$

together with the Bianchi identity $\mathrm{d}H_t + \kappa\,\langle \mathcal{R}^{g_t,H_t} \wedge \mathcal{R}^{g_t,H_t}\rangle_{g_t} = 0$.

As in the case of the renormalization group flow equations at first order, the evolution equation for the dilaton ϕ_t decouples from the evolution equations of (g_t, H_t) and therefore can be considered separately.

Proceeding as in the generalized Ricci-flow case, we define the *Heterotic-Ricci flow equations* as the renormalization group flow equations of the NS-NS sector of the Heterotic string at second order in κ, that is:

$$\frac{\partial g_t}{\partial t} = -2\left(\mathrm{Ric}^{g_t} - \frac{1}{4} H_t \circ H_t\right) - 2\kappa\, \mathfrak{v}_t(\mathcal{R}^{g_t,H_t} \circ_{g_t} \mathcal{R}^{g_t,H_t})\,, \quad \frac{\partial H_t}{\partial t} = -\mathrm{d}\delta^{g_t} H_t\,,$$
$$\mathrm{d}H_t + \kappa\, \mathfrak{v}_t(\mathcal{R}^{g_t,H_t} \wedge \mathcal{R}^{g_t,H_t}) = 0\,,$$

for pairs (g_t, H_t). When $\kappa = 0$ we recover the generalized Ricci flow.

The Heterotic-Ricci flow (g_t, H_t) can be completed into a solution of the two-loop RG flow equations of the NS-NS sector of the Heterotic string iff there exists a $\varphi_0 \in \Omega^1_{cl}(M)$ and a family of functions ϕ_t such that:

$$\frac{\partial \phi_t}{\partial t} = \frac{1}{2}\left(|H_t|^2_{g_t} - \delta^{g_t}\varphi_t - \kappa\, |\mathcal{R}^{g_t,H_t}|^2_{g_t}\right),$$

where we have set $\varphi_t = \varphi_0 + \mathrm{d}\phi_t$.

Some notation

We have defined the bilinear maps:

$$(H \circ_g H)_{ij} = H_{ilm} H_j^{\ lm} , \quad (\mathcal{R}_\nabla \circ_g \mathcal{R}_\nabla)_{ij} = (\mathcal{R}_\nabla)_{iklm} (\mathcal{R}_\nabla)_j^{\ klm} ,$$

The latter term is therefore a higher order term in the curvature of g. Recall:

$$\mathcal{R}^{g,H}(v_1, v_2, v_3, v_4) = \mathrm{R}^g(v_1, v_2, v_3, v_4) + \frac{1}{2} g(H(v_1, v_4), H(v_2, v_3)) \\ -\frac{1}{2} g(H(v_2, v_4), H(v_1, v_3)) - \frac{1}{2}(\nabla^g_{v_1} H)(v_2, v_3, v_4) + \frac{1}{2}(\nabla^g_{v_2} H)(v_1, v_3, v_4)$$

$\mathcal{R}_\nabla \circ_g \mathcal{R}_\nabla$ results in a very complicated expression that can be simplified in 3d.

Let (g_t, H_t) be a Heterotic-Ricci flow and let $\sigma \in H^1(M, \mathbb{R})$ be a cohomology class. A dilaton for (g_t, H_t) with Lee class σ is a family of closed one-forms (φ_t) on M such that $\sigma = [\varphi_t]$ and such that writing $\varphi_t = \varphi_0 + \mathrm{d}\phi_t$ in terms of any closed one-form $\varphi_0 \in \Omega^1(M)$ then (ϕ_t) solves de dilaton flow equation.

Let (g_t, H_t) be a Heterotic-Ricci flow. Then, for each cohomology class $\sigma \in H^1(M, \mathbb{R})$ there exists a dilaton flow φ_t with Lee class σ associated to (g_t, H_t).

Plugging $\varphi_t = \varphi_0 + \mathrm{d}\phi_t$ in the dilaton flow equation we obtain:

$$\frac{\partial \phi_t}{\partial t} = \frac{1}{2}\left(-\Delta_{g_t}\phi_t + |H_t|^2_{g_t} - \kappa\, |\mathcal{R}^{g_t,H_t}|^2_{g_t} - \delta^{g_t}\varphi_0\right),$$

where $\Delta_{g_t} = \mathrm{d}\delta^{g_t} + \delta^{g_t}\mathrm{d}$. This is a linear parabolic equation for ϕ_t which admits a unique smooth solution such that $\phi_0 = 0$.

By the previous result it is not necessary to be concerned with the dilaton equation in the following, although it is important to define the solitons of the flow!

The proper *gauge theoretic* formulation of the Heterotic-Ricci flow in terms of the *b-field* instead of H requires defining the Heterotic-Ricci flow as a differential system on a string structure on the frame bundle of M or as a generalized Ricci flow on a string Courant algebroid: Baraglia, García-Fernández, Hekmati, Rubio.

• Expectation: the Heterotic-Ricci flow is a particular type of generalized Ricci flow on a string Courant algebroid whose associated principal bundle is the frame bundle of M. For this to work, a natural condition on the evolving generalized metric needs to be found to guarantee that the naturally associated *gauge* connection is ∇^{g_t, H_t}.

Check the very recent preprint *Ricci flow on Courant algebroids* by Jeffrey Streets, Charles Strickland-Constable, and Fridrich Valach!

Compare with the recent preprint *Ricci flow on Courant algebroids* by Jeffrey Streets, Charles Strickland-Constable, and Fridrich Valach!

Example 7.8. A further example occurs in [19], where the explicit generalized Ricci flow equations are derived for a class of transitive Courant algebroids obtained by reduction (of equivariant exact Courant algebroids). The resulting equation can be expressed in terms of a metric g, a two-form B, a dilaton φ, and a principal G-connection A with curvature F as

$$\begin{aligned}
\frac{\partial}{\partial t} g_{ij} &= -2\,\mathrm{Rc}_{ij} - 4\nabla_i\nabla_j\varphi + \tfrac{1}{2} H_i{}^{kl} H_{jkl} + \tfrac{1}{2}\,\mathrm{Tr}\, F_{ik} F_j{}^k,\\
\frac{\partial}{\partial t} B_{ij} &= \nabla^k H_{kij} - 2(\nabla^k\varphi) H_{kij}, \qquad H = H_0 + dB\\
\frac{\partial}{\partial t}\varphi &= \Delta_g\varphi - 2(\nabla\varphi)^2 + \tfrac{1}{12} H_{ijk}H^{ijk} + \tfrac{1}{16}\,\mathrm{Tr}\, F_{ij}F^{ij},\\
\frac{\partial}{\partial t} A_i &= \nabla^k F_{ki} - 2(\nabla^k\varphi) F_{ki} - \tfrac{1}{2} H_{ijk} F^{jk},
\end{aligned}$$

where ∇ is the combination of the Levi-Civita connection with A, and $\mathrm{Tr}(XY)$ is an invariant inner product on $\mathfrak{g}$. Thus it is a further coupling of the generalized Ricci flow on exact Courant algebroids (1.1) to Yang-Mills flow.

When M is two-dimensional we have $H_t = 0$ and the Heterotic-Ricci flow reduces to the RG-2 flow [Gimre, Guenther and Isenberg]:

$$\frac{\partial g_t}{\partial t} = -2\,\mathrm{Ric}^{g_t} - 2\kappa\,\mathfrak{v}_{g_t}(\mathrm{R}^{g_t} \circ \mathrm{R}^{g_t})\,,$$

In this dimension the RG-2 flow equations simplify to:

$$\frac{\partial g_t}{\partial t} = -(s_{g_t} + \frac{\kappa}{2} s_{g_t}^2)\, g_t\,,$$

where s_{g_t} denotes the scalar curvature of g_t, see [Oliynyk 2009] for more details.

More generally, when $H_t = 0$ the Heterotic-Ricci flow reduces to the RG-2 flow supplemented with the condition $\langle \mathcal{R}^{g_t,H_t} \wedge \mathcal{R}^{g_t,H_t} \rangle_{g_t} = 0$: novel condition?.

In three dimensions the Heterotic-Ricci flow reduces to:

$$\partial_t g_t = -2\left(\mathrm{Ric}^{g_t} - \frac{1}{4} H_t \circ H_t\right) - 2\kappa\, \mathcal{R}^{g_t,H_t} \circ_{g_t} \mathcal{R}^{g_t,H_t}\,, \quad \partial_t H_t = -\mathrm{d}\delta^{g_t} H_t$$

If $H_t = H_0 + \mathrm{d}b_t$ then $\partial_t(g_t + b_t) = -2\mathrm{Ric}^{g_t,H_t} - 2\kappa\, \mathcal{R}^{g_t,H_t} \circ_{g_t} \mathcal{R}^{g_t,H_t}$.

Proposition

$(g_t, f_t) \in \mathrm{Conf}(M)$ *is a Heterotic-Ricci flow iff:*

$$\partial_t g_t = 2\kappa\, \mathrm{Ric}^{g_t} \circ \mathrm{Ric}^{g_t} - (2 + \kappa(2s^{g_t} - f_t^2))\mathrm{Ric}^{g_t}$$
$$+(f_t^2 + \kappa((s^{g_t})^2 - 2|\mathrm{Ric}^{g_t}|^2_{g_t} - \tfrac{1}{2}|\mathrm{d}f_t|^2_{g_t} - \tfrac{1}{4} f_t^4))g_t - \kappa[*_{g_t}\mathrm{d}f, \mathrm{Ric}^{g_t}] - \tfrac{1}{2}\kappa\, \mathrm{d}f_t \otimes \mathrm{d}f_t$$
$$\partial_t f_t + \tfrac{1}{2}\mathrm{Tr}_{g_t}(\partial_t g_t) f_t + \Delta_{g_t} f_t = 0$$

Left invariant flows?, parabolicity regime? short time existence?.

The Heterotic-Ricci solitons

Given a curvature flow, one of the first issues that need to be addressed is the classification of its solitons, namely its *self-similar solutions*: for the Heterotic-Ricci flow this leads to the notion of the Heterotic-Ricci soliton system.

Rather than considering the general Heterotic-Ricci soliton system, we consider in the following those Heterotic-Ricci solitons that arise as solutions of Heterotic supergravity with trivial gauge bundle: simplied system of soliton equations.

Heterotic supergravity can be thought of as the *supersymmetrization* of the Einstein-Yang-Mills system $(\mathrm{R}^g - |F_A|^2_{g,\mathfrak{c}})$ in ten dimensions compatibly with Heterotic string theory: Heterotic supergravity encodes the low-energy dynamics of the *massless* modes of Heterotic string theory. Basic ingredients:

- A principal bundle P with semi-simple compact structure group G and Lie algebra $\mathfrak{g}$ over an oriented four-manifold M.
- Positive definite inner product $\mathfrak{c}$ on the adjoint bundle $\mathfrak{g}_P$ of P.
- A positive constant $\kappa > 0$, the string slope parameter.

We will formulate bosonic Heterotic supergravity on a fixed tuple $(M, P, \mathfrak{c}, \kappa)$.

Fix $(M, P, \mathfrak{c}, \kappa)$. (Bosonic) Heterotic supergravity associated to $(M, P, \mathfrak{c}, \kappa)$ is defined through the following differential system [Bergshoeff,Roo]:

$$\mathrm{Ric}^g + \nabla^g \varphi - \frac{1}{2} H \circ H - \kappa\, \mathfrak{c}(\mathcal{F}_A \circ \mathcal{F}_A) + \kappa\, \mathfrak{v}(\mathcal{R}_{\nabla^H} \circ \mathcal{R}_{\nabla^H}) = 0\,,$$
$$\delta^g H + \iota_\varphi H = 0\,, \quad \mathrm{d}_A * \mathcal{F}_A - \varphi \wedge * \mathcal{F}_A + \mathcal{F}_A \wedge * H = 0\,,$$
$$\delta^g \varphi + |\varphi|^2_g - |H|^2_g - \kappa\, |\mathcal{F}_A|^2_{g,\mathfrak{c}} + \kappa\, |\mathcal{R}_{\nabla^H}|^2_{g,\mathfrak{v}} = 0\,,$$

together with the *Bianchi identity*:

$$\mathrm{d}H = \kappa(\mathfrak{c}\,(\mathcal{F}_A \wedge \mathcal{F}_A) - \mathfrak{v}(\mathcal{R}_{\nabla^H} \wedge \mathcal{R}_{\nabla^H}))\,,$$

for variables (g, φ, H, A), where g is a Riemannian metric on M; $\varphi \in \Omega^1(M)$, $H \in \Omega^3(M)$, A connection on P and $\nabla^H = \nabla^g - \frac{1}{2} H^\sharp$.

Suppose that M is spin. Given (g, φ, H, A), let S_g denote a bundle of irreducible real spinors on (M, g).

Definition

A tuple $(g, \varphi, H, A) \in \mathrm{Sol}_\kappa(M, P, \mathfrak{c})$ is *supersymmetric* if there exists a spinor bundle S_g and a spinor $\epsilon \in \Gamma(S_g)$ such that:

$$\nabla^{-H}\epsilon = 0\,, \qquad \varphi\cdot\epsilon = H\cdot\epsilon\,, \qquad \mathcal{F}_A\cdot\epsilon = 0\,.$$

These are *Killing spinor equations* of Heterotic supergravity.

By a theorem of S. Ivanov, $(g, \varphi, H, A, \epsilon)$ satisfying the KSE and the BI belongs to $\mathrm{Sol}_\kappa(M, P, \mathfrak{c})$ if and only if the connection ∇^H is an *instanton*.

Existence of compatible KSE is a consequence of supersymmetry: the KSE are obtained by imposing the vanishing of the Heterotic supersymmetry transformations on a given bosonic background.

The Heterotic Killing spinor equations together with the Bianichi identity conform the celebrated Hull-Strominger system, written in its spinorial form. In even dimensions, it has an equivalent description in terms of polystable holomorphic vector bundle over a locally conformally balanced complex manifold [Fernández,Fei,Fu,Ivanov,Tseng,Ugarte,Yau].

In three dimensions solutions to the Hull-Storminger system are flat. In four dimensions, compact solutions of the Hull-Strominger system correspond to anti-self-dual instantons over either flat complex tori, K3 surfaces [Strominger 1986] or quaternionic complex Hopf surfaces [García-Fernández, Rubio, Tipler and CSS, 2018].

What about non-supersymmetric solutions: much harder to obtain and classify.

Not known examples?

Assume that $P = M$ is trivial. Heterotic supergravity $\to$ Heterotic soliton system:

$$\mathrm{Ric}^{g,H} + \nabla^{g,H}\varphi + \kappa\,\mathcal{R}^{g,H} \circ_g \mathcal{R}^{g,H} = 0\,, \quad \delta^g\varphi + |\varphi|_g^2 - |H|_g^2 + \kappa\,|\mathcal{R}^{g,H}|_g^2 = 0$$

together with the *Bianchi identity*:

$$\mathrm{d}H + \kappa\langle\mathcal{R}^{g,H} \wedge \mathcal{R}^{g,H}\rangle_g = 0$$

Heterotic solitons: Heterotic soliton system. The limit $\kappa \to 0$:

$$\mathrm{Ric}^{g,H} + \nabla^{g,H}\varphi = 0\,, \quad \delta^g\varphi + |\varphi|_g^2 - |H|_g^2 = 0\,, \qquad \mathrm{d}H = 0$$

recovers a particular case of generalized Ricci soliton, or NS-NS supergravity.

Therefore, we can think of Heterotic solitons as a certain extension of generalized Ricci solitions in a Heterotic string theory framework.

A remarkable identity

Proposition

The following formula holds:

$$\nabla^{g*}\mathcal{E}^s_{\mathrm{E}}(g,\varphi,H) + \varphi\lrcorner\mathcal{E}^s_{\mathrm{E}}(g,\varphi,H) + \frac{1}{2}\mathrm{d}\mathrm{Tr}_g(\mathcal{E}^s_{\mathrm{E}}(g,\varphi,H)) = \langle\mathcal{R}^{g,H}_v, \nabla^{g,H*}\mathcal{R}^{g,H} + \varphi\lrcorner\mathcal{R}^{g,H}\rangle_g$$
$$+\frac{1}{2}v\lrcorner\mathrm{d}\mathcal{E}_{\mathrm{D}}(g,\varphi,H) - \langle\mathcal{E}^a_{\mathrm{E}}(g,\varphi,H), H(v)\rangle_g - \frac{1}{2}\langle H, v\lrcorner\mathcal{E}_{\mathrm{B}}(g,H)\rangle_g$$

for every $(g,\varphi,H)\in\mathrm{Conf}(M)$.

Every $(g,\varphi,H)\in\mathrm{Conf}(M)$ satisfying $\mathcal{E}^a_{\mathrm{E}}(g,\varphi,H)=0$, $\mathcal{E}_{\mathrm{D}}(g,\varphi,H)=0$, $\mathcal{E}^a_{\mathrm{E}}(g,\varphi,H)=0$ and $\nabla^{g,H*}\mathcal{R}^{g,H}+\varphi\lrcorner\mathcal{R}^{g,H}=0$ satisfies:

$$\nabla^{g*}\mathcal{E}_{\mathrm{E}}(g,\varphi,H) + \varphi\lrcorner\mathcal{E}_{\mathrm{E}}(g,\varphi,H) + \frac{1}{2}\mathrm{d}\mathrm{Tr}_g(\mathcal{E}_{\mathrm{E}}(g,\varphi,H)) = 0\,.$$

Related to the existence of a variational principle for the Heterotic soliton system.
Strong condition: $\nabla^{g,H*}\mathcal{R}^{g,H}+\varphi\lrcorner\mathcal{R}^{g,H}=0 \to$ Strong Heterotic soliton system.

Proposition

A triple $(g, \varphi, H) \in \mathrm{Conf}(M)$ is a strong Heterotic soliton iff the associated triple $(\bar{g}_\kappa, \bar{\varphi}, \bar{H})$ satisfies:

$$\mathrm{Ric}^{\bar{g}_\kappa, \bar{H}_\kappa} + \nabla^{\bar{g}_\kappa, \bar{H}_\kappa} \bar{\varphi} = 0\,, \qquad \delta^{\bar{g}_\kappa} \bar{\varphi} + |\bar{\varphi}|^2_{\bar{g}_\kappa} - |\bar{H}_\kappa|^2_{\bar{g}_\kappa} = 0\,, \qquad \mathrm{d}\bar{H}_\kappa = 0$$

$\bar{\varphi} = \pi^* \varphi$, $\bar{H}_\kappa = H + \kappa\, \mathrm{CS}(\mathcal{A}_{g,H})$ *and* $\bar{g}_\kappa = (\pi^* g)(-,-) - \kappa\, \mathrm{Tr}(\mathcal{A}_{g,H}(-), \mathcal{A}_{g,H}(-))$.

The previous quations correspond to the equations of motion of NS-NS supergravity for a pseudo-Riemannian metric of split signature and a three-form $\bar{H}_k$ in a *string class*. The fact that the signature of the metric $\bar{g}$ on the total space of $\mathrm{Fr}_g(M)$ is of split signature is not accidental and can be traced back to the sign of the higher order term $\mathcal{R}^{g,H} \circ_g \mathcal{R}^{g,H}$. Had this term appeared with the opposite sign then the corresponding $\bar{g}$ would be positive define.

Heterotic solitons $\sim$ NS-NS supergravity/generalized RS in split signature.

Proposition

A triple $(g, \varphi, f) \in \mathrm{Conf}(M)$ *is a Heterotic soliton iff (below* $f = *_g H$*):*

$$-\kappa\,\mathrm{Ric}^g \circ_g \mathrm{Ric}^g + (1 + \kappa s_g - \tfrac{\kappa}{2} f^2)\mathrm{Ric}^g + \left(\kappa|\mathrm{Ric}^g|^2_g - \tfrac{\kappa}{2} s_g^2 + \tfrac{\kappa}{4}|\mathrm{d}f|^2_g - \tfrac{1}{2} f^2 + \tfrac{\kappa}{8} f^4\right) g$$
$$+ \tfrac{\kappa}{2}\,[*_g \mathrm{d}f, \mathrm{Ric}^g] + \tfrac{\kappa}{4}\mathrm{d}f \otimes \mathrm{d}f + \nabla^g \varphi = 0$$
$$f\varphi = \mathrm{d}f\,, \qquad s_g = 3\,\delta^g \varphi + 2|\varphi|^2 - \tfrac{1}{2} f^2$$

Proposition

Let $(g, \varphi, f) \in \mathrm{Sol}_\kappa(M)$*.* $\exists\, \{\Psi_t\}_{t\in\mathcal{I}} \,|\, (g_t, , f_t) = (\psi_t^* g, f \circ \psi_t)$ *is a Heterotic-Ricci flow.*

Proof.

Using $\partial_t(\psi_t^* g) = \psi_t^* \mathcal{L}_\varphi g = 2\psi_t^* \nabla^g \varphi$ we compute:

$$\partial_t(f \circ \psi_t) + \frac{1}{2}\mathrm{Tr}_{\psi_t^* g}(\partial_t \psi_t^* g) f \circ \psi_t + \Delta_{\psi_t^* g}(f \circ \psi_t) = \psi_t^*(\mathrm{d}f(\varphi) + \mathrm{Tr}_g(\nabla^g \varphi) + \delta^g \mathrm{d}f) = 0$$

Here we have used that $\mathrm{d}f = f\varphi$, which implies $\delta^g \mathrm{d}f = -\mathrm{d}f(\varphi) + \delta^g \varphi$. □

Proposition

A three-dimensional Heterotic soliton is trivial if and only if $f = 0$.

Proposition

Let (g, φ, f) be a non-trivial three-dimensional Heterotic soliton. Then, there exists a function $\phi \in C^\infty(M)$ such that $\varphi = \mathrm{d}\phi$ and $f = c\, e^{\phi}$ for a non-zero constant $c \in \mathbb{R}^*$.

Proposition

Let $(g, f) \in \mathrm{Sol}_\kappa(M)$ be a non-trivial Heterotic solition. Then, the scalar curvature s_g of g is strictly negative in some open set of M.

Examples of three-dimensional Heterotic solitons

Proposition

$(g, f) \in \mathrm{Sol}_\kappa(M)$ *has constant principal Ricci curvatures* (μ_1, μ_2, μ_3) *if* f *is a non-vanishing constant, in which case:*

1. $\kappa f^2 = 1$ *and* $(\mu_1 = \mu_2 = -\frac{1}{2\kappa}, \mu_3 = \frac{1}{2\kappa})$.
2. $\kappa f^2 = 2$ *and* $(\mu_1 = \mu_2 = 0, \mu_3 = -\frac{1}{2\kappa})$. *In particular, the universal cover of* M *is isometric to either* $\widetilde{\mathrm{Sl}}(2, \mathbb{R})$ *or* $\mathrm{E}(1,1)$ *equipped with a left-invariant metric.*
3. $\kappa f^2 = 3$ *and* $(\mu_1 = \mu_2 = \mu_3 = -\frac{1}{2\kappa})$. *In particular* (M, g) *is a hyperbolic three-manifold endowed with a metric of scalar curvature* $-\frac{3}{2\kappa}$.

Proof.

With the given assumptions the problem reduces to an algebraic equation for Ric^g:

$$-\kappa\, \mathrm{Ric}^g \circ_g \mathrm{Ric}^g + (1 - \kappa f^2)\mathrm{Ric}^g + \frac{1}{2}(f^2 - \frac{\kappa f^4}{4})g = 0\,, \qquad s_g = -\frac{1}{2} f^2$$

The discriminant is one! □

What about Heterotic solitons with non-constant dilaton?

Theorem

All Einstein three-dimensional Heterotic solitons have constant dilaton.

If we weaken the previous condition we can rapidly encounter important obstructions.

Proposition

Let $(g, f) \in \mathrm{Sol}_\kappa(M)$ be such that $\mathrm{d}f \neq 0$ and:

$$\mathrm{d}s_g = 0\,, \qquad \mathrm{d}|\mathrm{Ric}^g|^2_g = 0$$

Then M is diffeomorphic to the sphere S^3.

Proof.

Suppose (g, f) is a Heterotic soliton with non-constant dilaton f. We evaluate the Heterotic soliton equations at a critical point, obtain a quadratic equation for f_c whose coefficients are constant. Hence, the function f can have at most two critical values. □

We note that the existence of three-dimensional Heterotic solitons with non-constant dilaton is currently an open problem.

Let (g, f) be a non-trivial three-dimensional Einstein Heterotic soliton with constant dilaton. Recall that in this case the Heterotic soliton system implies:

$$\mathrm{Ric}^g = -\frac{f^2}{6} g\,, \qquad \kappa f^2 = 3$$

The existence of a natural slice for the action of the diffeomorphism group around every point $(g, f) \in \mathrm{Conf}(M)$ implies:

$$T_{g,f}\mathfrak{M}(M) \subset \mathrm{Ker}(\mathrm{d}_{g,f}\mathcal{E}) \cap \mathrm{Ker}(\mathrm{d}_e \Psi^*_{g,f})$$

where $T_{g,f}\mathfrak{M}(M)$ the tangent space of $\mathfrak{M}(M)$ at the class $[g, f] \in \mathfrak{M}$ determined by (g, f) and:

$$\mathrm{d}_{g,f}\mathcal{E} \colon T_{g,f}\mathrm{Conf}(M) \to T_{g,f}\mathrm{Conf}(M)$$

is the differential of $\mathcal{E}$ at (g, f).

Definition

$\mathbb{E}_{g,f} = \mathrm{Ker}(\mathrm{d}_{g,f}\mathcal{E}) \cap \mathrm{Ker}(\mathrm{d}_e\Psi^*_{g,f})$ is the vector space *essential deformations* of (g, f).

Lemma

Then, the following equations hold:

$$\Delta_g \mathrm{Tr}_g(h) = f^2 \mathrm{Tr}_g(h)\,, \qquad \sigma = \frac{7f}{12}\mathrm{Tr}_g(h)$$

for every $(h, \sigma) \in \mathbb{E}_{g,f}$ *.*

Lemma

The following equations hold:

$$\nabla^{g*}\nabla^g h = \frac{f^2}{6} h + f^2 \mathrm{Tr}_g(h) g + \frac{13}{6}\nabla^g \mathrm{d}\mathrm{Tr}_g(h)\,, \quad \Delta_g \mathrm{Tr}_g(h) = f^2 \mathrm{Tr}_g(h)$$

for every $(h, \sigma) \in \mathbb{E}_{g,f}$*.*

Theorem

$$\mathbb{E}_{g,f} = 0.$$

Proof.

Our starting point is the celebrated Weitzenböck formula:

$$(\mathrm{d}^g \mathrm{d}^{g*} + \mathrm{d}^{g*}\mathrm{d}^g)(\alpha \otimes v) = \nabla^{g*}\nabla^g(\alpha \otimes v) + q^g(\alpha \otimes v)$$

where $\alpha \in \Omega^1(M)$, $v \in \mathfrak{X}(M)$ and q_g is the linear operator given explicitly by:

$$q^g(\alpha \otimes v) = \mathrm{Ric}^g(\alpha) \otimes v + e_i \otimes \mathcal{R}^g_{\alpha e_i} v$$

in terms of an orthonormal basis (e_i). We compute:

$$q^g(h) = q^g(h(e_k) \otimes e_k) = \mathrm{Ric}^g(h(e_k) \otimes e_k + e_i \otimes \mathcal{R}^g_{e_k e_i} h(e_k) = -\frac{f^2}{4} h_0$$

where h_0 is the traceless projection of h and we have used $\mathrm{Ric}^g = -\frac{f^2}{6} g$ and $\mathcal{R}^g_{v_1 v_2} = \frac{f^2}{12} v_1 \wedge v_2$. □

Proof.

We obtain:

$$\mathrm{d}^{g*}\mathrm{d}^g h = \nabla^{g*}\nabla^g h - \frac{f^2}{4}h_0 = -\frac{f^2}{12}h + \frac{13f^2}{12}\mathrm{Tr}_g(h)g + \frac{13}{6}\nabla^g \mathrm{d}\mathrm{Tr}_g(h)$$

We apply now $\mathrm{d}^{g*}\colon \Omega^1(M, TM) \to \mathfrak{X}(M)$ to both sides of the previous equation: the result of applying d^{g*} to each monomial in the previous equation is a constant times $\mathrm{Tr}_g(h)$ and that the combination does not vanish. Hence $\mathrm{d}\mathrm{Tr}_g(h) = 0$, which in turn implies $\mathrm{Tr}_g(h) = 0$ since $\Delta_g \mathrm{Tr}_g(h) = f^2\mathrm{Tr}_g(h)$. Hence, $\sigma = 0$ and $\mathrm{d}^{g*}\mathrm{d}^g h = -\frac{f^2}{12}h$. □

The existence of a slice for together with the previous theorem implies:

Corollary

Three-dimensional compact Einstein Heterotic solitons with constant dilaton are rigid.

We are not aware about any rigidity result for a compact solution of a supergravity theory, especially in the non-supersymmetric case.

Many potential future research lines!

- Develop the higher gauge-theoretic formulation of the Heterotic-Ricci flow on a string Courant algebroid or string structure.
- Weakly parabolicity regime, local existence, gradient formulation?
- Moduli space *á la Kuranishi* in low dimensions. Stability?
- Examples of Heterotic solitons with non parallel torsion.
- Classify all left-invariant solitons on simply connected Lie groups.
- Study adapted Heterotic-Ricci flows/solitons on complex surfaces; relation to [García-Fernandez, Jordan, Streets,Ustinovskiy].
- Study, using Riemannian submanifold theory, the relation between the moduli of supersymmetric solutions and the moduli of Heterotic solitons.
- Study the analog Lorentzian problem and associated constraint and evolution equations on a Cauchy hypersurface.
- T-duality of Heterotic solitons? [Baraglia,Crtoés,Hekmati,García-Fernández]

Thanks!

Theorem

Let M be a compact and oriented four-manifold and $\kappa > 0$. A non-flat pair $(g, \alpha) \in \mathrm{Conf}_\kappa(M)$ is a null Heterotic soliton with parallel torsion if and only if:

1. *Relations $\kappa|\alpha|_g^2 = 1$ and $(\mu_1 = -\frac{1}{4\kappa}, \mu_2 = \frac{1}{4\kappa})$ hold. In particular, there exists a double cover of (Σ, h) that admits a prescribed Sasakian structure.*
2. *Relation $\kappa|\alpha|_g^2 = 1$ holds and the lift $(\hat{g}, \hat{\alpha})$ of (g, α) to the universal cover $\hat{M}$ of M is isometric to either $\mathbb{R} \times \widetilde{\mathrm{Sl}}(2, \mathbb{R})$ or $\mathbb{R} \times \mathrm{E}(1, 1)$ equipped with a left-invariant metric with constant principal Ricci curvatures given by $(0, 0, -\frac{1}{2\kappa})$ and $\hat{\alpha} = |\alpha|_g \mathrm{d}t$.*
3. *Relation $\kappa|\alpha|_g^2 = \frac{3}{2}$ holds and the lift $(\hat{g}, \hat{\alpha})$ of (g, α) to the universal cover $\hat{M}$ of M is isometric to $\mathbb{R} \times \mathbb{H}$ equipped with the standard product metric of scalar curvature $-\frac{3}{4\kappa}$ and $\hat{\alpha} = |\alpha|_g \mathrm{d}t$.*

1. Denote by ξ the unit-norm simple eigenvector of Ric^h and define the endomorphism $\mathcal{C}(v) = \nabla^h_v \xi$, $v \in T\Sigma$. Decompose $\mathcal{C} = \mathcal{A} + \mathcal{S}$, show that h_S is positive definite, $\mathcal{C}$ is everywhere regular in $\xi^\perp$ is such that (h_S, ξ_S) is Sasakian, where:

$$\xi_S := \sqrt{\frac{\mu_2}{2}}\xi\,, \quad \mathrm{Ric}^h(\xi) = \frac{1}{4\kappa}\xi\,, \quad |\xi|^2_h = 1\,, \quad \xi \in \mathfrak{X}(M)\,,$$

as well as:

$$h_S(v_1, v_2) = \begin{cases} -2h(\mathcal{A}\circ\mathcal{C}(v_1), v_2) & \text{if} \quad v_1, v_2 \in \mathcal{H} \\ 0 & \text{if} \quad v_1 \in \mathcal{H},\ v_2 \in \mathrm{Span}(\xi) \\ \frac{\mu_2}{2} h(v_1, v_2) & \text{if} \quad v_1, v_2 \in \mathrm{Span}(\xi) \end{cases}$$

2. Show existence of a global frame $[e_i, e_j] = c_{ij}{}^k\, e_k$ with $c_{ij}{}^k$ constant, using compactness of M. Use Milnor results on the curvature of Riemannian three-dimensional Lie groups.

3. Follows directly from the characterization of the principal Ricci curvatures.

The previous theorem can be used to construct families of Heterotic solitons. These are, to the best knowledge of the authors, the first solutions in the literature that are not locally isomorphic to a supersymmetric Heterotic compactification background.

Corollary

Every mapping torus of a complete hyperbolic three-manifold admits a solution (g, α) of the Heterotic system.

Corollary

Let (h_S, ξ_S) be a Sasakian structure on Σ with contact one-form η_S satisfying:

$$\mathrm{Ric}^{h_S} = -\frac{1}{2} h_S + \eta_S \otimes \eta_S \, .$$

Then, the mapping torus of $(\Sigma, c^2 h_S)$ admits a null Heterotic soliton with parallel torsion for $c^2 = 2\kappa$.

Potential mechanism of *discrete topology change* depending on κ in fixed *flux units*. Set $|\alpha|^2_g = 1/2$ and assume for simplicity that we consider Heterotic solitons which are suspensions. Then, $\kappa \in \{1, 2, 3\}$ and:

- If $\kappa = 1$, (M, g, α) is the suspension of a Sasakian three-manifold.
- If $\kappa = 2$ then a *transition* occurs and (M, g, α) is the suspension of a quotient of $\widetilde{\mathrm{Sl}}(2)$ or a solvable three-manifold of type $\mathrm{E}(1, 1)$.
- If $\kappa = 3$ then another *transition* occurs and (M, g, α) is the suspension of a hyperbolic three-manifold.

In particular, κ is only allowed to take discrete values. Also, if $|\alpha|^2_g \neq 0$ then the limit $\kappa \to 0$ is not well-defined (no generalized Ricci soliton limit!).