# What can abelian gauge theories teach us about kinematic algebras ?

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Based on work done in Arxiv: *2205.02136, 2401.10750*

# Outline

- Introduction: Gravity and Gauge Theories
- The Classical Double Copy
- Kinematic Algebras
- Outlook

# Introduction

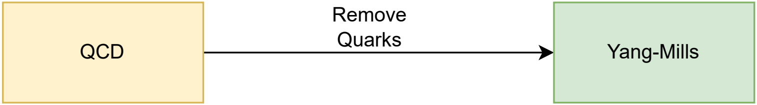
- An open problem in theoretical physics is the relationship between theories of *gravity* and theories of *particle physics*.
- The last 50 years have seen various attempts to resolve this problem: String Theory, Quantum Loop Gravity, etc ...
- Though none of these attempts have completely solidified the relationship between gravity and particle physics, we have learned a great deal from these investigations.

# Particle Physics and Gauge Theories

The standard model of particle physics is described by *gauge theories*; quantum field theories which incorporate *local symmetries* defined at every point in spacetime.

The most relevant of these theories for us is the theory of quarks, gluons and the strong nuclear force: **Quantum Chromodynamics**.

We can also remove the quarks from QCD to receive a non-abelian gauge theory known as Yang-Mills theory.

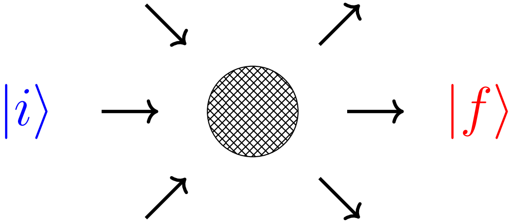


# Gravity

- Currently, the best theory we have to describe gravity is *general relativity*.
- Recent experimental results on black holes and gravitational waves has increased interest in the subject.
- Gravity is the only fundamental force that we cannot comfortably combine with quantum mechanics.
- Gravity is an example of a **non-renormalisable** theory.

# Quantum Field Theory and Interactions

***Scattering amplitudes*** in quantum field theory are quantities related to the probability for an interaction (also known as a *scattering process*) between particles to happen.




- Number of (external legs) points $\rightarrow$ Number of arrows going in **and** out.
- Number of Loops $\rightarrow$ Number of "self interactions".

# Developing a Double Copy for Scattering amplitudes

We can write down (*2203.13013*) for a Yang-Mills like *non-abelian gauge theory*, a scattering amplitude with $L$ loops and $m$ points (in $D$ dimensions).

$$\mathcal{A}_m^{(L)} = i^{L-1} \underbrace{\overbrace{g^{m-2+2L}}}_{\text{Coupling Constant}} \sum_i \int \prod_{l=1}^{L} \frac{d^D \ell_l}{(2\pi)^D} \frac{1}{S_i} \frac{c_i n_i}{\prod_{i_j} d_{i_j}} \,, \qquad (1)$$

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# BCJ Duality

It turns out the kinematic numerators ($n_i$) can be made to obey (mirroring the colour factors $c_i$):

$$c_i + c_j + c_k = 0 \tag{2}$$

$$n_i + n_j + n_k = 0 \tag{3}$$

This has become known as BCJ Duality (*0805.3993, 1004.0476*). Importantly, equation (3) implies the existance of structures known as ***kinematic algebras***!

This allows us to write down relevant scattering amplitudes in quantum gravity.

# Developing a Double Copy for Scattering amplitudes

We can promote gravity to a quantum field theory (ignoring issues with renormalizability), to write down a scattering amplitude with $L$ loops and $m$ points:

$$\mathcal{M}_m^{(L)} = i^{L-1}\left(\frac{\kappa}{2}\right)^{m-2+2L} \sum_i \int \prod_{l=1}^{L} \frac{d^D \ell_l}{(2\pi)^D} \frac{1}{S_i} \frac{n_i \tilde{n}_i}{\prod_{i_j} d_{i_j}}\,, \qquad (4)$$

For:

$$\kappa = \sqrt{32\pi G_N}$$

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# Developing a Double Copy for Scattering amplitudes

Comparing both expressions:

$$\mathcal{A}_m^{(L)} = i^{L-1} g^{m-2+2L} \sum_i \int \prod_{l=1}^{L} \frac{d^D \ell_l}{(2\pi)^D} \frac{1}{S_i} \frac{c_i n_i}{\prod_{i_j} d_{i_j}} \,,$$

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# Double Copy for scattering amplitudes

We can turn the gauge theory amplitude into the gravity amplitude via the following replacements:

$$\mathcal{M}_m^{(L)} = \mathcal{A}_m^{(L)}\Big|_{\substack{c_i \to \tilde{n}_i \\ g \to \kappa/2}} = i^{L-1}\left(\frac{\kappa}{2}\right)^{m-2+2L} \sum_i \int \prod_{l=1}^{L} \frac{d^D \ell_l}{(2\pi)^D} \frac{1}{S_i} \frac{n_i \tilde{n}_i}{\prod_{i_j} d_{i_j}}, \tag{5}$$

We then make the choice $n_i \equiv \tilde{n}_i$.

This is known as the **Double Copy**.

# The Classical Double Copy

- The double copy can be extended to produce dualities between exact classical solutions (*classical yang-mills and general relativity*).
- The first instance of the classical double copy is the so-called Kerr-Schild Double Copy (*1410.0239, 1606.04724*).
- This relates gauge fields in classical yang mills ($A_\mu^a$) with *exact linearized vacuum solutions* to the Einstein equations ($h_{\mu\nu}$). Where:

$$g_{\mu\nu} = \eta_{\mu\nu} + \kappa h_{\mu\nu} \tag{6}$$

# Kerr-Schild Double Copy

In this case, our gauge and graviton fields are constructed by combining some selected vector $k_\mu$ (*known as Kerr-Schild vector*) with some harmonic scalar $\phi \equiv \phi(x)$:

$$A^a{}_\mu = k_\mu \phi c^a \tag{7}$$

where $c^a$ is some constant *color* vector that "dresses" $\phi$.

$$h_{\mu\nu} = k_\mu k_\nu \phi \tag{8}$$

Where our Kerr-Schild vectors are constrained by the following:

$$k^\mu k_\mu = 0, \quad k_\nu \partial^\nu k_\mu = 0. \tag{9}$$

# Kerr-Schild Operator Formalism

The Kerr-Schild Double Copy can be expressed in terms of **differential operators** $\hat{k}_\mu$ (*known as Kerr-Schild operators*) acting on some harmonic scalar $\phi \equiv \phi(x)$:

$$A_\mu = \hat{k}_\mu \phi, \tag{10}$$

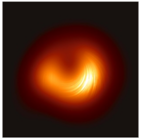
$$h_{\mu\nu} = \hat{k}_\mu \hat{k}_\nu \phi \tag{11}$$

Where our operators are constrained by the following:

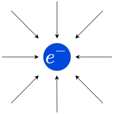
$$\hat{k}^2 = 0, \quad \partial \cdot \hat{k} = 0. \tag{12}$$

# The Kerr-Schild Double Copy

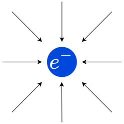
A classic example of the Kerr-Schild double copy in action is the relationship between the Schwarzschild solution and the Coulomb Solution:



Schwarzschild Solution




Coulomb Solution




Coulomb Solution

# What are Kinematic Algebras ?

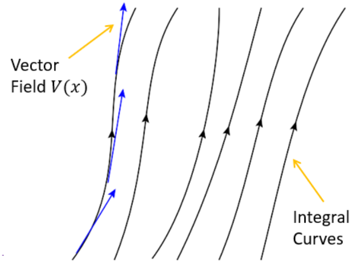
- Gauge theories are now known to possess mysterious structures known as **Kinematic Algebras**.
- One downside to BCJ duality currently is that is defined *order by order* in perturbation theory.
- We currently believe that kinematics algebras are in general *not* Lie algebras, but some more general mathematical structures such as *strong* homotopy or $L^{\infty}$ algebras. *(Reiterer; then Borsten, Jurco, Kim, Macrelli, Saemann, Wolf; Bonezzi, Chiaffrino, Diaz-Jaramilo, Hohm, Plefka)*.
- These are examples of so-called "higher" bracket theories, which require a higher-order generalisation of the Lie bracket to satisfy the Jacobi identity.

# What are Kinematic Algebras ?

- The first examples of *Lie algebra based* kinematic algebras were found for the cases of *(Monteiro, O'Connell, Fu, Krasnov, Ben-Shahar, Johansson)* :
    1. Self-Dual Yang-Mills theory in *lightcone gauge* (in the form of *Area Preserving Diffeomorphisms*)
    2. Non-abelian Chern-Simons Theory *in Lorenz gauge* (*Volume Preserving Diffeomorphisms*)
- It was previously thought that only non-abelian theories could possess kinematic algebras, *however*, it has been recently discovered that kinematic algebras exist for **linear (abelian) theories** as well.

# What are Diffeomorphisms ?

- Diffeomorphism are simultaneous translation along all integral curves (field lines) of the vector field $v(x)$.
- It turns out we can understand the Kinematic Algebra of Electromagnetism in terms of diffeomorphisms.



# Diffeomorphisms Algebras

A given vector field $V^\mu$ on a manifold generates infinitesimal diffeomorphisms:

$$V^\mu(x)\partial_\mu, \tag{13}$$

Then the set of possible all vector fields on a manifold forms a **closed *diffeomorphism* algebra** under the *Lie bracket*:

$$[V^{(1)\mu}\partial_\mu, V^{(2)\nu}\partial_\nu] = V^{(3)\mu}\partial_\mu. \tag{14}$$

$\rightarrow$ i.e the bracket of two vector fields is itself a vector field.

# Volume Preserving Diffeomorphism

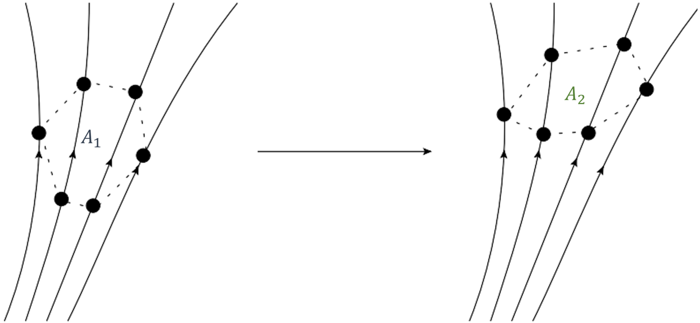
- A vector field $V(X)$ is said to be **Volume preserving** if:

$$\partial \cdot V = 0 \tag{15}$$

- For *Abelian* gauge fields, this corresponds to the **Lorenz** gauge fixing condition.
- The full set of volume-preserving diffeomorphism indeed corresponds to an algebra.
- We can take **lower dimensional** slices of the volume-preserving algebra to yield subalgebras (*transformations*) which act sole in this lower dimension.

# Area Preserving Diffeomorphism

An example of such transformations are *Area Preserving Diffeomorphisms*.




$$A_1 = A_2$$

# Gauge Fields at a Closer Look

Consider any gauge field **A** can be expressed as being "valued" in terms of diffeomorphism and so-called gauge (local symmetries) symmetries:

$$\mathbf{A} = A^a_\mu \underbrace{\partial_\mu}_{\substack{\textit{Infinitesimal} \\ \textit{Diffeomorphisms}}} \overbrace{\mathbf{T}^a}^{\substack{\textit{Infinitesimal} \\ \textit{gauge} \\ \textit{transformations}}} \tag{16}$$

Where $\mathbf{T}^a$ are the generators of the associated Lie algebra.

$$[\mathbf{T}^a, \mathbf{T}^b] = if^{abc}\mathbf{T}^c, \tag{17}$$

# Abelian Kinematic Algebras from Non-Linear Theories

- By making **certain** restrictions to a non-linear interacting gauge theory, we can obtain its **self-dual (anti-self dual)** ***abelian*** counterpart in lightcone gauge.
- The kinematic algebra of this self-dual theory corresponds to an algebra of **area-preserving diffeomorphisms** (*2205.02136*).
- This suggests that the kinematic algebra of an interacting theory – is somehow related to the self-dual diffeomorphisms found in the abelian theory.
- We can extend this approach to derive the kinematic algebra for non-self dual abelian theories. e.g electromagntism

# Self-Dual Linearised Fields in Lightcone Gauge

We restrict the (in Euclidean signature) non-linear theory to linearised (abelian) self-dual solutions (in light-cone gauge) by choosing:

$$A_\mu = \hat{k}_\mu \phi, \tag{18}$$

Where:

$$\hat{k}_\mu = B_i \bar{\eta}^i_{\mu\nu} \partial_\nu, \tag{19}$$

For $B_i$ being a constant 3-vector such that $\vec{B}^2 = 0$.
$\bar{\eta}^i_{\mu\nu}$ are the so-called *'t Hooft symbols*:

$$\bar{\eta}^1_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \quad \bar{\eta}^2_{\mu\nu} = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

$$\bar{\eta}^3_{\mu\nu} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \tag{20}$$

# Self-Dual Linearised Fields in Lightcone Gauge

The infinitesimal diffeomorphisms of the gauge field:

$$A_\mu \partial_\mu = (\hat{k}_\mu \phi)\partial_\mu = \left( b^{(1)}_{[\mu} b^{(2)}_{\nu]} \partial_\nu \phi \right) \partial_\mu, \tag{21}$$

with (making the choice $B_1 \neq 0$):

$$b^{(1)}_\mu = (B_1, B_2, B_3, 0), \quad b^{(2)}_\mu = \left( 0, \frac{B_3}{B_1}, -\frac{B_2}{B_1}, -1 \right). \tag{22}$$

$b^{(1)}_\mu$ and $b^{(2)}_\mu$ are tangent bivectors, which define **null planes** where the diffeomorphisms act. *These are known as $\alpha$- and $\beta$-planes.*

# A Possible Objection

- Within the literature, kinematic algebras are usually associated with **interaction terms** in a *non-linear theory*.
- However, we can then use abelian gauge theories to **clarify** aspects of more general kinematic algebras.
- Given that any interacting theory (including a non-abelian gauge theory) must have a non-interacting linearisation, we can ask **which** of our "special" subgroups of diffeomorphisms can be *preserved* by the inclusion of interactions.

# Lightcone Gauge Electromagnetism

To see the kinematic algebra for electromagnetism, we need to constrict our algebra of volume-preserving diffeomorphisms to a *closed* subgroup of transformations known as **Symplectomorphisms**.

Which for a given scalar field $\phi^i$, we define $A_\mu$ to be a *Hamiltonian Vector Field*:

$$A_\mu^{(i)} = \Omega_{\mu\nu}\partial_\nu\phi_i. \tag{23}$$

where $\Omega_{\mu\nu}$ is the *symplectic form*.

In Euclidean signature, this takes a familiar form:

$$\Omega_{\mu\nu} = B_i\bar{\eta}^i_{\mu\nu} \tag{24}$$

# Lightcone Gauge Electromagnetism

We then restrict to real solutions of $A_\mu$ in **Lorentzian signature**:

$$A_\mu = \hat{k}_\mu \phi + \hat{k}_\mu^\dagger \phi^\dagger. \tag{25}$$

We may choose a particular lightcone gauge defined through the **lightcone coordinates**:

$$u = \frac{t-z}{\sqrt{2}}, \quad v = \frac{t+z}{\sqrt{2}}, \quad X = \frac{x+iy}{\sqrt{2}}, \quad Y = \frac{x-iy}{\sqrt{2}}, \tag{26}$$

where $(t, x, y, z)$ are Cartesian coordinates in Lorentzian signature. With line element:

$$ds^2 = dt^2 - dx^2 - dy^2 - dz^2 = 2dudv - 2dXdY. \tag{27}$$

# Lightcone Gauge Electromagnetism

- Our Kerr-Schild Operators then take the following form in the $(u, v, X, Y)$ system:

$$\hat{k}_\mu = (0, \partial_Y, \partial_u, 0) \quad \Rightarrow \quad \hat{k}^\mu = (\partial_Y, 0, 0, -\partial_u), \tag{28}$$

$$\hat{k}^\dagger_\mu = (0, \partial_X, 0, \partial_u) \quad \Rightarrow \quad \hat{k}^{\dagger\mu} = (\partial_X, 0, -\partial_u, 0) \tag{29}$$

  which corresponds to a choice: $(B_1, B_2, B_3) = (-i, 1, 0)$.
- The gauge field then generates a combination of two area-preserving diffeomorphisms, in $(u, Y)$ and $(u, X)$ planes respectively.

# Lightcone Gauge Electromagnetism

The kinematic algebra must then subgroup of the product group

$$\mathrm{Diff}_{(u,Y)} \times \mathrm{Diff}_{(u,X)}, \tag{30}$$

As a Hamiltonian vector field, $A_\mu$ is given by:

$$(A^u, A^v, A^X, A^Y) = (\partial_Y \phi + \partial_X \phi^\dagger, 0, -\partial_u \phi^\dagger, -\partial_u \phi). \tag{31}$$

Restricting $\phi \in \mathbb{R}$, we find:

$$(A^u, A^v, A^X, A^Y) = ((\partial_X + \partial_Y)\phi, 0, -\partial_u \phi, -\partial_u \phi)\,. \tag{32}$$

Or the case $\phi = i\xi$ i.e purely imaginary (for $\xi \in \mathbb{R}$):

$$(A^u, A^v, A^X, A^Y) = \Big( i(\partial_Y - \partial_X), 0, i\partial_u \xi, -i\partial_u \xi \Big) \tag{33}$$

# Lightcone Gauge Electromagnetism

Transforming from $(u, v, X, Y)$ to $(u, v, x, y)$, the non-zero components of the gauge field for each choice of $\phi$ are:

Real $\phi$

$$A^u = \sqrt{2}\partial_x \phi, \quad A^x = -\sqrt{2}\partial_u \phi. \tag{34}$$

Imaginary $\phi$:

$$A^u = -\sqrt{2}\partial_y \xi, \quad A^y = \sqrt{2}\partial_u \xi, \tag{35}$$

With **infinitesimal diffeomorphisms**:

Real $\phi$

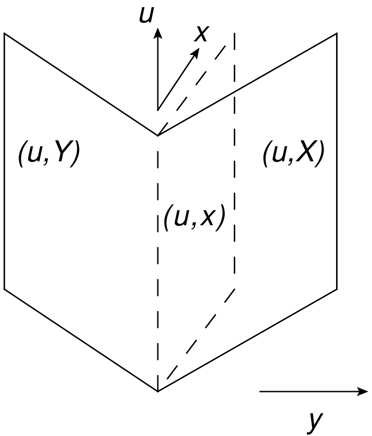
$$\partial_u A^u + \partial_x A^x = 0, \tag{36}$$

Imaginary $\phi$:

$$\partial_u A^u + \partial_y A^y = 0. \tag{37}$$

# Lightcone Gauge Electromagnetism

- For **REAL** $\phi$, $A^{\mu}$ generates area-preserving diffeomorphisms in the $(u, x)$ plane.
- For **IMAGINARY** $\phi$, $A_{\mu}$ generates area-preserving diffeomorphisms in the $(u, y)$ plane.



# Gauge Dependence of the Diffeomorphism Algebra

If $A_\mu$ generates a symplectomorphism, then a *general gauge transformation*:

$$A_\mu = \hat{k}_\mu \phi + \partial_\mu \chi, \qquad (38)$$

will produce a vector field that **does not** preserve the symplectic form.

It will **remain** in *Lorenz gauge*, provided $\chi$ is harmonic ($\partial^2 \chi = 0$)

**Varying** $\chi$ will gradually move out of the **special subgroups** of the diffeomorphism algebra.

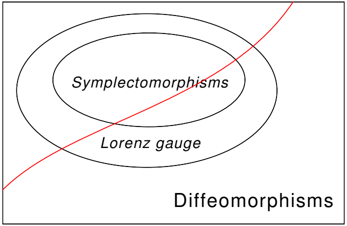



Figure: The set of all physically equivalent abelian gauge fields (related by a gauge transformation) shows up as a line – shown in red – in the space of all possible diffeomorphisms.

# Scalar QED From First Principles

- As mentioned before, the kinematic algebra for (non-linear) interacting theories (such as self-dual Yang-Mills) is **inherited** at least in part from the kinematic algebra for a related linear theory.
- We can **exploit** this idea to construct non-linear gauge theories from their abelian counterparts.
- Start with an **abelian** gauge field $A_\mu$ that is restricted to the subset of Hamiltonian fields:

$$A_\mu = \Omega_{\mu\nu}\partial^\nu \phi, \tag{39}$$

# Scalar QED From First Principles

- The general vacuum field equation for $A_\mu$ is

$$\partial^2 A_\mu - \partial_\mu(\partial \cdot A) = 0 \tag{40}$$

- For Hamiltonian vector fields we find:

$$\partial^2 \phi = 0. \tag{41}$$

# Possion Brackets and Hamiltonian Vector Fields

- Hamiltonian vector fields come equipped with some additional structure known as a **Poisson Bracket**.
- The *Poisson Bracket* which acts on our scalar fields $\phi_i$:

$$\{\phi_1, \phi_2\} = \Omega_{\mu\nu}(\Omega_{\mu\alpha}\partial_\alpha\phi_1)(\Omega_{\nu\beta}\partial_\beta\phi_2) = \Omega_{\mu\nu}(\partial_\mu\phi_1)(\partial_\nu\phi_2), \tag{42}$$

- This implies that some scalar field $\phi_3$ is related to $\phi_1$ and $\phi_2$ via:

$$\phi_3 = -\{\phi_1, \phi_2\}, \tag{43}$$

# Scalar QED From First Principles

- Let us start with an abelian gauge field $A_\mu$ we will restricted to the subset of Hamiltonian fields:

$$A_\mu = \Omega_{\mu\nu} \partial^\nu \phi, \tag{44}$$

- Since we are dealing with Abelian gauge fields, the Poisson bracket ends up being trival:

$$\{\phi, \phi\} = 0. \tag{45}$$

# Scalar QED From First Principles

- Thus, in order to consider extensions to non-linear theories, we need to introduce an *additional* gauge field $\psi$, which at linear level satisfies the Klein-Gordon equation (*e.g Harmonic*)

$$\partial^2\psi = 0. \tag{46}$$

- We can then extended the E.O.M non-linearly by adding a Poisson bracket composed of $\phi$ and $\psi$:

$$\partial^2\psi + c_1\{\psi, \phi\} = 0, \tag{47}$$

# Scalar QED From First Principles

- We now wish to see whether equation (47) is a physically consistant.
- If we want to consider $\psi$ interacting with the gauge field, then equation (47) (or it's *generalisation*) must be **gauge-covariant**.
- The Hamiltonian nature of $A_\mu$ is **preserved** by the gauge transformations:

$$A_\mu \to A'_\mu = A_\mu + \partial_\mu \chi, \tag{48}$$

- This implies the corresponding gauge transformation for $\psi$:

$$\psi \to \psi' = e^{-ie\chi}\psi, \tag{49}$$

# Scalar QED From First Principles

- $\chi$ (for some $\alpha$) is then restricted by:

$$\partial_\mu \chi = \Omega_{\mu\nu} \partial^\nu \alpha \tag{50}$$

  Using the definition for Hamiltonian $A_\mu$, one may rewrite the E.O.M for $\psi$ as:

$$\partial^2 \psi + c_1 A_\mu \partial^\mu \psi = 0, \tag{51}$$

- Under a gauge transformation this satisfies:

$$\partial^2 \psi' + A'_\mu \partial^\mu \psi' = 0 \quad \rightarrow \quad \partial^2 \psi + A_\mu \partial^\mu \psi + \Delta = 0, \tag{52}$$

- Where

$$\Delta = (c_1 - 2ie)(\partial_\mu \chi)(\partial^\mu \psi) \\ -iec_1 A_\mu (\partial^\mu \chi)\psi - (iec_1 + e^2)(\partial_\mu \chi)(\partial^\mu \chi)\psi. \tag{53}$$

- There is **no** solution for $c_1$ that yields $\Delta = 0$.
- This follows from that one must add a **seagull vertex** to scalar QED in order to make it gauge-invariant.

# Scalar QED From First Principles

- We correct this by adding an additional term to equation (51):

$$\partial^2\psi + c_1 A_\mu \partial^\mu \psi + c_2 A^\mu A_\mu \psi = 0. \tag{54}$$

- Now when performing the gauge transformation, the "difference" $\Delta$ is given by:

$$\begin{aligned} \Delta &= (c_1 - 2ie)(\partial_\mu \chi)(\partial^\mu \psi) \\ &+(2c_2 - iec_1)A_\mu(\partial^\mu \chi)\psi + (c_2 - iec_1 - e^2)(\partial_\mu \chi)(\partial^\mu \chi)\psi. \end{aligned} \tag{55}$$

# Scalar QED From First Principles

- The unique solution for $\Delta = 0$ (e.g gauge invariance) is $(c_1, c_2) = (2ie, -e^2)$, so that the gauge-invariant scalar field equation is

$$\partial^2\psi + 2ie\{\psi, \phi\} - e^2 A^\mu A_\mu \psi = 0. \tag{56}$$

- This has a cubic term, (coming from a quartic interaction in the Lagrangian).
- Therefore, it is **not** true in general that there is a straightforward kinematic Lie algebra, **i.e. such that there are up-to-quadratic terms in the field equation only.**

# Scalar QED From First Principles

- However, it is possible to find a the subsector of solutions where the cubic term **vanishes**.
- The cubic in equation (56) will vanish provided

$$\Omega_{\mu\alpha}\Omega^{\mu}{}_{\beta} = 0. \quad (57)$$

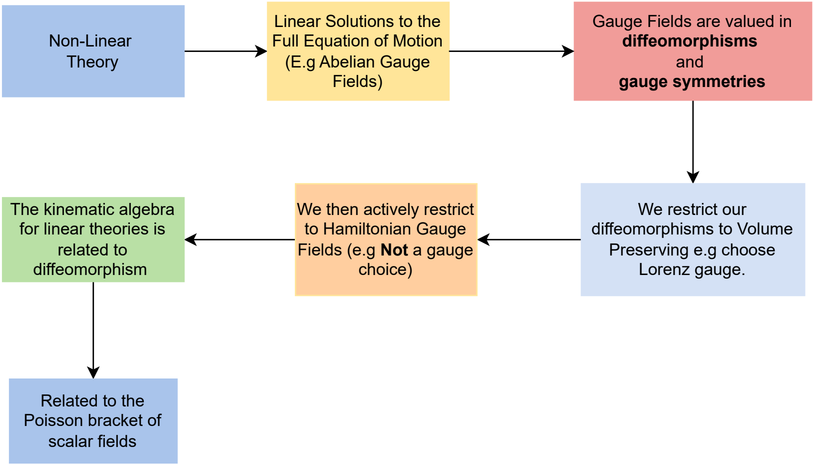
- This corresponds to self-dual field configurations in the light-cone gauge.
- Applying this to equation (56), we receive:

$$\partial^2\psi + 2ie\{\psi, \phi\} = 0. \quad (58)$$

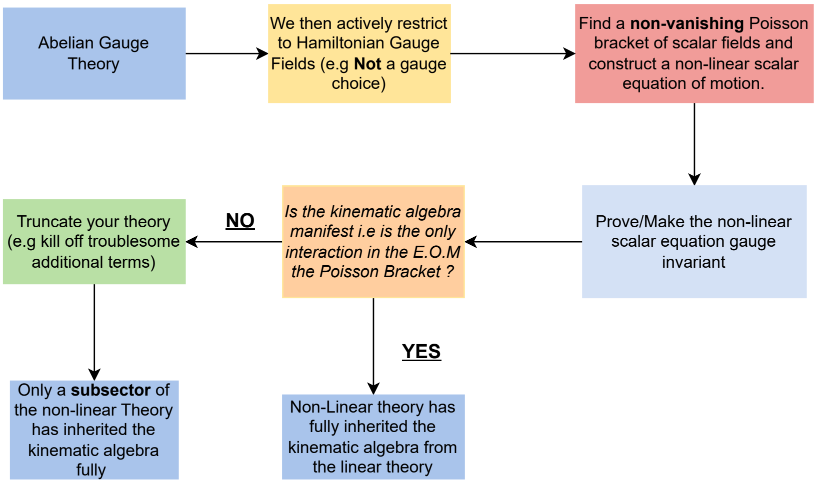
# Scalar QED From First Principles

- We have shown that an interacting non-linear theory can be derived from considerations of the "kinematic algebra" associated with its abelian (linear) counterpart.
- In order to do this, we must restrict ourselves to the use of Hamiltonian gauge fields as well as Light Cone gauge.
- It is possible to go the **other way round**: start with the full lagrangian of a non-linear theory, and then make a series of **restrictions** (Lorenz gauge and Hamiltonian vector fields) to yield equation (58).

# Overview: Kinematic Algebras for Linear Theories



# Overview: Building Non-Linear Theories



# Conclusion

- The Double Copy for scattering amplitudes and classical solutions has revealed to us the existence of mysterious structures known as ***kinematic algebras***, which are related to interactions in a gauge theory.
- Kinematic algebras are *not* thought to be Lie algebras in general, but more structured objects such as **homotopy algebras**.
- Kinematic algebras can be made manifest (in principle) for non-interacting classical theories if we **truncate** the non-linear theory to consist solely of **Hamiltonian vector fields**.
- If we have a kinematic algebra for an Abelian gauge theory, we can **construct** from first principles equations of motion for a non-linear theory.

# Further Work

- How are these ideas related to the study of homotopy algebras?
- If a kinematic algebra is not Lie, is there some alternative description of gauge theories (not fibre-bundle based) that gives a **geometric meaning** to the kinematic algebra?
- Kinematic algebras have been found for theories of **fluid mechanics**; can we get useful physical insights about kinematic algebras by looking at fluid mechanics?
- Can we make interesting **new interacting theories** out of abelian building blocks, that have geometrically visualisable kinematic algebras?
- Might some of these theories be useful for (Non-abelian Chern Simons Theory) **condensed matter physics**?