A double copy in twistor space

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Based on 2406.04539 with T. Adamo

Two organizing principles of tree-level amplitudes

- Double copy

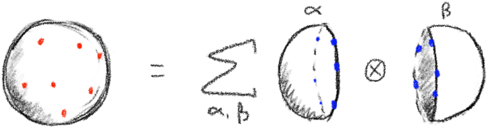
$$\text{gravity} = \text{gauge} \otimes \text{gauge}$$

- External helicity configuration (in 4d)

The double copy

$$\text{gravity} = \text{gauge} \otimes \text{gauge}$$

- The original double copy relation was discovered by Kawai-Lewellen-Tye (KLT) in 1986, relating closed and open string amplitudes



- Taking the field theory limit this amounts to

$$\mathcal{M}^{\text{GR}}_{\text{tree}}(1,\ldots,n) = \sum_{\substack{\alpha\in S_{n-3}\\ \beta\in\tilde{S}_{n-3}}} \mathcal{A}^{\text{YM}}_{\text{tree}}(\alpha)\,\underbrace{S[\alpha|\beta]}_{\otimes}\,\mathcal{A}^{\text{YM}}_{\text{tree}}(\beta)$$

for two bases $S_{n-3}, \tilde{S}_{n-3}$ of colour-orderings of size $(n-3)!$

A closer look at the KLT kernel

- The field theory KLT kernel is given by e.g.

$$S[123\alpha|213\beta] = \prod_{j=4}^{n} \sum_{\substack{i<_{3\alpha}j \\ i<_{3\beta}j}} s_{ij}, \qquad s_{ij} = 2k_i \cdot k_j$$

- Remarkably, the matrix inverse of this object is related to binary tree graphs (cubic vertices) [CHY, Frost, Mafra, Mason, Mizera, . . .]

$$S^{-1}[213\beta|123\alpha] = \pm \sum_{BT\in\mathcal{BT}_{213\alpha,123\beta}} \prod_{E\in BT} \frac{1}{s_E}, \qquad s_E = \sum_{i,j\in E} s_{ij}$$

- Equivalently: doubly colour-ordered amplitudes in biadjoint scalar theory (up to sign)

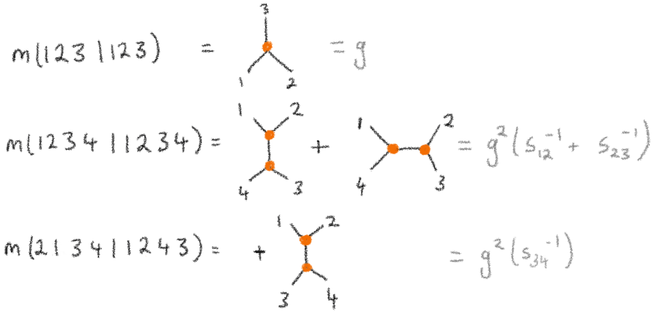
Bi-adjoint scalar theory

- This is a theory of a massless scalar valued in the adjoint representations of two Lie algebras $g, \bar{g}$ with Lagrangian

$$\mathcal{L} = \partial_\mu \phi_{a\bar{a}} \partial^\mu \phi^{a\bar{a}} + igf^{abc} \bar{f}^{\bar{a}\bar{b}\bar{c}} \phi_{a\bar{a}} \phi_{b\bar{b}} \phi_{c\bar{c}}$$

- Amplitudes can be expanded in terms of colour orderings

$$m = \sum_{\beta, \bar{\beta}} \mathrm{Tr}(T^{\beta}) \mathrm{Tr}(\bar{T}^{\bar{\beta}})\, m(\beta|\bar{\beta})$$

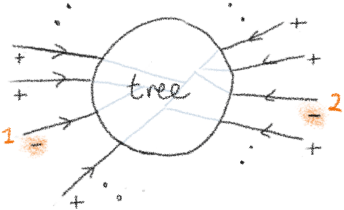


$$m(123|123) = \quad = g$$

$$m(1234|1234) = \quad + \quad = g^2(s_{12}^{-1} + s_{23}^{-1})$$

$$m(2134|1243) = + \quad = g^2(s_{34}^{-1})$$

Tree-level amplitudes: maximally helicity violating (MHV)



- Surprisingly beautiful structures in tree-level amplitudes were first seen with the Parke-Taylor (PT) ['86] formula, at *all multiplicity*

$$\mathcal{A}^{\text{YM}}_{\text{tree}}\underbrace{(1^{-}2^{-}3^{+}\dots n^{+})}_{\text{MHV}} = \delta^4(\cdots)\frac{\langle 12\rangle^4}{\langle 12\rangle\langle 23\rangle\cdots\langle n1\rangle}, \quad p_i^{\alpha\dot{\alpha}} = |i\rangle^{\dot{\alpha}}[i|^{\alpha}$$

- In gravity, there is a corresponding expression (Hodges formula) ['12, Nguyen-Spradlin-Volovich-Wen:'09,...]

$$\mathcal{M}^{\text{GR}}_{\text{tree}}(1^{-}2^{-}3^{+}\dots n^{+}) = \delta^4(\cdots)\langle 12\rangle^8\det{}'(\text{H})$$

From twistor space

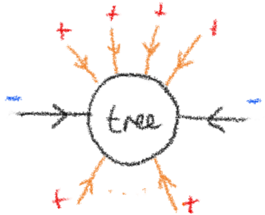
- This simplicity comes from the integrability of the self-dual sector in gravity and Yang-Mills
- Positive particles → self-dual background
- Twistor theory can be used to trivialise self-dual backgrounds in gravity and Yang-Mills [Penrose:'76, Ward:'77]
- Here twistor space is defined as

$$Z^I = (\mu^{\dot{\alpha}}, \lambda_\alpha), \qquad Z^I \sim r\, Z^I \quad \forall r \in \mathbb{C}^*$$
$$\mathbb{PT} = \left\{ Z^I \in \mathbb{CP}^3 \,\middle|\, \lambda_\alpha \neq 0 \right\}$$

- Self-dual backgrounds can be encoded by deforming the complex structure on $\mathbb{PT}$ (gravity) or introducing a holomorphic vector bundle over $\mathbb{PT}$ (Yang-Mills)

From twistor space

- View negative particles as perturbations on this background
- MHV amplitude can be derived from the two-point correlation function on this background [Mason-Skinner:'08]



- Can also extend this to describe scattering on non-perturbative self-dual backgrounds (radiative, Taub-NUT + dyon) [Adamo, Bogna, Mason, Sharma]
- Generic helicity formulae can be derived using twistor string theory [Witten:'04, Berkovits:'04, Cachazo-Skinner:'12, Skinner:'13]

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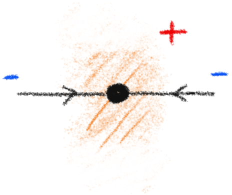

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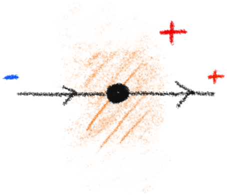

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Tree-level amplitudes: N^{d-1}MHV in twistor space

- At N^{d-1}MHV level (with $d+1$ negative helicity gluons), we have [Witten:'04; Roiban-Spradlin-Volovich:'04]

$$\mathcal{A}^{\mathrm{YM}}_{n,d}[\rho] = \int \mathrm{d}\mu_d \, |\tilde{\mathbf{g}}|^4 \, \mathrm{PT}_n[\rho] \prod_{i\in\mathbf{g}} a_i \prod_{j\in\tilde{\mathbf{g}}} \bar{a}_j$$

where $\mathrm{PT}_n[\rho] = \frac{1}{(\rho(1)\rho(2))\ldots(\rho(n)\rho(1))}$

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- And for gravity the Cachazo-Skinner formula ['12]

$$\mathcal{M}^{\mathrm{GR}}_{n,d} = \int \mathrm{d}\mu_d \, |\tilde{\mathbf{h}}|^8 \, \det{}'(\mathbb{H}) \, \det{}'(\mathbb{H}^\vee) \prod_{i\in\mathbf{h}} h_i \prod_{j\in\tilde{\mathbf{h}}} \bar{h}_j$$

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- External kinematics are encoded in the twistor representatives of momentum eigenstates

$$\xi_i \in H^{0,1}(\mathbb{PT}, \mathcal{O}(2h-2)), \qquad h \text{ helicity}$$

$$\text{MHV} \to \text{N}^{d-1}\text{MHV}$$

$$\frac{\langle 12\rangle^4}{\langle 12\rangle\langle 23\rangle\cdots\langle n1\rangle} \quad\to\quad \frac{|\tilde{\mathbf{g}}|^4}{(12)(23)\cdots(n1)}$$

$$\langle 12\rangle^8 \det{}'(\mathrm{H}) \quad\to\quad |\tilde{\mathbf{h}}|^8\,\det{}'(\mathbb{H})\,\det{}'(\mathbb{H}^\vee)$$

and then integrate these over

$$\mathrm{d}\mu_d \prod_{\substack{i\in\text{pos.helicity}\\ \mathbf{g}/\mathbf{h}}} \xi_i(Z) \prod_{\substack{j\in\text{neg.helicity}\\ \tilde{\mathbf{g}}/\tilde{\mathbf{h}}}} \bar{\xi}_i(Z)$$

$$\text{YM amplitude} \xrightarrow[\text{Double copy}]{\qquad\qquad} \text{GR amplitude}$$

$$\begin{tikzcd}
\text{YM amplitude} \arrow[r, "\text{Double copy}"'] \arrow[d, "\text{helicity grading}"'] & \text{GR amplitude} \arrow[d, "\text{helicity grading}"] \\
\mathcal{A}_{n,d}[\alpha] = \int \mathrm{d}\tilde{\mu}_d \, \mathcal{I}_n^{\text{YM}}[\alpha] & \mathcal{M}_{n,d} = \int \mathrm{d}\tilde{\mu}_d \, \mathcal{I}_n^{\text{GR}}
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$$\downarrow$$

Double copy on non-trivial backgrounds??

Further understanding of double copy?

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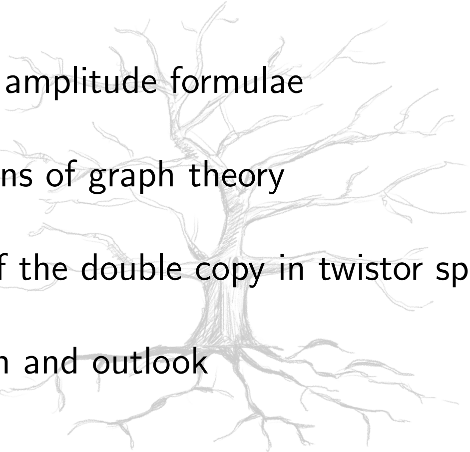
Double copy on non-trivial backgrounds??

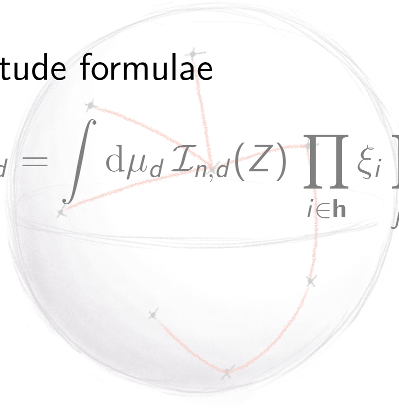
Further understanding of double copy?

This talk



- Tree-level amplitude formulae
- Applications of graph theory
- Aspects of the double copy in twistor space
- Conclusion and outlook

Tree-level amplitude formulae



$$\mathcal{M}_{n,d} = \int \mathrm{d}\mu_d \, \mathcal{I}_{n,d}(Z) \prod_{i\in\mathbf{h}} \xi_i \prod_{j\in\tilde{\mathbf{h}}} \bar{\xi}_j$$

Map moduli integrals

$$\mathcal{M}_{n,d} = \int \mathrm{d}\mu_d \, \mathcal{I}_{n,d}(Z) \prod_{i \in \mathbf{h}} \xi_i \prod_{j \in \tilde{\mathbf{h}}} \bar{\xi}_j$$

- We consider maps of degree d

$$\mathcal{Z} : \mathbb{CP}^1 \to \mathbb{PT}, \qquad \mathbb{PT} \overset{\text{open}}{\subset} \mathbb{CP}^3$$

with coordinates on $\mathbb{CP}^1$ given by $\sigma = (\sigma^0, \sigma^1) \sim r(\sigma^0, \sigma^1)$, and

$$\mathcal{Z}(r\sigma) = r^d \mathcal{Z}(\sigma), \quad \mathcal{Z}((u, 1)) = U_d u^d + U_{d-1} u^{d-1} + \ldots + U_0$$

- Each map has $4(d+1)$ degrees of freedom up to proj. scalings
- The integration measure of the moduli space of these maps and n marked points

$$\mathrm{d}\mu_d := \frac{\mathrm{d}^{4(d+1)} U}{\mathrm{vol}\, \mathbb{C}^* \times \mathrm{SL}(2, \mathbb{C})} \prod_{i=1}^{n} (\sigma_i \, \mathrm{d}\sigma_i)$$

External states - twistor representatives

$$\mathcal{M}_{n,d} = \int \mathrm{d}\mu_d \, \mathcal{I}_{n,d}(Z) \prod_{i \in \mathbf{h}} \xi_i \prod_{j \in \tilde{\mathbf{h}}} \bar{\xi}_j$$

- The Penrose transform: equates solutions of the zero-rest-mass equations on spacetime to cohomology classes on twistor space

$$\xi^h(Z) \in H^{0,1}(\mathbb{PT}, \mathcal{O}(2h-2)), \qquad h = \text{helicity}$$

- The twistor representatives momentum eigenstates take the form

$$\xi_i^h(Z(\sigma_i)) = \int_{\mathbb{C}^*} \mathrm{d}t_i \, t_i^{1-2h} \, \bar{\delta}^2(\kappa_i - t_i \, \lambda(\sigma_i)) \, e^{it_i[\mu(\sigma_i) \, \tilde{\kappa}_i]}$$

where $Z = (\mu^{\dot{\alpha}}, \lambda_\alpha) : \mathbb{CP}^1 \to \mathbb{PT}$ and $k_i^{\alpha\dot{\alpha}} = \kappa_i^\alpha \tilde{\kappa}_i^{\dot{\alpha}}$

Localising the map integrals

$$\frac{\mathrm{d}^{2(d+1)}\lambda\,\mathrm{d}^{2(d+1)}\mu}{\mathrm{vol}\,\mathrm{GL}(2,\mathbb{C})}\prod_{i=1}^{n}\bar{\delta}^2(\kappa_i - t_i\lambda(\sigma_i))\exp\left(i\sum_{i=1}^{n}[\mu(\sigma_i)\,\tilde{\kappa}_i]\right)\prod_{i=1}^{n}(\sigma_i\,\mathrm{d}\sigma_i)$$

- The integrals entirely localise on a set of solutions and 4-momentum conservation
- At $d = 1$, the map is a line, with coefficients fixed by the external momenta (1 solution)
- At higher degree there are $E(n-3, d-1)$ solutions for the map moduli, where the Eulerian numbers satisfy

$$\sum_{d=1}^{n-3} E(n-3, d-1) = (n-3)!$$

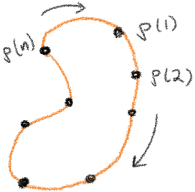
Integrands in twistor space

$$\mathcal{M}_{n,d} = \int \mathrm{d}\mu_d \, \mathcal{I}_{n,d}(Z) \prod_{i\in\mathbf{h}} \xi_i \prod_{j\in\tilde{\mathbf{h}}} \bar{\xi}_j$$

$$|\mathbf{A}| = \prod_{i<j\in\mathbf{A}} (i\,j), \qquad (i\,j) = \epsilon_{ab}\sigma_i^a\sigma_j^b$$

Yang-Mills:

$$\frac{|\tilde{\mathbf{g}}|^4}{(\rho(1)\rho(2))\cdots(\rho(n-1)\rho(n))(\rho(n)\rho(1))}$$



Gravity:

$$|\tilde{\mathbf{h}}|^8 \det{}'(\mathbb{H}) \det{}'(\mathbb{H}^\vee)$$

Integrands in twistor space

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Integrands in twistor space

$$\text{Gravity:} \quad |\tilde{\mathbf{h}}|^8 \, \det{}'(\mathbb{H}) \, \det{}'(\mathbb{H}^\vee)$$

- Hodges matrix has entries for $i, j \in \mathbf{h}$ $(+\ \text{hel.})$

$$\mathbb{H}_{ij} = \frac{[\frac{\partial}{\partial\mu(\sigma_i)}\ \frac{\partial}{\partial\mu(\sigma_j)}]}{(ij)}, \qquad \mathbb{H}_{ii} = -\sum_{\substack{j\in\mathbf{h}\\ j\neq i}} \mathbb{H}_{ij} \prod_{l\in\tilde{\mathbf{h}}} \frac{(jl)}{(il)}$$

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- Dual Hodges matrix has entries for $i, j \in \tilde{\mathbf{h}}$ (- hel.)

$$\mathbb{H}^\vee_{ij} = \frac{\langle \lambda(\sigma_i) \, \lambda(\sigma_j) \rangle}{(ij)}, \qquad \mathbb{H}^\vee_{ii} = -\sum_{\substack{j \in \tilde{\mathbf{h}} \\ j \neq i}} \mathbb{H}^\vee_{ij} \prod_{k \in \tilde{\mathbf{h}} \setminus \{i,j\}} \frac{(ki)}{(kj)}$$

Integrands in twistor space

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- The reduced determinants are

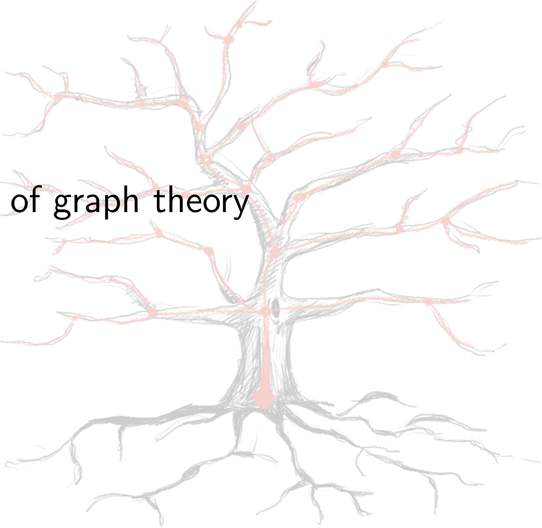
$$\det{}'(\mathbb{H}) = \frac{|\mathbb{H}^b_b|}{|\tilde{\mathbf{h}} \cup \{b\}|^2}, \qquad \det{}'(\mathbb{H}^\vee) = \frac{|\mathbb{H}^{\vee\,a}_{a}|}{|\tilde{\mathbf{h}} \setminus \{a\}|^2}$$

Integrands:

Yang-Mills: $$\frac{|\tilde{\mathbf{g}}|^4}{(\rho(1)\rho(2))\cdots(\rho(n-1)\rho(n))(\rho(n)\rho(1))}$$

Gravity: $$|\tilde{\mathbf{h}}|^8 \det{}'(\mathbb{H}) \det{}'(\mathbb{H}^\vee)$$

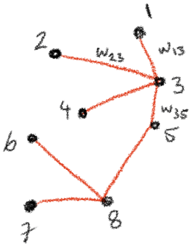
Applications of graph theory



Tree graphs

Def: A tree graph is a set of edges E over vertices V, that is connected and has no loops.

- It's possible to associate a weight w_{ij} with each possible edge $(i-j)$



Weighted Matrix-Tree Theorem

$$\sum_{\substack{\text{tree graphs}\\ \text{on}\,V}} \left(\prod_{(i-j)} w_{ij} \right) = |W(V)^a_a|$$

where the weighted Laplacian matrix is

$$W(V)_{ij} = \begin{cases} \sum_{(k-i)} w_{ik} & \text{if } i = j \\ -w_{ij} & \text{if } i \neq j \end{cases}$$

Can rewrite the Hodges' reduced determinants in this language! For the positive helicity piece:

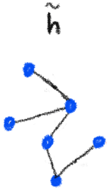
$$\det{}'(\mathbb{H}) = \frac{1}{|\tilde{\mathbf{h}}|^2}\prod_{\substack{k\in\mathbf{h}\\ l\in\tilde{\mathbf{h}}}}\frac{1}{(kl)^2}\times \underbrace{\sum_{\substack{T\\ \text{spanning } \mathbf{h}}}\prod_{(i-j)} B_{ij}}_{|B^a_a|}$$

where

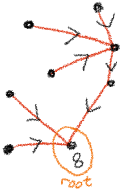
$$B_{ij} = \frac{\left[\frac{\partial}{\partial\mu(\sigma_i)}\,\frac{\partial}{\partial\mu(\sigma_j)}\right]}{(ij)}\prod_{l\in\tilde{h}}(il)(jl)$$

and similarly for the negative helicity piece

$$\det{}'(\mathbb{H}^\vee) = |\tilde{\mathbf{h}}|^2\times\prod_{(i-j)}\sum_{\substack{\tilde{T}\\ \text{spanning } \tilde{\mathbf{h}}}} B^\vee_{ij}$$



Orderings and tree graphs



- It's possible to direct a tree graph by giving it a root. The edges now obtain a direction $(i \to j)$
- Taking b as the root, we can associate a set of orderings to each tree graph

(b12) (b12), (b21)

Proposition (Frost '21)

For a directed tree T_b on vertices V and generic $x \in \mathbb{CP}^1$

$$\prod_{(i \to j) \in T_b} \frac{(xj)}{(ij)(ix)} = \sum_{\substack{\text{compatible} \\ \text{ords. } b\rho \text{ of } T}} \mathrm{PT}(b\rho x)(bx)(xb)$$

$$\det{}'(\mathbb{H}) = \frac{1}{|\tilde{\mathbf{h}}|^2} \prod_{l\in\tilde{\mathbf{h}}} \frac{1}{(bl)^2} \times \sum_{\substack{T_b \\ \text{spanning } \mathbf{h}}} \prod_{(i\to j)} B_{ij}$$

For each specific tree T_b rooted at b

$$\prod_{(i\to j)} B_{ij} \equiv \prod_{(i\to j)} \frac{[\partial_\mu(\sigma_i)\,\partial_\mu(\sigma_j)]}{(ij)} \prod_{l\in\tilde{\mathbf{h}}} (jl)(il)$$

$$= \sum_{b\rho} \mathrm{PT}(b\rho x)(bx)(xb) \prod_{(i\to j)} [\partial_\mu(\sigma_i)\,\partial_\mu(\sigma_j)] \frac{(ix)}{(xj)} \prod_{l\in\tilde{\mathbf{h}}} (jl)(il)$$

$$= \sum_{b\rho, b\sigma} \mathrm{PT}(b\rho x)\mathrm{PT}(b\sigma y)(bx)^2(by)^2 \prod_{(i\to j)} [\partial_\mu(\sigma_i)\,\partial_\mu(\sigma_j)](ij) \frac{(ix)(iy)}{(xj)(yj)}$$

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Weighted tree → $(\text{Parke-Taylor})^2$ × something

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Doing the sum over the weighted trees

$$\sum_{\substack{T_b\\ \text{spanning } \mathbf{h}}} \prod_{(i\to j)} B_{ij} = \sum_{\substack{T_b\\ \text{spanning } \mathbf{h}}} \sum_{\substack{b\rho, b\sigma\\ \text{comp. } T_b}} \mathrm{PT}(b\rho x)\mathrm{PT}(b\sigma y) \times \text{▨}$$

$$= \sum_{b\rho, b\sigma} \mathrm{PT}(b\rho x)\mathrm{PT}(b\sigma y) \sum_{\substack{\text{trees compatible}\\ \text{with } b\rho, b\sigma}} \text{▨}$$

So we've found that

$$\det{}'(\mathbb{H}) = \sum_{b\rho, b\sigma} \mathrm{PT}(b\rho x)\mathrm{PT}(b\sigma y) \times \text{something}[\rho|\sigma]$$

Repeat the same for $\det{}'(\mathbb{H}^\vee)$

KLT kernel in twistor space [Adamo, SK; '24]

- Combining the contributions from trees over $\mathbf{h}$ and $\tilde{\mathbf{h}}$ (gluing together the PT factors) the gravity integrand can be rewritten as

$$\sum_{\substack{a\tilde{\rho}b\rho \\ \tilde{\omega}^T ab\omega}} |\tilde{\mathbf{h}}|^8 \mathrm{PT}[a\tilde{\rho}b\rho]\, \underbrace{S_{n,d}[\rho,\tilde{\rho}|\omega,\tilde{\omega}]}_{\substack{\text{KLT kernel} \\ \text{in twistor space}}}\, \mathrm{PT}[\tilde{\omega}^T ab\omega]$$

where

$$S_{n,d}[\rho,\tilde{\rho}|\omega,\tilde{\omega}] = \mathcal{D}[\omega,\tilde{\omega}] \left[\sum_{\tilde{T}\in\mathcal{T}^a_{\tilde{\rho},\tilde{\omega}}} \prod_{(i\to j)} \tilde{\phi}_{ij} \right] \times \left[\sum_{T\in\mathcal{T}^b_{\rho,\omega}} \prod_{(i\to j)} \phi_{ij} \right]$$

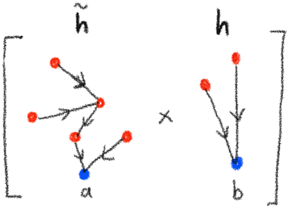
- The weights on each of the sets of trees are

$$\phi_{ij} := [\partial_\mu(\sigma_i)\,\partial_\mu(\sigma_i)](ij) \prod_{l\in\tilde{\mathbf{h}}\setminus\{a,y\}} (il)(jl), \qquad i,j\in\mathbf{h},$$

$$\tilde{\phi}_{ij} := \frac{\langle\lambda(\sigma_i)\,\lambda(\sigma_j)\rangle}{(ij)} \prod_{k\in(\tilde{\mathbf{h}}\cup\{b,t\})\setminus\{i,j\}} \frac{1}{(ki)(kj)} \qquad i,j\in\tilde{\mathbf{h}}$$

$$S_{n,d}[\rho,\tilde{\rho}|\omega,\tilde{\omega}] = \mathcal{D}[\omega,\tilde{\omega}] \left[\sum_{\tilde{T}\in\mathcal{T}^{a}_{a\tilde{\rho},a\tilde{\omega}}} \prod_{(i\to j)} \tilde{\phi}_{ij} \right] \times \left[\sum_{T\in\mathcal{T}^{b}_{b\rho,b\omega}} \prod_{(i\to j)} \phi_{ij} \right]$$

$$= \mathcal{D}(\omega,\tilde{\omega}) \sum_{\tilde{T},T}$$



$$\text{Note} \quad \sum_{T\in\mathcal{T}^{b}_{b\rho,b\omega}} \prod_{(i\to j)} \phi_{ij} = \prod_{j\neq b} \sum_{\substack{i<_{b\rho}j \\ i<_{b\omega}j}} \phi_{ij}$$

$$\mathcal{M}^{\mathrm{GR}}_{n,d} = \int \mathrm{d}\mu_d \sum_{\substack{a\tilde{\rho}b\rho \\ \tilde{\omega}^T ab\omega}} |\tilde{\mathbf{h}}|^8 \mathrm{PT}[a\tilde{\rho}b\rho] \underbrace{S_{n,d}[\rho,\tilde{\rho}|\omega,\tilde{\omega}]}_{\substack{\text{KLT kernel} \\ \text{in twistor space}}} \mathrm{PT}[\tilde{\omega}^T ab\omega] \prod_i h_i^{\pm}(Z)$$

$\mathbb{PT}$ formulae

$$\Big\downarrow \text{integrands!}$$

$$\det{}'(\mathbb{H})\det{}'(\mathbb{H}^{\vee}) = \sum_{\alpha,\beta} \mathrm{PT}[\alpha] \underbrace{\otimes}_{\text{kernel}} \mathrm{PT}[\beta]$$

✓

Helicity graded double copy kernel!

$$\begin{tikzcd}
\text{YM amplitude} \arrow[r, "\text{Double copy}"'] \arrow[d, "\text{helicity grading}"'] & \text{GR amplitude} \arrow[d, "\text{helicity grading}"] \\
\mathcal{A}_{n,d}[\alpha] = \int \mathrm{d}\mu_d \, \mathcal{I}_n^{\text{YM}}[\alpha] & \mathcal{M}_{n,d} = \int \mathrm{d}\mu_d \, \mathcal{I}_n^{\text{GR}}
\end{tikzcd}$$

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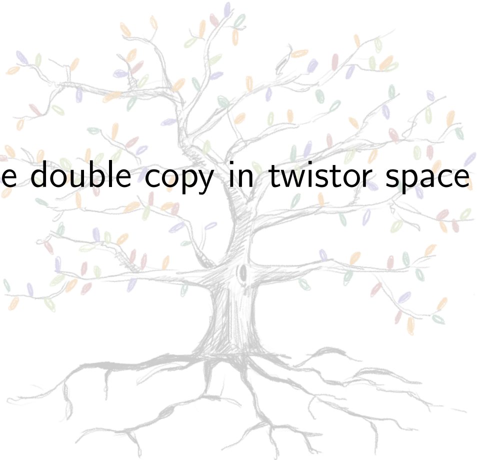
\text{YM amplitude} \arrow[r, "\text{Double copy}"'] \arrow[d, "\text{helicity grading}"'] & \text{GR amplitude} \arrow[d, "\text{helicity grading}"] \\
\mathcal{A}_{n,d}[\alpha] = \int \mathrm{d}\mu_d\, \mathcal{I}_n^{\text{YM}}[\alpha] \arrow[r, "\substack{\text{Chirally split}\\ \text{KLT kernel}}"'] & \mathcal{M}_{n,d} = \int \mathrm{d}\mu_d\, \mathcal{I}_n^{\text{GR}}
\end{tikzcd}$$

✓

$$\boxed{\mathcal{I}^{\text{GR}}_{n,d} = \sum_{\substack{a\tilde{\rho}b\rho \\ \tilde{\omega}^T ab\omega}} \mathcal{I}^{\text{YM}}_{n,d}[a\tilde{\rho}b\rho]\, S_{n,d}[\tilde{\rho},\rho|\tilde{\omega},\omega]\, \mathcal{I}^{\text{YM}}_{n,d}[\tilde{\omega}^T ab\omega]}$$

Aspects of the double copy in twistor space



Interpretation of twistorial KLT kernel

$$\mathcal{M}^{\mathsf{GR}}_{n,d} = \int \mathrm{d}\mu_d \sum_{\substack{a\tilde{\rho}b\rho \\ \tilde{\omega}^T ab\omega}} |\tilde{\mathbf{h}}|^8 \mathrm{PT}[a\tilde{\rho}b\rho] \, \underbrace{S_{n,d}[\rho,\tilde{\rho}|\omega,\tilde{\omega}]}_{\substack{\text{KLT kernel} \\ \text{in twistor space}}} \mathrm{PT}[\tilde{\omega}^T ab\omega] \prod_i h_i^{\pm}(Z)$$

- A matrix on orderings of $\mathbf{h}$ and $\tilde{\mathbf{h}}$: basis has $(n-d-2)! \times (d)!$ elements - graded by helicity, where $\#$ neg. gravitons $= d+1$
- On the other hand the number of solutions we're summing over is $E(n-3, d-1)$

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- A matrix on orderings of **h** and $\tilde{\mathbf{h}}$: basis has $(n-d-2)! \times (d)!$ elements - graded by helicity, where $\#$ neg. gravitons $= d+1$
- On the other hand the number of solutions we're summing over is $E(n-3, d-1)$
- CHY: basis 1, solutions $(n-3)!$
  KLT: basis $(n-3)!$, solutions 1

Inverse of the twistorial KLT kernel

$$\mathcal{M}_{n,d}^{\mathrm{GR}} = \int \mathrm{d}\mu_d \sum_{\substack{a\tilde{\rho}b\rho \\ \tilde{\omega}^T ab\omega}} |\tilde{\mathbf{h}}|^8 \mathrm{PT}[a\tilde{\rho}b\rho] \underbrace{S_{n,d}[\rho,\tilde{\rho}|\omega,\tilde{\omega}]}_{\substack{\text{KLT kernel} \\ \text{in twistor space}}} \mathrm{PT}[\tilde{\omega}^T ab\omega] \prod_i h_i^{\pm}(Z)$$

- It has been proven [CHY:'13;Mizera:'16; Mafra:'20;Frost,Mafra,Mason:'21] that the matrix inverse of the usual field theory kernel is equal to the scattering amplitudes of bi-adjoint scalar theory (BAS)
- Can we extract the BAS amplitude from this object? Idea:

$$m_n(a\tilde{\rho}b\rho|\tilde{\omega}^T ab\omega) \overset{?}{=} \int \mathrm{d}\mu_d \, S_{n,d}^{-1}[\rho,\tilde{\rho}|\omega,\tilde{\omega}] \prod_i \phi_i(Z)$$

where $\phi_i(Z)$ is a scalar wavefunction representative.

Inverse of the twistorial KLT kernel

- Recall that

$$S_{n,d}[\rho,\tilde{\rho}|\omega,\tilde{\omega}] = \mathcal{D}[\omega,\tilde{\omega}]\left[\sum_{\tilde{T}\in\mathcal{T}^{a}_{\tilde{\rho},\tilde{\omega}}}\prod_{(i\to j)}\tilde{\phi}_{ij}\right]\times\left[\sum_{T\in\mathcal{T}^{b}_{\rho,\omega}}\prod_{(i\to j)}\phi_{ij}\right]$$

Inverse of the twistorial KLT kernel

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$$S_{n,d}[\rho,\tilde\rho|\omega,\tilde\omega]=\mathcal{D}[\omega,\tilde\omega]\left[\sum_{\tilde T\in\mathcal{T}^a_{\tilde\rho,\tilde\omega}}\prod_{(i\to j)}\tilde\phi_{ij}\right]\times\left[\sum_{T\in\mathcal{T}^b_{\rho,\omega}}\prod_{(i\to j)}\phi_{ij}\right]$$

Theorem (CHY '13)

For symmetric w_{ij}, consider the matrix on colour orderings

$$S[\alpha,\beta]=\sum_{T\in\mathcal{T}^a_{a\alpha,b\beta}}\prod_{i\to j}w_{ij}.$$

Then the inverse is

$$S^{-1}[\alpha,\beta]=\frac{\pm1}{w_{\text{total}}}\sum_{BT\in\mathcal{BT}_{xa\alpha,xa\beta}}\prod_{E\in BT}\frac{1}{w_E},$$

where for any internal edge E in the binary tree, $w_E=\sum_{i,j\in E}w_{ij}$, the sum over the momenta flowing into that edge

Inverting the kernel

- The inverse of the kernel is therefore

$$S^{-1}_{n,d}[\rho,\tilde{\rho}|\omega,\tilde{\omega}] = \frac{\pm 1}{\mathcal{D}[\rho,\tilde{\rho}]}\left[\frac{1}{\tilde{\phi}_{\text{total}}}\sum_{\mathcal{BT}_{xa\tilde{\rho},xa\tilde{\omega}}}\prod_{E\in BT}\frac{1}{\tilde{\phi}_E}\right] \times \left[\frac{1}{\phi_{\text{total}}}\sum_{\mathcal{BT}_{xb\rho,xa\omega}}\prod_{E\in BT}\frac{1}{\phi_E}\right]$$

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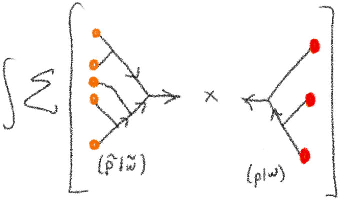
- We prove in [Adamo, SK: '24] (using amplitude recursion relations in twistor space) a new representation of BAS amplitudes

$$m_n(a\tilde{\rho}b\rho|\tilde{\omega}^T ab\omega) = \int \mathrm{d}\mu_d \, S^{-1}_{n,d}[\rho,\tilde{\rho}|\omega,\tilde{\omega}] \prod_i \phi_i(Z)$$



BAS theory in twistor space

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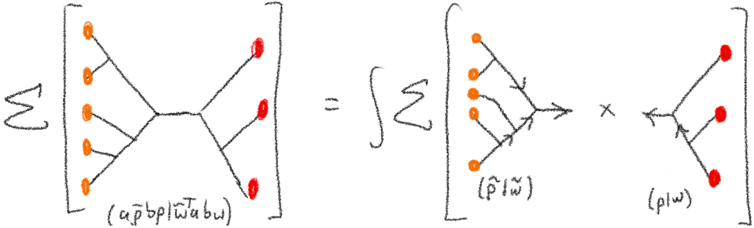

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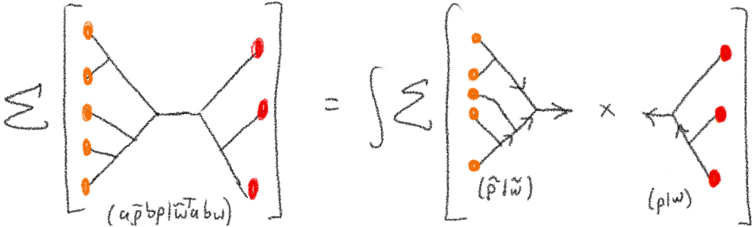
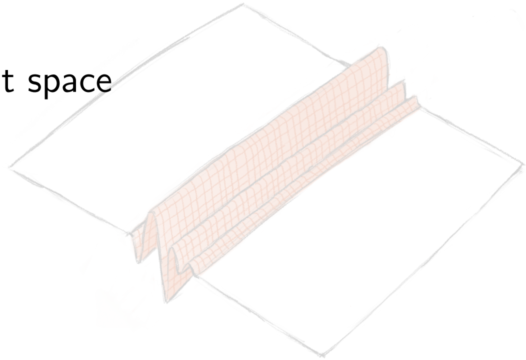


- This suggests that biadjoint scalar theory can have an attributed ‘helicity violating’ degree dependent on the colour orderings
- Applying similar methods to YMS (WIP) suggests that a similar structure exists for other theories containing scalars

Beyond flat space

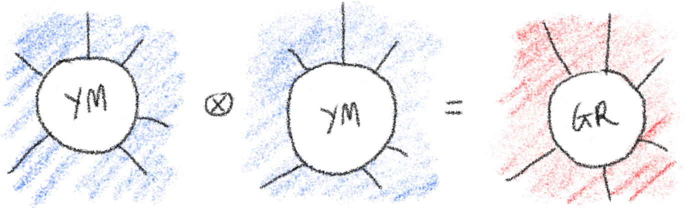


Amplitudes on non-trivial backgrounds

- Q: To what extent do 'nice' properties of scattering amplitudes survive if we put them on a non-trivial background?
- Amplitudes are defined via e.g. the perturbiner [Arefeva, Faddeev, Slavnov] and a Lagrangian where fields have a background value

$$\text{e.g. } \mathcal{L}^{\mathrm{YM}}_{\eta}[a] \to \mathcal{L}^{\mathrm{YM}}_{\eta}[A+a], \qquad \mathcal{L}^{\mathrm{GR}}[\eta+h] \to \mathcal{L}^{\mathrm{GR}}[g+h]$$

- Is there a notion of double copy? Three-point on plane waves: [Adamo, Casali, Mason, Nekovar: '17]



Amplitudes on self-dual backgrounds

- On radiative self-dual backgrounds in gauge theory and gravity we have all-multiplicity formulae [Adamo, Mason, Sharma]

$$\mathcal{A}_{n,d}[\rho] = \int \mathrm{d}\mu_d |\tilde{\mathbf{g}}|^4 \mathrm{PT}_n[\rho] \prod_{i=1}^{n} a_i^{\pm} e^{e_i\, g(U,\sigma_i)}$$

$$\mathcal{M}_{n,d} = \sum_{t=1}^{n-d-3} \sum_{p1,\ldots,pt} \int \mathrm{d}\mu_d |\tilde{\mathbf{h}}|^8 \det{}'(\mathbb{H}^{\vee}) \left(\prod_{m=1}^{t} \mathbf{N}(p_m) \circ \right) \det{}'(\mathcal{H}) \prod_{i=1}^{n} h_i^{\pm}$$

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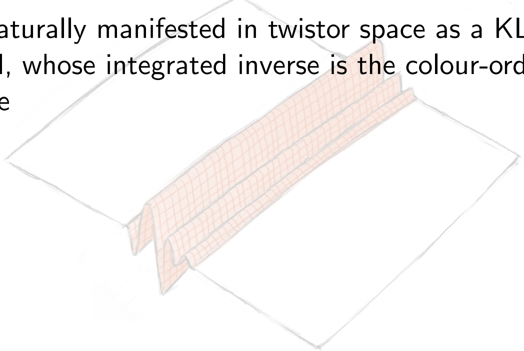
- Repeating the story from before, we can find a structure resembling a KLT kernel for these formulae

$$\mathcal{M}_{n,d} = \sum_{a\tilde{\rho}b\rho,\tilde{\omega}^T ab\omega} \int \mathrm{d}\mu_d \mathcal{I}^e[a\tilde{\rho}b\rho] S^{\mathbf{N}}_{n,d}[\rho,\tilde{\rho}|\omega,\tilde{\omega}] \mathcal{I}^{-e}[\tilde{\omega}^T ab\omega] \prod_{i=1}^{n} h_i^{\pm}$$

- Interpretation of $S^{\mathbf{N}}_{n,d}[\rho,\tilde{\rho}|\omega,\tilde{\omega}]$ is unclear!

Summary and outlook

- We found a new double copy structure between gravity and YM tree amplitudes labelled by helicity
- This is naturally manifested in twistor space as a KLT kernel integrand, whose integrated inverse is the colour-ordered BAS amplitude



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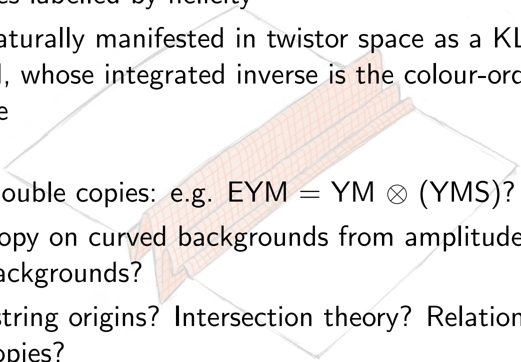

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- Web of double copies: e.g. $\text{EYM} = \text{YM} \otimes (\text{YMS})$?
- Double copy on curved backgrounds from amplitude formulae on curved backgrounds?
- Twistor string origins? Intersection theory? Relations to other double copies?

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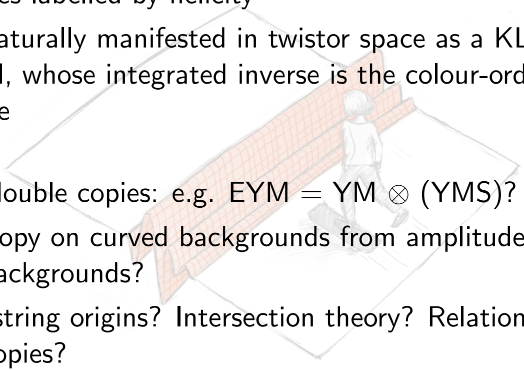

Thank you!