# Principal groupoid bundles with connections (and a bit beyond)

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# Outline



- Motivation
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- Background on Lie groupoids
- Principal bundles with connections revisited
- Severa differentiation
- Principal groupoid bundles with connections

# Motivation

- Principal groupoid bundles with connections.
- First step towards higher principal bundles with connections.

# Principal groupoid bundles with connections

Provides kinematics for gauge theories in the most general form (i.e. gauged sigma models).

Example: Given an action of a Lie group on a manifold $\mathsf{G} \curvearrowright M$, principal $\mathscr{G}$-bundles with connections, for $\mathscr{G}$ the action groupoid $\mathsf{G} \times M \rightrightarrows M$, provide the kinematics of non-linear gauged sigma models.

# First step towards higher principal bundles with connections



Understanding the two case: Lie groupoids and strict Lie 2-groups, leads us to the general definition

Most of the definition of connections on higher principal bundles must satisfy the fake flatness condition. This condition also appears in the case of Lie groupoids. The final goal is to define a general theory of higher principal bundles with connections avoiding fake flatness condition.

# Fake flatness and physics



Most of the examples of higher gauge theories do not satisfy the fake flatness condition.

- Higher gauge theories are the analogue of gauge theories for higher dimensional objects like string and branes. They appear within many contexts in string theory, e.g. gauged supergravities and M-theory.
- Gravity can be formulated as a pure 2-gauge theory using the Poincaré 2-group and a generalisation of the spinfoam quantisation is considered.
- Higher Lax connections appear in higher-dimensional integrable field theories.

# Fake flatness and mathematics



- The fake flatness condition is inconsistent. Only fully flat connections respect the inclusion of principal bundles into principal 2-bundles.

- Fake flat connections are not as general as one would expect for differential cohomologies. Locally, a connection must be given by a map $\mathscr{A} : \mathrm{T}[1]X \to \mathfrak{g}$ for a dg-manifold $\mathfrak{g}$ ($L_\infty$-algebra valued differential forms for $\mathfrak{g}$ an $L_\infty$-algebra).

## Adjusted principal 2-bundles with connections



In the case of strict Lie 2-groups, adjustments have been introduced to define principal 2-bundles with connections avoiding the fake flatness condition. The same pattern is seen for Lie groupoids.

For a given Lie 2-group $\mathscr{G}$, if some additional datum $\kappa$ (adjustment datum) is given, one can define consistent theory avoiding fake flatness in the cocycle pictute. Note that $\kappa$ is not unique and the corresponding theory depends on $\kappa$.

The origin and meaning of adjustments are not been clear.
We will argue that the pair $(\mathscr{G}, \kappa)$ is a differential refinement of $\mathscr{G}$.

# Results

# Results



- A principal $\mathscr{G}$-bundle with connection, for a Lie groupoid $\mathscr{G}$ , over a manifold $M$ is a principal $\mathscr{A}^{\mathsf{W}}(\mathscr{G})$-bundle over $\mathrm{T}[1]M$.

- The $NQ$-Lie groupoid $\mathscr{A}^{\mathsf{W}}(\mathscr{G})$ (adjustment groupoid) is defined axiomatically exploiting all the conditions we expect.

- There is a one-to-one correspondence between $NQ$-Lie groupoid $\mathscr{A}^{\mathsf{W}}(\mathscr{G})$ and Cartan connection $\mathsf{W}$ on the Lie groupoid $\mathscr{G}$. Adjustment is differential refinement by a Cartan connection. Adjustment groupoids are not unique and the theory depends on the choice of adjustment groupoid.

# Results



- There is a unique Cartan connection on Lie groups. The definition reproduces the usual cocycle picture.
- Adopting the definition to Lie 2-groups, it reproduces adjustments known for Lie 2-groups.

# Bicategory of Lie groupoids

### Lie Groupoid

A Lie groupoid $\mathscr{G} = \mathsf{G} \rightrightarrows M$ consists of smooth manifolds $\mathsf{G}$ and $M$, surjective submersions $\mathsf{s}, \mathsf{t} : \mathsf{G} \to M$, composition map $\bullet : \mathsf{G} \times_M \mathsf{G} \to \mathsf{G}$ and inverse map $(-)^{-1} : \mathsf{G} \to \mathsf{G}$ subject to axioms.

If $M$ is trivial, you get Lie groups, if $G = M$ you get manifolds.

### Lie Groipoid Action

A right action of a Lie groipoid $\mathscr{G} = \mathsf{G} \rightrightarrows M$ on a smooth manifold $X$ is a pair $(\psi, \triangleleft)$ of smooth maps $\psi : X \to M, \ \triangleleft : X \times_M \mathsf{G} \to X$ subject to axioms.

# Bicategory of Lie groupoids

## Principal Groipoid-Bundle

A principal $\mathscr{G}$-bundle over a manifold $X$ is a triple $(P, \pi, \triangleleft)$, where $P$ is a smooth manifold, $\pi : P \to X$ is a surjection submersion, and $\triangleleft$ is a right action of $\mathscr{G}$ on $P$ such that it leaves $\pi$ invariant and the shear map $\tau : P \times_M \mathsf{G} \to P \times_X P$ is an isomorphism.

## Morphism of Lie Groupoids

A bibundle (1-morphism) $F : \mathscr{G}_1 \to \mathscr{G}_2$ is a smooth manifold with commuting left and right action of $\mathscr{G}_1$ and $\mathscr{G}_2$ such that $F$ is principal $\mathscr{G}_2$-bundle over $M_1$. A 2-morphism $\phi : F_1 \Rightarrow F_2$ between bibundles is an equivariant smooth map.

In particular, a 1-morphim from a manifold $X$ to a Lie groupoid $\mathscr{G}$ is a principal $\mathscr{G}$-bundle over $X$. As a result, principal bundles are the very heart of the theory of higher geometry.

# background on Lie groupoids

**Theorem**

Lie groupoids, bibundles and their 2-morphisms form a bicategory which is called the bicategory of Lie groupoids Grp.

**Theorem**

The bicategroy of Lie groupoids is euivalent to the bicategory of geometric stacks over smooth manifolds.

**Remark**

All definitions above are for the category of smooth manifolds, but they work for other categories like the category of $NQ$-manifolds to define $NQ$-Lie groupoids.

# Principal bundles with connections revisited

# Principal bundles with connections revisited



Fix an open cover $U = \{U_i\}$ of a manifold $X$ and a Lie group $\mathsf{G}$.

- A cocycle $g$ consists of smooth maps $g_{ij} : U_i \times_X U_j \to \mathsf{G}$ such that $g_{ij}g_{jk} = g_{ik}$
- A coboundary $a : g \to \hat{g}$ consists of smooth maps $a_i : U_i \to \mathsf{G}$ such that $a_i\hat{g}_{ij} = g_{ij}a_j$.
- Cocycles and coboundaries form a groupoid $(\check{C}(U), \mathsf{G})$.
- It is well known that principal $\mathsf{G}$-bundles are cocycles up to coboundaries. More precisely

$$\mathsf{Bun}_{\mathsf{G}}(X) \cong \check{H}^*(M, \mathsf{G}) := \varinjlim_{U}(\check{C}(U), \mathsf{G}) \ ,$$

# Principal bundles with connections revisited



- $H^*(M, \mathsf{G})$ is the Giraud non-abelian cohomology. Principal bundles are geometric realization of non-abelian cohomologies

- The cocycle picture of principal bundles is a generic phenomenon in the theory of stacks and is not specific to groups nor smooth manifolds. In particular, it works for $NQ$-manifolds too. There is some duality between cocycle and principal bundle picture.

- Principal bundles with connections, also have cocycle descriptions. As a result, it is expected that principal bundles with connections can be seen principal bundles without connection but in a different category.

# Principal bundles with connections revisited



**Theorem**

*A principal* $\mathsf{G}$*-bundle with a connection over a manifold* $X$*, is a principal* $\mathscr{A}^{\mathsf{W}}(\mathsf{G})$*-bundle over* $\mathrm{T}[1]X$*, where* $\mathscr{A}^{\mathsf{W}}(\mathsf{G})$ *is the* $NQ$*-Lie groupoid*

$$\begin{array}{c}\mathrm{T}[1]G \times \mathrm{T}[1]\mathfrak{g} \\ \downdownarrows \\ \mathrm{T}[1]\mathfrak{g},\end{array}$$

*and* $\mathfrak{g}$ *is the Lie algebra corresponding to* $\mathsf{G}$*. In other words,* $\mathscr{A}^{\mathsf{W}}(\mathsf{G})$ *is the classification space.*

Where does this $NQ$-Lie groupoid come from?
We will see that this $NQ$-Lie groupoid has some properties that uniquely determine it. Then we use these properties as axioms for Lie groupoids.

# Severa differentiation

# Severa differentiation



Severa proposed a procedure to differentiate Kan simplicial manifolds (Lie quasi-groupoids) to $NQ$-manifolds which is generalisation of differentiating Lie groups to Lie algebras. We present a natural lift of this definition in the case of Lie groupoids that maps a Lie groupoid to an $NQ$-Lie groupoid.

**Definition**

Given any Lie groupoid $\mathscr{G}$, we define $\mathscr{A}(\mathscr{G})$ as the internal hom $\mathscr{A}(\mathscr{G}) := \underline{\text{hom}}(\check{\mathscr{C}}_\Theta, \mathscr{G})$ in the bicategory of $NQ$-Lie groupoids, where $\check{\mathscr{C}}_\Theta$ is the $NQ$-Lie groupoid $\Theta \times \Theta \rightrightarrows \Theta$ and $\Theta$ is the shifted real line.

The definition has a natural generalisation to higher Lie groupoids.

# Severa differentiation



**Theorem**

*For any Lie groupoid $\mathscr{G} : \mathsf{G} \rightrightarrows M$, the differentiation $\mathscr{A}(\mathscr{G})$ is given by*

$$\begin{array}{c} \mathrm{T}[1]G \times_{\mathrm{T}[1]M} \mathfrak{g} \\ \downdownarrows \\ \mathfrak{g}, \end{array}$$

*where $\mathfrak{g}$ is the Lie algebroid corresponding to $\mathscr{G}$.*

The Severa differentiation only produces the Lie algebroid $\mathfrak{g}$. What is the role of $\mathscr{A}(\mathsf{G})$ for Lie groups $\mathsf{G}$? The relation to $\mathscr{A}^{\mathsf{W}}(\mathsf{G})$ is the key.

# Principal groupoid bundles with connections
## flat connections

**Theorem**

*For a Lie group $\mathsf{G}$, a principal $\mathsf{G}$-bundle with a flat connection over a manifold $X$ is a principal $\mathscr{A}(\mathsf{G})$-bundle over $\mathrm{T}[1]X$.*

**Definition**

For a Lie groupoid $\mathscr{G}$, a principal $\mathscr{G}$-bundle with a flat connection over a manifold $X$ is a principal $\mathscr{A}(\mathscr{G})$-bundle over $\mathrm{T}[1]X$.

# Principal groupoid bundles with connections axiomatisation

For a Lie group $\mathsf{G}$, what is the relation between $\mathscr{A}^{\mathsf{W}}(\mathsf{G})$ and $\mathscr{A}(\mathsf{G})$?

There is a dg-functor $\Psi : \mathscr{A}(\mathsf{G}) \to \mathscr{A}^{\mathsf{W}}(\mathsf{G})$ such that

- $\Psi_0 : \mathscr{A}(\mathsf{G})_0 = \mathfrak{g} \to \mathscr{A}^{\mathsf{W}}(\mathsf{G})_0 = \mathrm{T}[1]\mathfrak{g}$ is the canonical inclusion,
- the diagram

$$\begin{array}{ccc} \mathscr{A}(\mathsf{G})_1 & \xrightarrow{\Psi_1} & \mathscr{A}^{\mathsf{W}}(\mathsf{G})_1 \\ \downarrow{\scriptstyle \mathrm{t}} & & \downarrow{\scriptstyle \mathrm{t}} \\ \mathfrak{g} & \xrightarrow{\Psi_0} & \mathrm{T}[1]\mathfrak{g} \end{array}$$

  is a pullback diagram (this is also true if $\mathrm{t}$ is replaced with $\mathrm{s}$).

The next theorem states that the $NQ$-Lie groupoid $\mathscr{A}^{\mathsf{W}}(\mathsf{G})$ is determined by these properties.

### Theorem

*Suppose* $\mathscr{A}^{\hat{\mathsf{W}}}(\mathsf{G})$ *is any* $NQ$*-Lie groupoid such that*

- *the space of objects* $\mathscr{A}^{\hat{\mathsf{W}}}(\mathsf{G})_0$ *is* $\mathrm{T}[1]\mathfrak{g}$
- *there exists a dg-functor* $\hat{\Psi} : \mathscr{A}(\mathsf{G}) \to \mathscr{A}^{\hat{\mathsf{W}}}(\mathsf{G})$
- $\hat{\Psi}_0 : \mathscr{A}(\mathsf{G})_0 \to \mathscr{A}^{\hat{\mathsf{W}}}(\mathsf{G})_0$ *is the canonical inclusion*
- *the diagram*

$$\begin{array}{ccc} \mathscr{A}(\mathsf{G})_1 & \xrightarrow{\hat{\Psi}_1} & \mathscr{A}^{\hat{\mathsf{W}}}(\mathsf{G})_1 \\ \downarrow \mathsf{t} & & \downarrow \mathsf{t} \\ \mathfrak{g} & \xrightarrow{\hat{\Psi}_0} & \mathrm{T}[1]\mathfrak{g} \end{array}$$

*is a pullback digram. Then, there is an isomorphism* $\mathscr{A}^{\hat{\mathsf{W}}}(\mathsf{G}) \cong \mathscr{A}^{\mathsf{W}}(\mathsf{G})$.

# Principal groupoid bundles with connections axiomatisation



We use this set of properties as axioms for Lie groupoids.

**Definition**

For any Lie groupoid $\mathscr{G}$, an adjustment groupoid $\mathscr{A}^{\mathsf{W}}(\mathscr{G})$ is an $NQ$-Lie groupoid satisfying the set of properties in the last slide.

**Definition**

For a given adjustment groupoid $\mathscr{A}^{\mathsf{W}}(\mathscr{G})$, a principal $\mathscr{G}$-bundle with a connection over a manifold $X$ is a principal $\mathscr{A}^{\mathsf{W}}(\mathscr{G})$-bundle over $\mathrm{T}[1]X$.

# Principal groupoid bundles with connections discussion



Why these axioms? Let me translate the axioms:

- Locally, a connection is a differential graded map $\mathrm{T}[1]X \to \mathrm{T}[1]\mathfrak{g}$ or, equivalently, is a graded map $\mathrm{T}[1]X \to \mathfrak{g}$, In other words, a Lie algebroid valued differential form.
- There a fully faithful inclusion $\Psi^* : \mathsf{Bun}(X)^{\flat}_{\mathscr{G},\mathsf{Con}} \to \mathsf{Bun}(X)^{\mathsf{W}}_{\mathscr{G},\mathsf{Con}}$.
- If two connections are equivalent and one of them is flat. Then, the other is also flat.

In fact, these properties are equivalent to the axioms.

The axioms, also, have a geometric meaning:

**Theorem**

*For any Lie groupoid $\mathscr{G}$, there is a one-to-one correspondence between Cartan connections $\mathsf{W}$ on $\mathscr{G}$ and adjustment groupoids $\mathscr{A}^{\mathsf{W}}(\mathscr{G})$.*

In other words, an adjustment groupoid $\mathscr{A}^{\mathsf{W}}(\mathscr{G})$ is a differential refinement of $\mathscr{G}$ by a Cartan connection $\mathsf{W}$.
For a Lie groupoid $\mathscr{G} : \mathsf{G} \rightrightarrows M$, a Cartan connection $\mathsf{W}$ is a multiplicative distribution of $\mathrm{T}\mathsf{G}$ which is complementry to $\ker(\mathrm{dt})$.

Why we have not seen this before?

- There exists a unique Cartan connection for any Lie groups.
- There exists a canonical Cartan connection action Lie groupoids.

Thanks for your attention.