Pluriclosed flow and the Hull-Strominger system

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Joint work with R. González Molina and J. Streets, arXiv:2408.11674

Pluriclosed flow

Let (M^{2n}, J) be a complex manifold. A Riemannian metric g on M is pluriclosed if

$$g(J\cdot, J\cdot) = g, \qquad \partial\overline{\partial}\omega = 0,$$

where $\omega = g(J\cdot, \cdot) \in \Lambda^{1,1}$ (Hermitian form).

Remarks:

- Natural linear integrability condition, replacing Kähler condition.
- By Gauduchon's Theorem, any compact complex surface admits a pluriclosed metric.
- Locally, they admit a potential $(1,0)$-form: $\omega = \overline{\partial}\alpha + \partial\overline{\alpha}$.

Example: Take $(M, J) = \mathbb{C}^2 \backslash \{0\}/\mathbb{Z}$ (Hopf surface) and

$$\omega_F = \frac{\sqrt{-1}}{2} \frac{dz_1 \wedge d\overline{z}_1 + dz_2 \wedge d\overline{z}_2}{|z_1|^2 + |z_2|^2} \qquad (\textit{Boothby Metric})$$

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Definition (Streets-Tian '10)

A one parameter family of pluriclosed metrics ω_t is a solution to <u>pluriclosed flow</u> if

$$\frac{\partial}{\partial t}\omega = -\partial\partial^*_\omega\omega - \overline{\partial\partial^*_\omega}\omega - \sqrt{-1}\overline{\partial}\partial \log \det g$$

- Well-posed problem (locally in time): Streets-Tian '10.
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Theorem (Streets-Tian '12)

Let ω_t be a solution to pluriclosed flow. Then, the metric $g_t = \omega_t(\cdot, J\cdot)$, the <u>torsion</u> $H_t = -d^c\omega_t$, and the <u>Lee form</u> $\theta_t = -d^*\omega_t J$ satisfy

$$\frac{\partial}{\partial t}g = -\operatorname{Rc} + \tfrac{1}{4}H^2 - \tfrac{1}{2}L_{\theta^\sharp}g$$
$$\frac{\partial}{\partial t}H = -\tfrac{1}{2}\Delta_g H - \tfrac{1}{2}L_{\theta^\sharp}H$$

This flow corresponds to a De Turck version of Generalized Ricci flow

- Streets, *Regularity and expanding entropy for connection Ricci flow*, J. Geom. Phys., 2008.
- Garcia-Fernandez, Streets, *Generalized Ricci flow*, University Lecture Series, 2021, AMS.

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Formally taking $\theta^\sharp$ to be gradient, we recover the Renormalization Group Flow in Type II String Theory:

$$\frac{\partial}{\partial t}g = -\operatorname{Rc} + \tfrac{1}{4}H^2 - \tfrac{1}{2}L_{\nabla f}g$$
$$\frac{\partial}{\partial t}H = -\tfrac{1}{2}\Delta_g H - \tfrac{1}{2}L_{\nabla f}H$$

Using the Bismut connection of a pluriclosed Hermitian metric ω

$$\nabla^B = \nabla - \tfrac{1}{2} g^{-1} d^c \omega$$

we can alternatively write the pluriclosed flow as

$$\frac{\partial}{\partial t}\omega = -\rho_B^{1,1}$$

where ρ_B denotes the Bismut-Ricci form

$$\rho_B(X, Y) = \frac{1}{2}\sum_{j=1}^{2n} g(R_{\nabla^B}(X, Y)Je_i, e_i)$$

Motivated by this, it is natural to consider a coupled flow by introducing a family of $(2, 0)$-forms $\beta_t \in \Lambda^{2,0}$

$$\frac{\partial}{\partial t}\omega = -\rho_B^{1,1}, \qquad \frac{\partial}{\partial t}\beta = -\rho_B^{2,0}$$

Alternatively, $\frac{\partial}{\partial t} \operatorname{Re} \sqrt{-1}\beta = -\frac{1}{2} d^* H + \frac{1}{2} d\theta - \frac{1}{2}\iota_{\theta^\sharp} H$.

Geometrization of complex surfaces

An important motivation for pluriclosed flow is a missing piece in Kodaira's classification of compact complex surfaces (Streets-Tian '10):

Conjecture (Kato '78)

All class VII surfaces ($b_1 = 1$, $b_2 > 0$, $Kod = -\infty$) are Kato

$$X \cong (S^3 \times S^1) \sharp_k \overline{\mathbb{CP}^2}.$$

Idea: pluriclosed flow should exist for all time on X, class VII, and there should be a limit X_∞ which admits a *Bismut Hermitian-Einstein metric*

$$\rho_B(\omega) = 0, \qquad dd^c\omega = 0.$$

Then $\theta_\omega^\sharp$ is holomorphic (K.-H. Lee '24), and $\mathrm{Ker}\,\theta_\omega$ should have a leaf which determines a *global spherical shell*, that is,

$$\varphi \colon A \to X, \qquad \overline{\partial}\varphi = 0, \qquad X \backslash \varphi(A) \text{ connected},$$

where $A = \{z \in \mathbb{C}^2 | 0 < r < |z| < R\}$.

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Geometrization of Reid's fantasy

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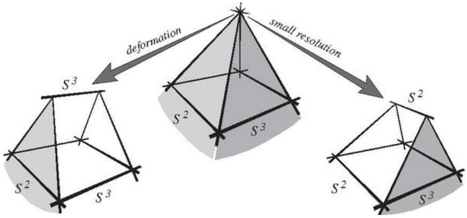
In six real dimensions there is a remarkable geometrization problem, related to the moduli space of algebraic Calabi-Yau threefolds:

Reid's Fantasy for algebraic Calabi-Yau threefolds:

There should exist a 'master moduli space' associated to the manifolds $\sharp_{k\geqslant 2}(S^3 \times S^3)$, such that every algebraic Calabi-Yau threefold arises as boundary phenomena for elements in this family.

- M. Reid, The moduli space of 3-folds with K=0 may nevertheless be irreducible, Math. Ann. **278** (1987) 329–334



The Hull-Strominger system

In 2005, J. Li and S.-T. Yau proposed to study supersymmetry equations in heterotic supergravity, known as the **Hull-Strominger system**, to geometrize *conifold transitions* in the passage from Kähler to non-Kähler Calabi-Yau threefolds.

$$F \wedge \omega^2 = 0 \qquad F^{0,2} = 0$$

$$d(\|\Omega\| \omega^2) = 0 \qquad dd^c\omega - \operatorname{tr} R \wedge R + \operatorname{tr} F \wedge F = 0$$

- Strominger, Nucl. Phys. B 274 (1986).
- Hull, Turin 1985 Proc. (1986).

Li-Yau's proposal is supported by a potential local model solution ...

The local smoothing of a conifold point can be identified with the Lie group $X_t \cong SL(2, \mathbb{C})$

$$X_0 = \{z_1^2 + z_2^2 + z_3^2 + z_4^2 = 0\} \rightsquigarrow X_t = \{z_1^2 + z_2^2 + z_3^2 + z_4^2 = t\} \subset \mathbb{C}^4,$$

and carries a natural left-invariant solution.

- Fei-Yau, Comm. Math. Phys. 338 (2015)
- Otal-Ugarte-Villacampa, Nuclear Phys. B 920 (2017)

... and several partial existence results:

• Fu, Li, Yau, JDG **90** (2012), • Chuan, Comm. Anal. Geom. **20** (2014), • Collins, Picard, Yau, arXiv:2102.11170 (2021), • Giusti, Spotti, arXiv:2301.11636 (2023), • Friedman, Picard, Suan, arXiv:2404.11840 (2024).

Geometric flows and Reid's fantasy

Question

Can we define a version of pluriclosed flow which helps us to address the geometrization of Reid's fantasy?

Idea: higher version of Donaldson's flow in gauge theory, via the unifying language of generalized geometry (*à la Hitchin*).

- N. Hitchin, Quart. J. Math. Oxford Ser. 54 (2003)

Alternative approach: anomaly flow, by Phong, Picard, Zhang

$$\partial_t(\|\Omega\|_\omega \omega^2) = dd^c\omega - \operatorname{tr} R^2 + \operatorname{tr} F^2, \qquad h_t^{-1}\partial_t h_t = -\Lambda_{\omega_t} F_{h_t}.$$

- Phong, Picard, Zhang, Math. Z. (2017)

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Gauge theory and Donaldson functional

Instantons in gauge theory are special solutions of the Yang-Mills equations on a smooth compact manifold M

$$d_A^* F_A = 0$$

which provide absolute minima for the Yang-Mills energy functional

$$A \to \int_M |F_A|^2 \,\mathrm{Vol}_g \,.$$

Here, A is a connection on a principal K-bundle $P \to M$, for a compact Lie group K. The norm of its curvature $F_A \in \Omega^2(\mathrm{ad}\, P)$ is

$$|F_A|^2 = \langle F_A \wedge * F_A \rangle,$$

calculated using the Hodge star operator of the Riemannian manifold (M, g) and a positive definite biinvariant product on the Lie algebra $\mathfrak{k}$

$$\langle , \rangle : \mathfrak{k} \otimes \mathfrak{k} \to \mathbb{R}.$$

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Hermitian-Yang-Mills connections

On a Hermitian manifold (M, J, g) ($\dim_{\mathbb{R}} M = 2n$) with Hermitian form $\omega = g(J,)$, natural candidates are the Hermitian-Yang-Mills connections

$$F_A \wedge \omega^{n-1} = 0, \qquad F_A^{0,2} = 0.$$

The instanton condition strongly depends on the torsion classes of the $\mathrm{U}(n)$-structure

$$-\int_M \langle F_A \wedge F_A \rangle \wedge \omega^{n-2} = \int_M |F_A|^2 \,\mathrm{Vol}_g - \int_M (|\Lambda_\omega F_A|^2 + 4|F_A^{0,2}|^2)\,\mathrm{Vol}_g$$

due to the fact that the LHS may not be topological.

Observe: restrict to $F_A^{0,2} = 0$: then $A^{0,1}$ determines a holomorphic vector bundle $V \Rightarrow$ unitary connections obtained via Chern correspondence: $(V, h) \mapsto A_h =_{loc} h^{-1}\partial h$.

Remark: even for non-Kähler manifolds, relation to the theory of *stable bundles* in algebraic geometry via the Donaldson-Uhlenbeck-Yau theorem.

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Donaldson functional

Assume further that ω is is balanced, that is, $d\omega^{n-1} = 0$. Then, there exists a functional

$$h \to \mathcal{D}(h, h_0) = \int_X R(h, h_0) \wedge \omega^{n-1},$$

defined using Bott-Chern secondary classes

$$R(h, h_0) \in \Omega^{1,1} / \operatorname{Im} \partial \oplus \bar{\partial}$$

whose gradient flow is

$$h_t^{-1} \partial_t h_t = -i \Lambda_{\omega_t} F_{h_t}.$$

Along the flow, the derivative is the norm square of the *moment map*

$$\partial_t \boldsymbol{\mathcal{D}} = -\|\Lambda_\omega F_h\|_{L^2}^2 \qquad (= -\|F_h\|_{L^2}^2 + p_1(P) \cdot [\omega]^{n-2} \quad \text{if } d\omega = 0).$$

- Donaldson, Proc. London Math. Soc. (1985)

Pluriclosed flow and generalized geometry

Let M be a manifold endowed with a principal K-bundle $P \to M$ and biinvariant pairing

$$\langle , \rangle \colon \mathfrak{k} \times \mathfrak{k} \to \mathbb{R}.$$

Assume that $p_1(P) = 0$, and hence there exists a solution (H_0, A_0), for $H_0 \in \Lambda^3$ and A_0 a connection on P, of the *heterotic Bianchi identity*

$$dH_0 = \langle F_A \wedge F_A \rangle.$$

This solution determines a Courant algebroid

$$E = T \oplus \operatorname{ad} P \oplus T^*,$$

where

$$\begin{aligned}
\pi(X + r + \xi) &= X \\
\langle X + r + \xi, X + r + \xi \rangle &= i_X \xi + \langle r, r \rangle \\
[V + r + \alpha, Y + s + \beta] &= [V, Y] + L_V \beta - \iota_Y d\alpha + \iota_Y \iota_V H_0 \\
&\quad - [r, s] - F_{A_0}(V, Y) + \iota_V d_{A_0} s - \iota_Y d_{A_0} r \\
&\quad + 2 \langle d_{A_0} r, s \rangle + 2 \langle \iota_V F_{A_0}, s \rangle - 2 \langle \iota_Y F_{A_0}, r \rangle .
\end{aligned}$$

Let M^{2n} be an even-dimensional manifold endowed with a solution (H_0, A_0) of the *heterotic Bianchi identity*.

Definition

A $(0,1)$-lifting on E is given by an isotropic subbundle

$$\bar{\ell} \subset E \otimes \mathbb{C}$$

s.t. $\exists\, J$ almost complex structure on M satisfying $\pi(\bar{\ell}) = T_J^{0,1}$.

More explicitly, $\bar{\ell} = \bar{\ell}(J, \omega, b, a)$, for $\omega \in \Lambda_J^{1,1}$, $b \in \Lambda^2$, $a \in \Lambda^1 \otimes adP$

$$\bar{\ell} = e^{(b + i\omega, a)} T_J^{0,1} \subset (T \oplus adP \oplus T^*) \otimes \mathbb{C}$$

Lemma

A $(0,1)$-lifting $\bar{\ell} = \bar{\ell}(J, \omega, b, a)$ is involutive for the Dorfman bracket iff

$$N_J = 0, \quad F_A^{0,2} = 0, \quad -d^c\omega = H := H_0 - db - CS(A_0) + CS(A) - d\langle A \wedge A_0 \rangle$$

where $A = A_0 + a$.

Let (M^{2n}, H_0, A_0) as before and let $\bar{\ell} = \bar{\ell}(J, \omega, b, a)$ an involutive positive ($\omega(,J) > 0$) $(0,1)$-lifting. Let

$$\ell = \overline{\bar{\ell}} \subset E \otimes \mathbb{C}.$$

Lemma (Álvarez-Cónsul, De Arriba de la Hera, GF '23)

For any choice of local holomorphic dual frames $\{\epsilon_j, \bar{\epsilon}_j\}_{j=1}^n$ of $\ell \oplus \bar{\ell}$ there exists a unique local $\varepsilon \in \Gamma(\operatorname{Ker} \pi) = \Lambda^1 \oplus adP$ solving the *D-term equation*:

$$\frac{1}{2} \sum_{j=1} [\bar{\epsilon}_j, \epsilon_j] = \pi_\ell \varepsilon - \varepsilon.$$

Furthermore, the section ε is a 'left symmetry' for the Dorfman bracket, that is, $[\varepsilon,] = 0$, iff (ω, A) solves the *coupled instanton equation*

$$\rho_B(\omega) + \langle \Lambda_\omega F_A, F_A \rangle = 0, \qquad d_A(\Lambda_\omega F_A) = 0, \qquad [\Lambda_\omega F_A,] = 0.$$

Proof: by direct calculations, taking $\bar{\epsilon}_j = e^{(b+i\omega,a)} \frac{\partial}{\partial \bar{z}_j}$, one has

$$\varepsilon = d \log \|\Omega\| + i(d^*\omega - d^c \log \|\Omega\|) + i\Lambda_\omega F_A.$$

Definition

A one-parameter family of positive $(0,1)$-liftings $\bar{\ell}_t = \bar{\ell}(J_t, \omega_t, b_t, a_t)$ on E satisfies the *pluriclosed flow* if

$$\partial_t \bar{\ell} = [\varepsilon, \bar{\ell}].$$

Lemma (GF, Gonzalez Molina, Streets '24)

The family $\bar{\ell}_t = \bar{\ell}(J_t, \omega_t, b_t, a_t)$ with involutive initial condition satisfies the *pluriclosed flow* iff $\partial_t J = 0$ and

$$\begin{aligned}
\partial_t \omega &= -\rho_B(\omega)^{1,1} - \langle \Lambda_\omega F_A, F_A \rangle, \\
\partial_t b^{1,1} &= -\frac{1}{2} \langle a \wedge J d_A(\Lambda_\omega F_A) \rangle, \\
\partial_t b^{0,2} &= -\rho_B(\omega)^{0,2} + \frac{i}{2} \langle a^{0,1} \wedge \overline{\partial}(\Lambda_\omega F_A) \rangle, \\
\partial_t A &= -\frac{1}{2} J d_A(\Lambda_\omega F_A).
\end{aligned}$$

Theorem (GF, Gonzalez Molina, Streets '24)

The pluriclosed flow is well-posed, that is, it admits a solution for short time given any initial data:

$$\partial_t \bar{\ell} = [\varepsilon, \bar{\ell}].$$

Remark: an alternative proof follows from work by Strickland-Constable, Valach, and Streets, via generalized Ricci flow.

- Streets, Strickland-Constable, Valach, Ricci flow on Courant algebroids, arXiv:2402.11069

Generalized Ricci flow

Proposition (GF, Gonzalez Molina, Streets '24)

If $\bar{\ell}_t = \bar{\ell}(J_t, \omega_t, b_t, a_t)$ satisfies the *pluriclosed flow*, then

$$\begin{aligned} \partial_t g &= -\operatorname{Rc} + \frac{1}{4}H^2 - \langle (F_A)_i, (F_A)_i \rangle - \frac{1}{2} L_{\theta_\omega^\sharp} g, \\ \partial_t b &= \frac{1}{2}(d^* H - \langle a, \partial_t A_t \rangle - d\theta_\omega + i_{\theta_\omega^\sharp} H), \\ \partial_t A &= -\frac{1}{2}(d_A^* F_A - F_A \lrcorner H + i_{\theta_\omega^\sharp} F_A), \end{aligned} \tag{1}$$

where $g_t = \omega_t(, J)$, $A_t = A_0 + a_t$, $\theta_{\omega_t} = J d^* \omega_t$ is the Lee form, and

$$-d^c \omega_t = H_t := H_0 - db_t - CS(A_0) + CS(A_t) - d\langle A_t \wedge A_0 \rangle.$$

Consequently, (g_t, b_t, A_t) satisfies *gauge-fixed generalized Ricci flow*.

Remark: Formally taking $\theta_\omega^\sharp$ to be gradient, we recover the Renormalization Group Flow in heterotic string String Theory.

Holomorphic gauge

The evolution equations for $b^{1,1}$ and A tell us that we can write the *universal pluriclosed flow* in a *holomorphic gauge*: given $\bar{\ell}$ involutive, the reduction

$$\mathcal{Q}_{\bar{\ell}} = \bar{\ell}^{\perp}/\bar{\ell}$$

is a Courant algebroid in the holomorphic category. More explicitly, for $\bar{\ell} = \bar{\ell}(J, \omega, b, a)$

$$\mathcal{Q}_{\bar{\ell}} \cong T^{1,0} \oplus adP \otimes \mathbb{C} \oplus T^*_{1,0}, \qquad \bar{\partial}_{\bar{\ell}} = \begin{pmatrix} \bar{\partial} & 0 & 0 \\ F_A & \bar{\partial}_A & 0 \\ -2i\partial\omega & 2\langle F_A, \rangle & \bar{\partial} \end{pmatrix}$$

Definition

Define a Hermitian metric on $\mathcal{Q}_{\bar{\ell}}$ by

$$G(\omega, b) = (e^{(b^{2,0}, a^{1,0})})_* \begin{pmatrix} g & 0 & 0 \\ 0 & -\langle , \rangle & 0 \\ 0 & 0 & g^{-1} \end{pmatrix}.$$

Proposition (GF, Gonzalez Molina, Streets '24)

The family $G_t = G(\omega_t, b_t^{2,0}, h_t)$ satisfies

$$G^{-1}\partial_t G = -\Lambda_\omega F_G$$

on the fixed holomorphic vector bundle $\mathcal{Q}_{\bar{\ell}_0}$ if and only if

$$\begin{aligned}
\partial_t \omega &= -\rho_B(\omega)^{1,1} - \langle \Lambda_\omega F_h, F_h \rangle, \\
\partial_t b^{2,0} &= -\rho_B(\omega)^{2,0} + \frac{i}{2}\langle a \wedge \partial_{A_h}(\Lambda_\omega F_h) \rangle, \\
h^{-1}\partial_t h &= -\Lambda_\omega F_h.
\end{aligned}$$

where $a = A_h - A_0$.

The dilaton functional

Assume that (M, J) admits a holomorphic volume form Ω, and consider:

$$\mathcal{W} = \{\bar{\ell} \text{ positive }, [\bar{\ell}, \bar{\ell}] \subset \bar{\ell}\}.$$

The dilaton functional $\mathcal{D}\colon \mathcal{W} \to \mathbb{R}$ is defined by

$$\mathcal{D}(\omega, b, A) = \int_M \|\Omega\|_\omega \frac{\omega^n}{n!}.$$

- Garcia-Fernandez, Rubio, Shahbazi, Tipler, PLMS 125 (2022)

Restrict further to variations where $A = h^{-1}\partial h$ and ω varies in a fixed *Aeppli class*

$$\mathfrak{a}_1 - \mathfrak{a}_0 = [\omega_1 - \omega_0 + R(h, h_0)] \in H_A^{1,1}(X, \mathbb{R}) = \frac{ker\ \partial\bar{\partial}}{Im\ \partial \oplus \bar{\partial}}.$$

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The Hull-Strominger system

Lemma

Critical points (ω, b, A) of the dilaton functional $\mathcal{D}\colon \mathcal{W} \to \mathbb{R}$ along the Aeppli class are solutions of the Hull-Strominger system

$$\begin{aligned} F_A \wedge \omega^{n-1} &= 0, \qquad & F_A^{0,2} &= 0 \\ d^*\omega - d^c \log \|\Omega\|_\omega &= 0, \qquad & H &= -d^c\omega \end{aligned}$$

where $A = h^{-1}\partial h$ and, for $a = A - A_0$,

$$H = H_0 - db + 2\langle a \wedge F_{A_0}\rangle + \langle a \wedge d_{A_0} a\rangle + \frac{1}{3}\langle a \wedge [a \wedge a]\rangle.$$

- Garcia-Fernandez, Rubio, Shahbazi, Tipler, PLMS 125 (2022)

Remark: the last equation implies the more familiar *Bianchi identity*

$$dd^c\omega + \langle F_A \wedge F_A\rangle = 0.$$

Gradient flow

Proposition

The gradient flow of the dilaton functional $\mathcal{D}\colon \mathcal{W} \to \mathbb{R}$ along the Aeppli class is the the pluriclosed flow in the holomorphic gauge

$$\frac{\partial}{\partial t}\omega = -\rho_B(\omega)^{1,1} - \langle \Lambda_\omega F_h, F_h \rangle,$$
$$h^{-1}\frac{\partial}{\partial t}h = -i\Lambda_\omega F_h.$$

Along a solution, the derivative of the dilaton functional is the norm square of a *higher moment map*

$$\frac{\partial}{\partial t}\mathcal{D} = -\|d^*\omega - d^c \log \|\Omega\|_\omega\|^2_{L^2} - \|\Lambda_\omega F_A\|^2_{L^2}.$$

- Garcia-Fernandez, González Molina, Streets, arXiv:2408.11674 (2024)
- Garcia1-Fernandez, Rubio, Tipler, J. Diff. Geom. (2024)

Main results

Higher regularity

Theorem (GF, Gonzalez Molina, Streets '24)

Let $G_t = G(\omega_t, b_t^{2,0}, h_t)$ be a solution of the flow $G^{-1}\partial_t G = -\Lambda_\omega F_G$ on the fixed holomorphic vector bundle $\mathcal{Q}_{\bar{\ell}_0}$ defined on $[0,\tau)$, $\tau \leq 1$. Suppose there are background data $\tilde{\omega}$, $\tilde{b}$, $\tilde{h}$ and constants $\lambda, \Lambda > 0$ so that

$$\lambda^{-1}\tilde{\omega} \leq \omega \leq \Lambda\tilde{\omega}, \qquad \lambda^{-1}\tilde{b} \leq b \leq \Lambda\tilde{b}, \qquad \lambda^{-1}\tilde{h} \leq h \leq \Lambda\tilde{h}.$$

Given $k \in \mathbb{N}$ there exists a constant $C > 0$ such that

$$\sup_{M\times[0,\tau)} t\Phi_k(\omega, \tilde{\omega}, h, \tilde{h}) \leq C.$$

where

$$\Phi_k(\omega, \tilde{\omega}, h, \tilde{h}) := \sum_{j=0}^{k} \left(|\nabla_g^j \Upsilon(\omega, \tilde{\omega})|_g^2 + |\nabla_{g,h}^j \Upsilon(h, \tilde{h})|_{g,h}^2 \right)^{\frac{1}{1+j}}$$

Long time existence and convergence

Theorem (GF, Gonzalez Molina, Streets '24)

Assume that the pairing $\langle , \rangle < 0$ is negative definite.

Let $G_t = G(\omega_t, b_t^{2,0}, h_t)$ be a solution of the flow $G^{-1}\partial_t G = -\Lambda_\omega F_G$ on the fixed holomorphic vector bundle $\mathcal{Q}_{\bar{\ell}_0}$ with initial condition G_0, that is,

$$\partial_t \omega = -\rho_B(\omega)^{1,1} - \langle \Lambda_\omega F_h, F_h \rangle,$$
$$\partial_t b^{2,0} = -\rho_B(\omega)^{2,0} + \frac{i}{2}\langle a \wedge \partial_{A_h}(\Lambda_\omega F_h)\rangle,$$
$$h^{-1}\partial_t h = -\Lambda_\omega F_h.$$

Assume that $\mathcal{Q}_{\bar{\ell}_0}$ admits a flat Hermitian metric G_F. Then, G_t exists for all time and converges to a flat coupled instanton $G_\infty = G(\omega_\infty, b_\infty^{2,0}, h_\infty)$

$$F_{G_\infty} = 0.$$

Problem

Given a Clemens-Friedman complex manifold $X \cong \sharp_{k \geqslant 2}(S^3 \times S^3)$ and a holomorphic vector bundle $V \to X$, is there a solution of the Bianchi identity

$$dd^c\omega = -trF_h \wedge F_h$$

with $\omega(, J) > 0$?

¡Muchas Gracias!