Higgs & Coulomb Branches from Superconformal Raviolo Vertex Algebras

Niklas Garner

Maths Institue, Univ. of Oxford

Univ. of Hertfordshire

Mathematical Physics Seminar

Dec 4th, 2024

arXiv: 2308.04414

with B. Williams

arXiv: 2310.08516

arXiv: 2310.08524

with B. Williams & S. Raghavendran

Overview

Symmetry is central to our understanding of QFT & String theory, allowing us to make sharp and robust constraints by way of algebra.

The prototypical example of 2d CFT exemplifies this point: the Virasoro and current algebra symmetries thereof are indispensible tools for constraining the spectrum of operators and the allowed form of their correlations.

Supersymmetries are especially potent and provide further structure to exploit. For example, SUSY theories often have rich moduli spaces that beget equally rich maths.

Today: use (∞-dim'l) symmetries to extract/understand moduli spaces of 3d $\mathcal{N}=4$ SUSY QFT's.

2d Holomorphic QFT

Holomorphicity and locality give us many symmetries! For every local operator $\mathcal{O}$, we can consider the topological defect

$$\oint_\gamma dz\, z^n\, \mathcal{O}(z)$$

We can use this to define an action of local operators on themselves via

$$\mathcal{O}_n : \mathcal{O}ps \longrightarrow \mathcal{O}ps$$
$$\tilde{\mathcal{O}}(0) \longmapsto \oint_{S^1} dz\, z^n\, \mathcal{O}(z)\, \tilde{\mathcal{O}}(0)$$

This action is equivalently encoded in the OPE

$$\mathcal{O}(z)\, \tilde{\mathcal{O}}(0) = \sum_{n\in\mathbb{Z}} z^{-n-1} \left(\mathcal{O}_n \cdot \tilde{\mathcal{O}}\right)(0)$$

Note: The modes $\mathcal{O}_n$ are labeled by holomorphic functions on $\mathbb{C}^\times$. Better: holomorphic functions on $\mathrm{Conf}_2^\circ(\mathbb{C})$. Yet better: Dolbeault cohomology of $\mathrm{Conf}_2^\circ(\mathbb{C})$.

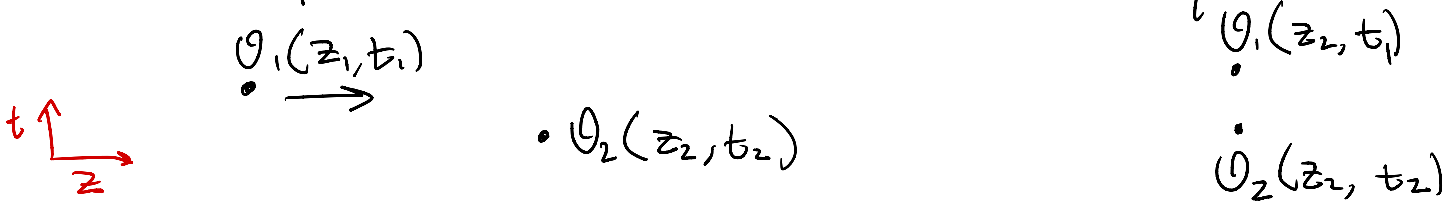
3d Holomorphic-Topological QFT

Now suppose we have a 3d QFT on $\mathbb{C}_{z,\bar{z}} \times \mathbb{R}_t$ that is holomorphic on $\mathbb{C}$ and topological along $\mathbb{R}$ (HT).
Concretely, this means correlation functions of local operators only depend on the holomorphic coordinates z_i of their insertions. Globally, such a QFT lives on 3-folds equipped with a transverse holomorphic foliation.

$$\rightsquigarrow \quad V \hookrightarrow T_M \otimes \mathbb{C} \quad \text{involutive complex subbundle}$$

$$\mathcal{F}_{HT} = \{ f \mid \mathcal{L}_X f = 0 \ \forall X \in V \}$$

Locality implies OPEs are necessarily non-singular:



topological on $\mathbb{R}$ $\Rightarrow$ indep. of $t_1 - t_2$ + locality $\Rightarrow$ only singular at coincident points = nonsingular OPE

This is a 3d HT analog of Hartog's Lemma: all HT functions on $\mathbb{C} \times \mathbb{R} \setminus \{0\}$ extend to $\mathbb{C} \times \mathbb{R}$.

The Raviolo

Unlike the 2d holomorphic setting, $\mathcal{F}_{HT}$ has higher cohomology!

$$\omega = \frac{t\,d\bar{z} - 2\bar{z}\,dt}{(|z|^2 + t^2)^{3/2}}$$

$$H^0(\mathbb{C}\times\mathbb{R}\setminus\{0\}, \mathcal{F}_{HT}) = \mathbb{C}[z] \qquad H^1(\mathbb{C}\times\mathbb{R}\setminus\{0\}, \mathcal{F}_{HT}) = \mathbb{C}[\partial_z]\,\omega$$

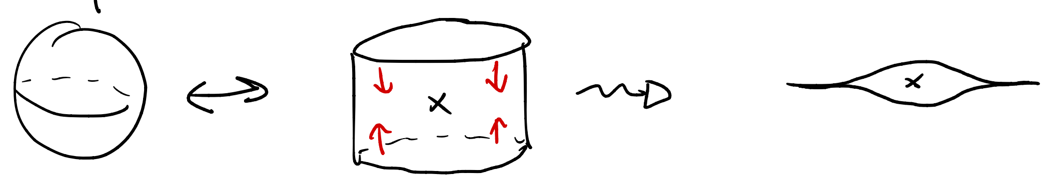
relation: $z\,\omega \equiv 0$, define $\Omega^n = \frac{(-1)^n}{n!}\partial_z^n\,\omega \rightsquigarrow z^n\Omega^m = \begin{cases} 0 & n > m \\ \Omega^{m-n} & \text{else} \end{cases}$

3d Cauchy formula: $\oint_{S^2} dz\,\Omega^n = 8\pi i\,\delta_{n,0}$

Alternatively, can consider tangential CR cohomology of $S^2 \hookrightarrow \mathbb{C}\times\mathbb{R}\setminus\{0\}$.

Equivalent algebraic description: the raviolo

$\mathrm{Rav} = \mathbb{D}_{z_1} \cup_{\mathbb{D}^\times} \mathbb{D}_{z_2}$



$z_1 = z_2$

$$H^\bullet(\mathbb{C}\times\mathbb{R}\setminus\{0\}, \mathcal{F}_{HT}) \simeq H^\bullet(\mathrm{Rav}, \mathcal{O})$$

Raviolo Vertex Algebras

see 2308.04414 for more details

Basic idea: build a vertex-algebraic object from $\mathbb{R}\mathrm{av}$.

Def: A raviolo field on V is an $\mathrm{End}(V)$-valued series

$$A(z) = \sum_{n \geq 0} A_{-n-1} z^n + A_n \Omega^n$$

such that for all $v \in V$ $A_n v = 0$, $n \gg 0$. $A(z)$, $B(w)$ are mutually local if there are fields $C^\ell(w)$ with

$$[A(z), B(w)] = \sum_{\ell \geq 0} \frac{1}{\ell!} \partial_w^\ell \Delta(z,w) C^\ell(w)$$

$\Delta(z,w) = \sum_{n \geq 0} w^n \Omega_z^n - z^n \Omega_w^n$ is the analogue of the δ-function.

Def: A raviolo vertex algebra is a tuple $(V, |0\rangle, \partial, Y)$ satisfying

1) $Y(a,z)$ is a raviolo field on V for all $a \in V$

2) $[\partial, Y(a,z)] = \partial_z Y(a,z)$

3) $\partial|0\rangle = 0$; $Y(|0\rangle, z) = \mathrm{id}_V$; $Y(a,z)|0\rangle \in V[[z]]$, $Y(a,z)|0\rangle\big|_{z=0} = a$

4) $Y(a,z)$, $Y(b,w)$ mutually local for all $a, b \in V$

Elementary Properties

Many of your favorite properties of vertex algebras extend to raviolo vertex algebras, with some surprises.

Def: Given raviolo fields $A(z)$, $B(w)$, their <u>normal-ordered product</u> is

$$:A(z)\,B(w): = A(z)_+ B(w) \pm B(w)\,A(z)_-$$

Prop: Let A, B, C be mutually local raviolo fields. Then

1) A and $\partial_z B$ are mutually local, as are A and $:BC:$
2) $:AB: = \pm :BA:$ and $:(:AB:)C: = :A(:BC:):$

Most importantly, there is a notion of OPE:

Thm: Let V be a RVA. For all $a, b \in V$

$$a(z)\,b(w) = \underbrace{\sum_{n\geq 0} (z-w)^n \,(a_{(-n-1)}b)(w)}_{:a(z)\,b(w):} + \Omega^n_{z-w}\,(a_{(n)}b)(w)$$

where $a(z) = \sum_{n\geq 0} z^n\, a_{(-n-1)} + \Omega^n_z\, a_{(n)}$.

Symmetries of Twisted $\mathcal{N} \geq 2$ Theories

see 2310.08516 for more details

Many examples of 3d HT theories arise from twisting SUSY theories. Symmetries of untwisted theory become enhanced upon twisting!

e.g. $\mathfrak{g}$ flavor symmetry $\rightsquigarrow$ raviolo current algebra

$$V_a(z)\, V_b(w) = \Omega^0_{z-w}\, f_{ab}^{\ \ c}\, V_c(w) + :V_a(z)\, V_b(w):$$

The same is true for supersymmetries! The 3d superconformal symmetry algebra is $\mathfrak{osp}(\mathcal{N}|4)$. Naively, expect the twisted theory to have

$$\mathfrak{osp}(\mathcal{N}|4)^{Q_{HT}} := H^\bullet(\mathfrak{osp}(\mathcal{N}|4), Q_{HT}) \simeq \mathfrak{osp}(\mathcal{N}-2|2)$$

Instead: raviolo analogs of superconformal algebras

e.g. 3d $\mathcal{N}=2$ $\rightsquigarrow$ raviolo analog of Virasoro

$$\Gamma(z)\,\Gamma(w) = \Omega^1_{z-w}\big(2\Gamma(w)\big) + \Omega^0_{z-w}\big(\partial\Gamma(w)\big) + :\Gamma(z)\,\Gamma(w):$$

Lightning Review of/Intro to 3d $\mathcal{N}=4$

SUSY algebra: $\{Q^{a\dot{a}}_{\alpha}, Q^{b\dot{b}}_{\beta}\} = \varepsilon^{ab}\varepsilon^{\dot{a}\dot{b}}(\sigma^{\mu})_{\alpha\beta}P_{\mu}$

$$\left.\begin{array}{l} a,b \sim \mathfrak{su}(2)_H \\ \dot{a},\dot{b} \sim \mathfrak{su}(2)_C \end{array}\right\} \mathfrak{so}(4)_R$$

$\alpha,\beta \sim \mathfrak{su}(2)_E$

$\mathbb{Z}_2$ mirror automorphism: $a, b, \ldots \leftrightarrow \dot{a}, \dot{b}, \ldots$

Often have complicated vacuum moduli. Two distinguished "branches":

$$\left.\begin{array}{llll} \mathcal{M}_H & \text{vac. inv't under} & \mathfrak{su}(2)_C \\ \mathcal{M}_C & \text{—''—} & \mathfrak{su}(2)_H \end{array}\right\} \text{hyperkähler!}$$

Twists:

$$Q_{HT} = Q^{++}_{+}$$

$Q_B = \delta^{\alpha}_{\dot{a}}\, Q^{+\dot{a}}_{\alpha}$

leads to Rozansky-Witten thy; $\mathrm{Ops}^B \simeq \mathbb{C}[\mathcal{M}_H]$

$Q_A = \delta^{\alpha}_{a}\, Q^{a\dot{+}}_{\alpha}$

reduction of 4d Donaldson twist; $\mathrm{Ops}^A \simeq \mathbb{C}[\mathcal{M}_C]$

3d Mirror Symmetry: Intriligator-Seiberg '96, de Boer-Hori-Oguri-Oz '96, ...

$$T \leftrightarrow T^{\vee} \quad \rightsquigarrow \quad T_A \simeq T^{\vee}_B \qquad T_B \simeq T^{\vee}_A$$

w/ SUSY action mirrored

$$\mathcal{M}_C \simeq \mathcal{M}^{\vee}_H \qquad \mathcal{M}_H \simeq \mathcal{M}^{\vee}_C$$

Superconformal RVAs

see 2310.08524 for more details

Very interesting case: $\mathcal{N}=4$ theories $\rightsquigarrow$ N=2 superconformal RVAs

Def: A RVA V is <u>N=2 superconformal</u> if it has Γ (Virasoro field), σ (abelian current), and $Q_\pm$ with OPEs

$$\Gamma(z)Q_\pm(w) \sim \Omega^1_{z-w}\left(\tfrac{3}{2}Q_\pm(w)\right) + \Omega^0_{z-w}\,\partial Q_\pm(w)$$

$$\sigma(z)Q_\pm(w) \sim \Omega^0_{z-w}\left(\pm Q_\pm(w)\right)$$

$$Q_\pm(z)\,Q_\pm(w) \sim 0 \qquad Q_+(z)\,Q_-(w) \sim \Omega^1_{z-w}(-\sigma(w)) + \Omega^0_{z-w}\left(-\Gamma(w) - \tfrac{1}{2}\partial\sigma(w)\right)$$

It <u>satisfies the BPS bound</u> if

$$V_{j,q} = 0 \quad \text{if} \quad j < \frac{|q|}{2}.$$

Def: $\mathcal{O} \in V$ is a <u>Higgs branch primary operator</u> if

$$\Gamma(z)\mathcal{O}(w) \sim \Omega^1_{z-w}(j\,\mathcal{O}(w)) + \Omega^0_{z-w}(\partial\mathcal{O}(w)) \qquad \sigma(z)\mathcal{O}(w) \sim \Omega^0_{z-w}(q\,\mathcal{O}(w))$$

$$Q_+(z)\mathcal{O}(w) \sim 0 \qquad\qquad Q_-(z)\mathcal{O}(w) \sim \Omega^0_{z-w}\,\Psi_{\mathcal{O}}(w)$$

for some $\Psi_{\mathcal{O}} \in V$. Coulomb branch primary operators defined by exchanging $Q_+ \leftrightarrow Q_-$.

Higgs & Coulomb Branches

We can exploit the analogy with 2d SCFT to extract moduli spaces from the HT twist, cf. Lerche-Vafa-Warner

Lemma:
HBP (CBP) operators satisfy $j = \frac{q}{2}$ $\left(j = \frac{-q}{2}\right)$.

Corollary:
The OPE of HBP (CBP) operators is regular. Moreover, the normal-ordered product gives them the structure of a graded-commutative algebra.

Thm: cf. Witten-Zweibach, Lian-Zuckerman, ...
The bilinear operation
$$\{\mathcal{O}_1, \mathcal{O}_2\} = \tfrac{1}{2}\,\mathrm{Res}\left(\Psi_{\mathcal{O}_1}(z)\,\mathcal{O}_2(0) \pm \Psi_{\mathcal{O}_2}(z)\,\mathcal{O}_1(0)\right)$$
defines a Poisson bracket of HBP (CBP) operators.

Thank you!