Democratic formulations and manifest duality

Karapet Mkrtchyan

Imperial College London

Seminar prepared for the Theoretical Physics Group at the University of Hertfordshire
January 8, 2025

References

Based on:

K.M. *JHEP* 1912 (2019) 076 [arXiv:1908.01789].

Zhirayr Avetisyan, Oleg Evnin and K.M.
Phys. Rev. Lett. 127 (2021) 271601 [arXiv:2108.01103],
JHEP 08 (2022) 112 [arXiv:2205.02522].

K.M. and Fridrich Valach
Phys. Rev. D 107 (2023) 6, 066027 [arXiv:2207.00626].

Oleg Evnin, Euihun Joung and K.M.
Phys. Rev. D 109 (2024) 6, 066003 [arXiv:2309.04625].

References

See also:

Sukruti Bansal, Oleg Evnin and K.M.
Eur. Phys. J. C 81 (2021) 3, 257 [arXiv:2101.02350].

Oleg Evnin and K.M.
Differ. Geom. Appl. 89 (2023), 102016 [arXiv:2207.01767].

Motivation

Some problems to solve during our lifetime

- $S-$duality in gauge theory: Montonen-Olive duality and its various reincarnations/extensions. Can we make it manifest?
- Field-theoretical (classical) description of magnetic charges in the same footing as electric ones (local, Lorentz covariant?).
- Quantization of gauge theory. Manifest S-duality is the key?
- Non-abelian interactions of (chiral) $p-$forms. In particular, the $6d$ two-forms (related to $M5$ branes and $(2,0)$ theory).
- Electric-magnetic duality in gravity. Key to quantization?

General Problem: Theories with extended symmetries

Extended space-time symmetries

Extended symmetries (ES) enlarge space-time isometry algebra by at least one more generator in a nontrivial representation of isometry.

General Problem: Theories with extended symmetries

Extended space-time symmetries

Extended symmetries (ES) enlarge space-time isometry algebra by at least one more generator in a nontrivial representation of isometry.

The problem

Five properties that are hard to combine in a classical field theory:

1. Extended (bosonic) space-time symmetries (as defined above),
2. Local action principle,
3. Unitarity,
4. Non-trivial bulk propagation,
5. Non-trivial interactions.

General Problem: Theories with extended symmetries

Extended space-time symmetries

Extended symmetries (ES) enlarge space-time isometry algebra by at least one more generator in a nontrivial representation of isometry.

The problem

Five properties that are hard to combine in a classical field theory:

1. Extended (bosonic) space-time symmetries (as defined above),
2. Local action principle,
3. Unitarity,
4. Non-trivial bulk propagation,
5. Non-trivial interactions.

Current status

Examples are available with **any four** of these properties.
No satisfactory example is available with all five so far.

Examples: Theories with extended symmetries

Five properties that are hard to combine in a classical field theory:

Removing one allows to have explicit examples:

1. Extended space-time symmetries,
    - $\times$ Einstein Gravity, SUGRA.

Examples: Theories with extended symmetries

Five properties that are hard to combine in a classical field theory:

Removing one allows to have explicit examples:

1. Extended space-time symmetries,
   - $\times$ Einstein Gravity, SUGRA.
2. Local action principle,
   - $\times$ (Vasiliev's) Higher-Spin Gravity.

Examples: Theories with extended symmetries

Five properties that are hard to combine in a classical field theory:

Removing one allows to have explicit examples:

1. Extended space-time symmetries,
   - $\times$ Einstein Gravity, SUGRA.
2. Local action principle,
   - $\times$ (Vasiliev's) Higher-Spin Gravity.
3. Unitarity,
   - $\times$ Conformal (Higher-Spin) Gravity.

Examples: Theories with extended symmetries

Five properties that are hard to combine in a classical field theory:

Removing one allows to have explicit examples:

1. Extended space-time symmetries,
   - × Einstein Gravity, SUGRA.
2. Local action principle,
   - × (Vasiliev's) Higher-Spin Gravity.
3. Unitarity,
   - × Conformal (Higher-Spin) Gravity.
4. Non-trivial bulk propagation,
   - × Higher-Spin/Colored (Chern-Simons) Gravities in 3d.

Examples: Theories with extended symmetries

Five properties that are hard to combine in a classical field theory:

Removing one allows to have explicit examples:

1. Extended space-time symmetries,
   - × Einstein Gravity, SUGRA.
2. Local action principle,
   - × (Vasiliev's) Higher-Spin Gravity.
3. Unitarity,
   - × Conformal (Higher-Spin) Gravity.
4. Non-trivial bulk propagation,
   - × Higher-Spin/Colored (Chern-Simons) Gravities in 3d.
5. Non-trivial interactions.
   - × Free Higher-Spin fields in $d \geq 4$.

Background: Lagrangian for equations

"Technical" problem

Lagrangian for Higher-Spin (Vasiliev) equations.

Background: Lagrangian for equations

"Technical" problem

Lagrangian for Higher-Spin (Vasiliev) equations.

Similar problem

Lagrangian for (twisted) self-duality equations.

Background: Lagrangian for equations

"Technical" problem

Lagrangian for Higher-Spin (Vasiliev) equations.

Similar problem

Lagrangian for (twisted) self-duality equations.

In other words, is it possible to make $S-$duality manifest?

S-duality

The simplest example of $S-$duality is the symmetry that rotates electric and magnetic fields in the free Maxwell equations:

$$\overrightarrow{\nabla} \times \overrightarrow{E} = -\frac{\partial \overrightarrow{B}}{\partial t}\,, \quad \overrightarrow{\nabla} \cdot \overrightarrow{E} = 0\,,$$

$$\overrightarrow{\nabla} \times \overrightarrow{B} = \frac{\partial \overrightarrow{E}}{\partial t}\,, \quad \overrightarrow{\nabla} \cdot \overrightarrow{B} = 0\,,$$

which are invariant with respect to the $SO(2)$ duality rotations:

$$\overrightarrow{E} \to \cos\alpha\,\overrightarrow{E} + \sin\alpha\,\overrightarrow{B}\,,$$

$$\overrightarrow{B} \to -\sin\alpha\,\overrightarrow{E} + \cos\alpha\,\overrightarrow{B}\,.$$

Duality symmetry of Maxwell equations

When the electromagnetic field is coupled to charged matter,

$$\overrightarrow{\nabla} \times \overrightarrow{B} = \frac{\partial \overrightarrow{E}}{\partial t} + \overrightarrow{j_e}\,, \quad \overrightarrow{\nabla} \cdot \overrightarrow{E} = 4\pi \rho_e\,,$$

the duality symmetry is broken

Duality symmetry of Maxwell equations

When the electromagnetic field is coupled to charged matter,

$$\overrightarrow{\nabla} \times \overrightarrow{B} = \frac{\partial \overrightarrow{E}}{\partial t} + \overrightarrow{j_e}\,, \quad \overrightarrow{\nabla} \cdot \overrightarrow{E} = 4\pi \rho_e\,,$$

the duality symmetry is broken, unless one introduces magnetic charges – monopoles. These form a magnetic current $\overrightarrow{j_m}$:

$$\overrightarrow{\nabla} \times \overrightarrow{E} = -\frac{\partial \overrightarrow{B}}{\partial t} - \overrightarrow{j_m}\,, \quad \overrightarrow{\nabla} \cdot \overrightarrow{B} = 4\pi \rho_m\,.$$

Duality symmetry of Maxwell equations

When the electromagnetic field is coupled to charged matter,

$$\vec{\nabla} \times \vec{B} = \frac{\partial \vec{E}}{\partial t} + \vec{j_e} \,, \quad \vec{\nabla} \cdot \vec{E} = 4\pi \rho_e \,,$$

the duality symmetry is broken, unless one introduces magnetic charges – monopoles. These form a magnetic current $\vec{j_m}$:

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} - \vec{j_m} \,, \quad \vec{\nabla} \cdot \vec{B} = 4\pi \rho_m \,.$$

The Maxwell equations remain duality invariant if the duality rotates also the four-vector currents $j_e^\mu = (\rho_e, \vec{j_e})$, $j_m^\mu = (\rho_m, \vec{j_m})$:

$$j_e^\mu \to \cos\alpha \, j_e^\mu + \sin\alpha \, j_m^\mu \,,$$

$$j_m^\mu \to -\sin\alpha \, j_e^\mu + \cos\alpha \, j_m^\mu \,.$$

Lorentz Covariant Approach

Lorentz covariant equations for electrodynamics

$$dF = \star j_m\,, \qquad d\star F = \star j_e\,.$$

When $j_m = 0$, the first equation is solved via Poincaré Lemma as $F = dA$, and a Lagrangian is formulated via gauge field A.

Lorentz Covariant Approach

Lorentz covariant equations for electrodynamics

$$dF = \star j_m\,, \qquad d\star F = \star j_e\,.$$

When $j_m = 0$, the first equation is solved via Poincaré Lemma as $F = dA$, and a Lagrangian is formulated via gauge field A.

Lagrangian

Lagrangian is not duality symmetric:

$$S = -\frac{1}{4}\int d^4x F_{\mu\nu}F^{\mu\nu} = \frac{1}{2}\int d^4x(\overrightarrow{E}^2 - \overrightarrow{B}^2)\,.$$

It changes the sign under discrete duality transformations.

Conventional non-linear electrodynamics

Lagrangian for general non-linear electrodynamics (NED)

$$\mathcal{L} = \mathcal{L}(s, p)\,, \quad s = \frac{1}{2}\, F_{\mu\nu}\, F^{\mu\nu}\,, \quad p = \frac{1}{2} F_{\mu\nu} \star F^{\mu\nu}\,,$$

Equations and duality transformations

$$d\,F = 0\,, \qquad d\,G = 0\,, \qquad G = \star \frac{\partial \mathcal{L}}{\partial F}\,,$$

Since now G is non-linearly related to F, the duality rotations:

$$F \to \cos\alpha\, F + \sin\alpha\, G\,,$$

$$G \to -\sin\alpha\, F + \cos\alpha\, G\,,$$

are not automatically a symmetry of the theory.

Duality symmetry in NED

Duality-symmetry in conventional NED

$SO(2)$ duality symmetry implies (Gaillard, Zumino '80, Bialynicki-Birula '83, Gibbons, Rasheed '95):

$$F \wedge F = G \wedge G$$

that is satisfied for Lagrangians $\mathcal{L}(s, p)$ solving the equation:

$$\mathcal{L}_s^2 - \frac{2\,s}{p}\mathcal{L}_s\,\mathcal{L}_p - \mathcal{L}_p^2 = 1\,,$$

where $\mathcal{L}_s = \frac{\partial \mathcal{L}}{\partial s}$, $\mathcal{L}_p = \frac{\partial \mathcal{L}}{\partial p}$. There are a few explicit solutions known: Maxwell, Born-Infeld, a few more solutions by M. Hatsuda, K. Kamimura and S. Sekiya '99. New solutions were found recently: M. Svazas '21 (master thesis), K.M. and M. Svazas '22.

Manifest duality-symmetry?

Different approaches

- Zwanziger '70 (manifest duality-symmetry, non-manifest Lorentz)
- Floreanini-Jackiw '87, Henneaux-Teitelboim '88, Schwarz-Sen '93 (manifest duality-symmetry, non-manifest Lorentz)
- Pasti-Sorokin-Tonin '95 (manifest duality-symmetry and Lorentz, reproduces non-covariant approaches in some gauge)

Manifest duality symmetry requires democracy.

General democracy for free $p-$forms

Equations

Free $p-$form dynamics is defined by the following equations:

$$\star F = G\,, \quad dF = 0\,(F = dA)\,, \quad dG = 0\,(G = dB)\,,$$

where F is a $(p+1)-$form and G is the dual $(d-p-1)-$form.

Standard approach: treat the last equation as the dynamical equation $d \star dA = 0$ derived from the Lagrangian:

$$\mathcal{L} = F \wedge \star F\,, \qquad F = dA\,.$$

This non-democratic approach cannot treat self-dual fields $(G = F)$.

Democracy treats F and G (or A and B) on an equal footing.

Free democratic equations

Twisted self-duality equations

The Maxwell equations are equivalent to first-order equations involving both dual potentials:

$$\star F = \pm G\,, \quad F = dA\,, \quad G = dB$$

Free democratic equations

Twisted self-duality equations

The Maxwell equations are equivalent to first-order equations involving both dual potentials:

$$\star F = \pm G\,, \quad F = dA\,, \quad G = dB$$

Duality-symmetric formulations

Zwanziger '70,..., Gaillard-Zumino '80, Bialynicki-Birula '83,..., Schwarz-Sen '93, Gibbons-Rasheed '95, Pasti-Sorokin-Tonin '96, Rocek-Tseytlin '99, Kuzenko-Theisen '00, Ivanov-Zupnik '02, ...

Self-dual (Chiral) fields

There are special representations of the Poincaré algebra which are described by self-dual forms. The covariant equations describing such representations are given as:

$$\star F = \pm F\,, \quad F = dA$$

which implies the regular "Maxwell equations" $d \star F = 0$.

Lagrangian?

Lagrangian formulation of the (free) chiral fields has a long history. Siegel '84, Kavalov-Mkrtchyan '87, Florianini-Jackiw '87, Henneaux-Teitelboim '88, Harada '90, Tseytlin '90, McClain-Yu-Wu '90, Wotzasek '91, ..., Pasti-Sorokin-Tonin '95,..., Sen '15,...

Chiral $p-$forms in $d = 4k + 2$ Minkowski space

Minkowski vs Euclidean

Since $\star^2 = (-1)^{\sigma+p+1}$ where σ is the number of time directions, only even-forms can be self-dual (chiral) in Minkowski space.

$p = 2k$ forms in $d = 4k + 2$ dimensions

For even $p-$form potentials in special dimensions the corresponding particles are not irreducible but contain two irreps — chiral and anti-chiral halves.

Pure Electrodynamics

Duality vs Lorentz symmetry?

Conventional Lagrangians manifest only one of the two.

"Democratic" equations for $3+1$ dimensional electrodynamics:

$$\star F^b = \epsilon^{bc} F^c\,, \quad F^b = dA^b\,, \quad b,c = 1,2\,.$$

Pure Electrodynamics

Duality vs Lorentz symmetry?

Conventional Lagrangians manifest only one of the two.

"Democratic" equations for $3+1$ dimensional electrodynamics:

$$\star F^b = \epsilon^{bc} F^c \,, \quad F^b = dA^b \,, \quad b, c = 1, 2 \,.$$

Lagrangian for these equations = manifest $S-$duality.

Pure Electrodynamics

Duality vs Lorentz symmetry?

Conventional Lagrangians manifest only one of the two.

"Democratic" equations for $3+1$ dimensional electrodynamics:

$$\star F^b = \epsilon^{bc} F^c\,, \quad F^b = dA^b\,, \quad b, c = 1, 2\,.$$

Lagrangian for these equations = manifest $S-$duality.

Related problem

Lagrangian for self-duality equation?

$$\star F = F\,, \quad F = dA\,,$$

self-dual $p-$forms in $d = 2p + 2$ dimensions.

Pure Electrodynamics

Duality vs Lorentz symmetry?

Conventional Lagrangians manifest only one of the two.

"Democratic" equations for $3+1$ dimensional electrodynamics:

$$\star F^b = \epsilon^{bc} F^c \,, \quad F^b = dA^b \,, \quad b,c = 1,2 \,.$$

Lagrangian for these equations = manifest $S-$duality.

Related problem

Lagrangian for self-duality equation?

$$\star F = F \,, \quad F = dA \,,$$

self-dual $p-$forms in $d = 2p+2$ dimensions.

"...As this field is non-Lagrangian..." Witten '96

Main problem

Lagrangian for (twisted) self-duality equation.

A new (democratic) approach

Spoiler

The problematic case turns into the simplest in a new formulation.

A new (democratic) approach

Spoiler

The problematic case turns into the simplest in a new formulation.

New action for self-dual $p-$forms $(d = 2p + 2)$

$$\mathcal{L} = \frac{1}{2}(F + a\,Q)^2 \pm a\,F \wedge Q\,,$$

where $F = dA$ and $Q = dR$.

Equations

On-shell a and R are pure-gauge and the e.o.m.'s can be gauged to:

$$\star F \pm F = 0\,.$$

K.M. *JHEP* '19

Democratic Electromagnetism

The Lagrangian for a single massless spin-one field in $d = 3 + 1$

$$\mathcal{L}_{Maxwell} = -\frac{1}{8} H^b_{\mu\nu} H^{b\mu\nu} + \frac{1}{8}\, \epsilon^{bc}\, \varepsilon^{\mu\nu\lambda\rho}\, a\, F^b_{\mu\nu}\, Q^c_{\lambda\rho}$$

where $H^b_{\mu\nu} \equiv F^b_{\mu\nu} + a\, Q^b_{\mu\nu}$, $b = 1, 2$, and

$$F^b_{\mu\nu} = \partial_\mu\, A^b_\nu - \partial_\nu\, A^b_\mu\,, \quad Q^b_{\mu\nu} = \partial_\mu\, R^b_\nu - \partial_\nu\, R^b_\mu\,.$$

Any solution of the e.o.m. is gauge equivalent to solutions of

$$Q^b_{\mu\nu} = 0, \qquad \star F^a_{\mu\nu} + \epsilon^{ab}\, F^b_{\mu\nu} = 0\,,$$

with a single propagating Maxwell field.

K.M. *JHEP* '19

The general democratic non-linear electrodynamics

Democratic non-linear electrodynamics

$$\mathcal{L} = \mathcal{L}_{Maxwell} + g(\lambda_1, \lambda_2)\,,$$

where

$$\lambda_1 = \tfrac{1}{2}\,G_{\mu\nu} \star G^{\mu\nu}\,, \quad \lambda_2 = -\tfrac{1}{2}\,G_{\mu\nu}\,G^{\mu\nu}\,, \quad G_{\mu\nu} \equiv \star H^1_{\mu\nu} - H^2_{\mu\nu}$$

The general democratic non-linear electrodynamics

Democratic non-linear electrodynamics

$$\mathcal{L} = \mathcal{L}_{Maxwell} + g(\lambda_1, \lambda_2)\,,$$

where

$$\lambda_1 = \frac{1}{2}\, G_{\mu\nu} \star G^{\mu\nu}\,, \quad \lambda_2 = -\frac{1}{2}\, G_{\mu\nu}\, G^{\mu\nu}\,, \quad G_{\mu\nu} \equiv \star H^1_{\mu\nu} - H^2_{\mu\nu}$$

Nonlinear electrodynamics with duality symmetry

Theories with $SO(2)$ duality symmetry will have:

$$\mathcal{L} = \mathcal{L}_{Maxwell} + h(w)\,, \qquad w = \sqrt{\lambda_1^2 + \lambda_2^2}$$

Z. Avetisyan, O. Evnin, **K.M.**, *PRL* '21.

Duality symmetry

Discreet duality symmetry

Under the discrete duality,

$$\lambda_1 \to -\lambda_1\,, \qquad \lambda_2 \to -\lambda_2$$

Theories with such symmetry will satisfy:

$$g(-\lambda_1, -\lambda_2) = g(\lambda_1, \lambda_2)$$

Duality symmetry

Continuous duality symmetry

Under continuous duality symmetry,

$$\lambda_1 \to \cos(2\alpha)\,\lambda_1 + \sin(2\alpha)\,\lambda_2\,,$$
$$\lambda_2 \to -\sin(2\alpha)\,\lambda_1 + \cos(2\alpha)\,\lambda_2$$

Theories with such symmetry will have:

$$g(\lambda_1, \lambda_2) = h(w)\,, \qquad w = \sqrt{\lambda_1^2 + \lambda_2^2}$$

The corresponding Lagrangian is given as:

$$\mathcal{L} = \mathcal{L}_{Maxwell} + h(w)\,,$$

where w can be also given as:

$$w = \sqrt{-\det \mathcal{H}}\,, \qquad \mathcal{H}^{ab} \equiv (\star H^a_{\mu\nu} - \epsilon^{ac} H^c_{\mu\nu})(\star H^{b\mu\nu} - \epsilon^{bd} H^{d\mu\nu})/2$$

Conformal symmetry

Requirement of conformal symmetry

Conformal invariance translates into:

$$\lambda_1 \frac{\partial g(\lambda_1, \lambda_2)}{\partial \lambda_1} + \lambda_2 \frac{\partial g(\lambda_1, \lambda_2)}{\partial \lambda_2} = g(\lambda_1, \lambda_2)$$

which can be solved, e.g. as:

$$g = \lambda_1 \, \tilde{g}(\lambda_1 / \lambda_2)$$

Conformal and duality-symmetric theory

Conformal symmetry for duality-symmetric theories

This case gives:

$$w\,\frac{\partial h(w)}{\partial w} = h(w)\,,$$

which is solved by a linear function:

$$h(w) = \delta\, w$$

General conformal and duality-symmetric electrodynamics is given by the one-parameter Lagrangian:

$$\mathcal{L} = -\frac{1}{2}\, H^b \wedge \star H^b + a\,\epsilon_{bc} F^b \wedge Q^c + \delta\, w$$

Equations of motion

Equations

E.o.m. imply in $R^a = 0$ gauge:

$$\star F^1 + F^2 = g_2 \left(\star F^1 - F^2\right) - g_1 \star \left(\star F^1 - F^2\right),$$

where $g_1 \equiv \partial g/\partial\lambda_1,\ g_2 \equiv \partial g/\partial\lambda_2$.
One can solve from here F^1 in terms of F^2:

$$F^1 = \alpha(s,p) F^2 + \beta(s,p) \star F^2\,,$$

where $s = \frac{1}{2} F^2_{\mu\nu} F^{2\mu\nu}\,,\ p = \frac{1}{2} F^2_{\mu\nu} \star F^{2\mu\nu}$. One can now make contact with the single-field formalism with Lagrangian $\mathcal{L}(s,p)$ via

$$\alpha(s,p) = -\frac{\partial \mathcal{L}}{\partial p}\,, \qquad \beta(s,p) = \frac{\partial \mathcal{L}}{\partial s}$$

Map between different formulations

The relation between single and double potential formulations

The relation between derivatives of Lagrangians in both formulations:

$$g_1 = \frac{2\,\alpha}{\alpha^2 + (\beta+1)^2}\,, \qquad g_2 = \frac{\alpha^2 + \beta^2 - 1}{\alpha^2 + (\beta+1)^2}\,,$$

where g is a function of λ_1, λ_2, which can also be expressed in terms of α, β, s, p:

$$\lambda_1 = 2\,\alpha\,(1+\beta)\,s - [\alpha^2 - (1+\beta)^2]\,p\,,$$
$$\lambda_2 = [\alpha^2 - (1+\beta)^2]\,s + 2\,\alpha\,(1+\beta)\,p\,,$$

while w is given as:

$$w \equiv \sqrt{\lambda_1^2 + \lambda_2^2} = (\alpha^2 + (\beta+1)^2)\,\sqrt{s^2+p^2}$$

Map for duality-symmetric theories

The $SO(2)$ invariant case

The relation between the two formulations is given in this case by:

$$\frac{\lambda_1}{w}h' = \frac{2\,\alpha}{\alpha^2 + (\beta+1)^2}\,, \quad \frac{\lambda_2}{w}h' = \frac{\alpha^2+\beta^2-1}{\alpha^2 + (\beta+1)^2}$$

which implies the duality-symmetry condition for the single-potential formulation

$$\beta^2 + \frac{2s}{p}\alpha\beta - \alpha^2 = 1\,,$$

and:

$$(\alpha s + (\beta+1)p)\; h'\Big|_{w=\sqrt{s^2+p^2}(\alpha^2+(\beta+1)^2)} = \alpha\sqrt{s^2+p^2}$$

Examples: ModMax and BB

The conformal duality-symmetric electrodynamics

The conformal and duality-symmetric Electrodynamics:

$$\mathcal{L} = -\frac{1}{2}\, H^b \wedge \star H^b + a\, \epsilon_{bc} F^b \wedge Q^c + \delta\, w$$

can be translated to single-potential formulation

$$L(s,p) = -\cosh\gamma\, s + \sinh\gamma \sqrt{s^2 + p^2}$$

using a parametrization: $\delta = \coth\frac{\gamma}{2}$. This is so-called ModMax theory (Bandos, Lechner, Sorokin, Townsend '20). In the special case of $\delta = 1$, the map breaks down. There, the single-field formulation does not exist.

Example: Generalized Born-Infeld theory

Generalized Born-Infeld theory

The conventional Lagrangian (T, γ are constants):

$$L_{GBI} = \sqrt{UV} - T, \quad U \equiv 2u + e^{\gamma}\, T, \quad V \equiv -2v + e^{-\gamma}\, T\,,$$

where $u \equiv (s + \sqrt{p^2 + s^2})/2$, $v \equiv (-s + \sqrt{p^2 + s^2})/2$.

Democratic formulation

The duality-symmetric Lagrangian is $\mathcal{L} = \mathcal{L}_{Maxwell} + h(w)$, where in this case $h(w)$ is implicitly given by:

$$h(\lambda) = 4\,T\, \sinh^2 \frac{\lambda}{2}\, \cosh(\lambda + \gamma)\,,$$

$$w(\lambda) = -4\,T\, \cosh^2 \frac{\lambda}{2}\, \sinh(\lambda + \gamma)\,.$$

Lorentz covariant equations for interacting self-dual forms

A comment on nonlinear theory of chiral two-forms in 6d

"It appears that not only is there no manifestly Lorentz invariant action, but *even the field equation lacks manifest Lorentz invariance*." Perry, Schwarz '96

Lorentz covariant equations for interacting self-dual forms

A comment on nonlinear theory of chiral two-forms in 6d

"It appears that not only is there no manifestly Lorentz invariant action, but *even the field equation lacks manifest Lorentz invariance*."
Perry, Schwarz '96

Equations of motion (Z. Avetisyan, O. Evnin, **K.M.** '22)

$$\star F + F = f(\star F - F)\,.$$

General equation for non-linear self-dual $p-$form in $d = 2p + 2$ dimensions, with arbitrary $f : \Lambda^- \to \Lambda^+$. In particular,

$$f(Y) = \frac{\partial g(Y)}{\partial Y}\,,$$

with arbitrary scalar $g(Y)$ of $Y \in \Lambda^-$.

Abelian interactions for (chiral) p-forms

Free Lagrangian

$$\mathcal{L} = \tfrac{1}{2}(F + a\,Q)^2 + a\,F \wedge Q\,, \qquad (F = dA\,,\ Q = dR)\,.$$

Self-interacting Lagrangian: general recipe

$$\mathcal{L} = \tfrac{1}{2}(F + a\,Q)^2 + a\,F \wedge Q + g(H^-)\,,$$

$$H^- = \star(F + a\,Q) - (F + aQ)\,.$$

Equations of motion

In the on-shell gauge $Q = 0$, the equations of motion are:

$$\star F + F = f(\star F - F)\,,$$

$$f(Y) = \frac{\partial g(Y)}{\partial Y}\,.$$

Z. Avetisyan, O. Evnin, **K.M.** *JHEP* '22

Comparison to other formulations

Formalism:	PST	Sen's	Our approach
Interactions by arbitrary functions	x	✓	✓
Auxiliary fields gauged away	✓	x	✓
Gauge potential as fundamental field	✓	x	✓

More details in:
Oleg Evnin and **K.M.**, "Three approaches to chiral form interactions", *Differ. Geom. and Appl.* 89 (2023), 102016.

New action for Chiral fields: more details

Lagrangian

$$\mathcal{L} = \frac{1}{2}(F + a\,Q)^2 + \,a\,F \wedge Q\,,$$

where $F = dA$ and $Q = dR$.

Symmetries

$$\delta A = dU\,; \qquad \delta R = dV\,;$$

$$\delta A = -\,a\,da \wedge W\,, \quad \delta R = da \wedge W\,;$$

$$\delta A = -\,\frac{a\,\varphi}{(\partial a)^2}\,\iota_{da}(Q + \star Q)\,,$$

$$\delta a = \varphi\,, \quad \delta R = \frac{\varphi}{(\partial a)^2}\,\iota_{da}(Q + \star Q)\,.$$

Equations and consequences

Equations

$$E_a \equiv \frac{\delta \mathcal{L}}{\delta a} \equiv (F + a\,Q) \wedge \star Q + F \wedge Q = 0\,,$$
$$E_A \equiv \frac{\delta \mathcal{L}}{\delta A} \equiv d\,[\star(F + a\,Q)] + da \wedge Q = 0\,,$$
$$E_R \equiv \frac{\delta \mathcal{L}}{\delta R} \equiv d\,[a \star (F + a\,Q)] - da \wedge F = 0\,.$$

Relations

$$E_R - a\,E_A = da \wedge [F + a\,Q - \star(F + a\,Q)] = 0$$

From here (for $(da)^2 \neq 0$):

$$F + a\,Q - \star(F + a\,Q) = 0$$

and $E_a \equiv [F + a\,Q - \star(F + a\,Q)] \wedge Q = 0$ follows from $E_A = 0 = E_R$.

The spectrum

Consequences of e.o.m.

From the equations of motion it follows that:

$$da \wedge dR = 0$$

which can be solved generally as:

$$R = d\lambda + da \wedge \rho$$

This implies that R is pure gauge. In the $R = 0$ gauge, we get:

$$\star F = F$$

Thus the propagating d.o.f. consist of a single self-dual $p-$form.

Derivation from Chern-Simons

Chern-Simons with a boundary term

The action:

$$S_{\text{free}} = \int_M H \wedge \mathrm{d}H - \frac{1}{2}\int_{\partial M} H \wedge \star H$$

Full variation:

$$\delta S_{\text{free}} = 2\int_M \delta H \wedge \mathrm{d}H - \frac{1}{2}\int_{\partial M} \delta H^+ \wedge H^- \,.$$

$$H^{\pm} = H \pm \star H$$

Arvanitakis, Cole, Hulik, Sevrin, and Thompson, arXiv:2212.11412

Reduction

Main idea

Decompose the field as ($v = da$ satisfies $v^2 \neq 0$ on the boundary):

$$H = \hat{H} + v \wedge \check{H}\,, \qquad dv = 0\,.$$

Then the field $\check{H}$ is a Lagrange multiplier enforcing a constraint on the field $\hat{H}$,

$$v \wedge \mathrm{d}\hat{H} = 0\,,$$

with a solution

$$H = \mathrm{d}A + v \wedge R$$

Plugging this back into the action gives the chiral Lagrangian discussed earlier.

Extensions to interacting chiral fields

General equations

General equations describing self-interactions of a chiral field are given as

$$H^- = f(H^+)\,, \qquad \mathrm{d}H = 0\,,$$

where $f : \Lambda^+ \to \Lambda^-$ is an antiselfdual form valued function of a selfdual variable.

Action

$$S = \int_M H \wedge \mathrm{d}H - \int_{\partial M} \frac{1}{2}\, H \wedge \star H + g(H^+)\,,$$

where $f(Y) = \partial g(Y)/\partial Y$.

Democratic $p-$forms

Generalization to democratic case

Action:

$$S = \int_M (-1)^{d-p}\, G \wedge \mathrm{d}F + \mathrm{d}G \wedge F$$
$$- \int_{\partial M} \frac{1}{2}\,(F \wedge \star F + G \wedge \star G) + g(F + \star G)\,,$$

with bulk equations $\mathrm{d}F = 0 = \mathrm{d}G$ and boundary equations:

$$F - \star G = f(F + \star G)\,,$$

where $f(Y) = \partial g(Y)/\partial Y$ for a $(p+1)-$form argument Y.

More

More details in: Evnin, Joung, K.M., arXiv:2309.04625.

Type II Supergravities: preliminaries

Some notations

Reflection operator: $\star\alpha = (-1)^{\lfloor \frac{\deg\alpha}{2} \rfloor + \deg\alpha} * \alpha\,,$

Mukai pairing: $(\alpha,\beta) := (-1)^{\lfloor \frac{\deg\alpha}{2} \rfloor} (\alpha\wedge\beta)^{top},$

Differential: $D\alpha = d\alpha + H\wedge\alpha\,.$

Properties

$$(\alpha,\star\beta) = (\beta,\star\alpha)\,, \qquad \star^2 = 1\,, \qquad D^2 = 0\,,$$

$$\int_M (\alpha, D\beta) = -\int_M (D\alpha,\beta) \qquad \text{(up to boundary terms)}\,,$$

$$D(f\alpha) = f D\alpha + df\wedge\alpha\,, \qquad \text{(for any function } f\text{)}.$$

K.M., F. Valach '22

Type II Supergravities

Action for Type II SUGRAS

$$S = S_{NS} + S_{RR}$$

where

$$S_{NS} = \frac{1}{2\kappa^2} \int \left[\sqrt{-g}\, e^{-2\varphi} \left(\mathcal{R} + 4(d\varphi)^2 - \frac{1}{12} H^2 \right) \right] ,$$

$$S_{RR} = \pm \frac{1}{8\kappa^2} \int \left[\frac{1}{2} (F + aQ, \star(F + aQ)) + (F, aQ) \right] ,$$

$$F = DA, \quad Q = DR\, .$$

Upper/lower sign corresponds to IIA/IIB.

Field content

$$F = F_2 + F_4 + F_6 + F_8 + F_{10}, \qquad \text{(IIA case)}$$

$$F = F_1 + F_3 + F_5 + F_7 + F_9. \qquad \text{(IIB case)}$$

Democratic type II SUGRA

On-shell reduction

On-shell one can gauge fix $Q = 0$,

$$DF = 0\,, \qquad \star F = F\,.$$

reproducing democratic equations for RR forms.

Manifest $SL(2,R)$-symmetric type IIB SUGRA

Action

$$S = \frac{1}{2\kappa^2}\int\sqrt{-g}\left\{\mathcal{R} - 2[(d\phi)^2 + e^{2\phi}(d\ell)^2] - \frac{1}{3}e^{-\phi}H^2 - \frac{1}{3}e^{\phi}(H' - \ell H)^2\right\} + S_{SD}\,,$$

where

$$S_{SD} = \frac{1}{16\kappa^2}\int\left[(F + aQ)\wedge *(F + aQ) + 2\,F\wedge aQ - 2\,(1 + *)(F + aQ)\wedge X + X\wedge *X\right]\,,$$

and

$$X = \frac{1}{2}(B\wedge H' - B'\wedge H)$$

11d SUGRA

Bulk action

$$S = \int_M G \wedge \mathrm{d}F + \mathrm{d}G \wedge F + \frac{2}{3}\,\lambda_3\, F \wedge F \wedge F$$
$$- \int_{\partial M} \frac{1}{2}\,(F \wedge \star F + G \wedge \star G) - g(\star G + F)\,.$$

11d SUGRA

Bulk action

$$S = \int_M G \wedge \mathrm{d}F + \mathrm{d}G \wedge F + \frac{2}{3}\lambda_3 \, F \wedge F \wedge F$$
$$- \int_{\partial M} \frac{1}{2}\,(F \wedge \star F + G \wedge \star G) - g(\star G + F)\,.$$

Boundary Lagrangian

After reduction we get:

$$\mathcal{L} = v \wedge S \wedge \mathrm{d}A - \mathrm{d}B \wedge v \wedge R - \frac{\lambda_3}{3} A \wedge \mathrm{d}A \wedge \mathrm{d}A$$
$$- \frac{1}{2}\,(F \wedge \star F + G \wedge \star G) - g(\star G + F)\,,$$

where

$$F = \mathrm{d}A + v \wedge R\,, \qquad G = \mathrm{d}B + v \wedge S - \lambda_3 \, A \wedge \mathrm{d}A\,.$$

Future directions: State of the art

Non-abelian twisted self-duality

Samtleben '11, Bandos, Samtleben, Sorokin '13

Future directions: State of the art

Non-abelian twisted self-duality

Samtleben '11, Bandos, Samtleben, Sorokin '13

Linearized gravity and higher spins

Henneaux, Teitelboim '04, Bunster, Henneaux '13,
Henneaux, Hörtner, Leonard '16

Future directions: State of the art

Non-abelian twisted self-duality

Samtleben '11, Bandos, Samtleben, Sorokin '13

Linearized gravity and higher spins

Henneaux, Teitelboim '04, Bunster, Henneaux '13,
Henneaux, Hörtner, Leonard '16

Covariant form for (linearized) gravity and higher spins

Work in progress with Calvin Chen and Euihun Joung

Outlook

A list of related (un)solvable problems

- 1) Democratic formulation for non-abelian gauge theory.
- 2,0) Non-abelian interactions for (chiral) $p-$forms.
- 4,0;...) Democratic GR, ... (higher spins on my mind).

Outlook

A list of related (un)solvable problems

- 1) Democratic formulation for non-abelian gauge theory.
- 2,0) Non-abelian interactions for (chiral) $p-$forms.
- 4,0;...) Democratic GR, ... (higher spins on my mind).

Thank you!

Democratic formulation for $p-$form in d dimensions

Lagrangian

$$\mathcal{L} = \frac{1}{2}(F + a\,P)^2 + \frac{1}{2}(G + a\,Q)^2 - \,a\,Q \wedge F + \,a\,G \wedge P$$

where $F = dA\,,\ G = dB\,,\ P = dS\,,\ Q = dR$. The fields A and S are $p-$forms, while B and R are $(d-p-2)-$forms.

This is a democratic formulation for a $p-$form field (together with dual $(d-p-2)-$form field) in d dimensions. The equations imply that S, R are pure gauge (as is the field a), and the only physical d.o.f. are in A, B, satisfying the duality relation:

$$\star\, dA + dB = 0\,.$$

Duality-symmetric Electromagnetism

The Lagrangian for a single massless spin-one field in $d = 4$

$$\mathcal{L}_{Maxwell} = -\frac{1}{4} H^b_{\mu\nu} H^{b\mu\nu} + \frac{1}{4}\, \epsilon_{bc}\, \varepsilon^{\mu\nu\lambda\rho}\, a\, F^b_{\mu\nu}\, Q^c_{\lambda\rho}$$

where $H^b_{\mu\nu} \equiv F^b_{\mu\nu} + a\, Q^b_{\mu\nu}$, $b = 1, 2$, and

$$F^b_{\mu\nu} = \partial_\mu\, A^b_\nu - \partial_\nu\, A^b_\mu\,, \quad Q^b_{\mu\nu} = \partial_\mu\, R^b_\nu - \partial_\nu\, R^b_\mu\,.$$

This Lagrangian describes a single Maxwell field, using 4 vectors and 1 scalar. Any solution of the e.o.m. is gauge equivalent to that of

$$R^b_\mu = 0, \qquad \star F^a_{\mu\nu} + \epsilon^{ab}\, F^b_{\mu\nu} = 0\,,$$

with a single propagating Maxwell field.

Non-linear electrodynamics: Ansatz

Ansatz for the consistent non-linear Lagrangian

$$\mathcal{L} = a\,\epsilon_{bc} F^b \wedge Q^c + f(U^{ab}, V^{ab})$$

where

$$U^{ab} \equiv \frac{1}{2}\, H^a_{\mu\nu}\, H^{b\mu\nu}\,, \quad V^{ab} \equiv \frac{1}{2}\, H^a_{\mu\nu} \star H^{b\mu\nu}$$

All symmetries are built in, except for the shift of a. The latter will fix the form of $f(U, V)$.

Equations of motion

$$E_{A^b} \equiv d[(f^U_{bc} + f^U_{cb}) \star H^c - (f^V_{bc} + f^V_{cb})\, H^c + a\,\epsilon_{bc}\, Q^c] = 0\,,$$

$$E_{R^b} \equiv d[a\{(f^U_{bc} + f^U_{cb}) \star H^c - (f^V_{bc} + f^V_{cb})\, H^c - \epsilon_{bc}\, F^c\}] = 0\,.$$

Imposing the missing symmetry

Shift symmetry $\delta a = \varphi$

Equations of motion for a:

$$E_a \equiv Q^b \wedge K_b = 0\,,$$

where

$$K_a \equiv (f^U_{ab} + f^U_{ba}) \star H^b - (f^V_{ab} + f^V_{ba})\, H^b - \epsilon_{ab}\, H^b\,,$$

and $f^U_{ab} \equiv \partial f / \partial U_{ab}$, $f^V_{ab} \equiv \partial f / \partial V_{ab}$ ($f^U_{21} \equiv 0 \equiv f^V_{21}$).

Note, that $E_{R^b} - a\, E_{A^b} = da \wedge K_b = 0\,$, which implies $K_b = 0$ iff

$$K_a \pm \epsilon_{ab} \star K_b \equiv 0$$

Then, the $E_a = 0$ is redundant, implying the shift symmetry for a.

The general democratic non-linear electrodynamics

Solution

The equation $K_a \pm \epsilon_{ab} \star K_b \equiv 0$ implies

$$\pm\, \delta^{ac}\,(f^U_{cb} + f^U_{bc}) - \epsilon^{ac}\,(f^V_{cb} + f^V_{bc}) + \delta^a_b = 0$$

The general solution gives the following Lagrangian:

$$\mathcal{L} = \mathcal{L}_{Maxwell} + g(\lambda_1, \lambda_2)\,,$$

where

$$\lambda_1 = \tfrac{1}{2}\, G_{\mu\nu} \star G^{\mu\nu}\,, \quad \lambda_2 = -\tfrac{1}{2}\, G_{\mu\nu}\, G^{\mu\nu}\,, \quad G_{\mu\nu} \equiv \star H^1_{\mu\nu} - H^2_{\mu\nu}$$

Reminder: non-linear electrodynamics in the conventional language

$$S = \int \mathcal{L}(s,p)\, d^4x, \quad s \equiv \frac{1}{2} F_{\mu\nu} F^{\mu\nu}, \quad p \equiv \frac{1}{2} F_{\mu\nu} \star F^{\mu\nu}$$

Nonlinear theories of chiral $p-$forms

Free Lagrangian

$$\mathcal{L} = \tfrac{1}{2}(F + a\,Q)^2 + a\,F \wedge Q\,, \qquad (F = dA\,,\ Q = dR)\,.$$

Self-interacting Lagrangian: general recipe

$$\mathcal{L} = \tfrac{1}{2}(F + a\,Q)^2 + a\,F \wedge Q + g(H^-)\,,$$

$$H^- = F + a\,Q - \star(F + aQ)\,.$$

Equations of motion

In the on-shell gauge $Q = 0$, the equations of motion are:

$$F + \star F = f(F - \star F)\,,$$

$$f(Y) = \frac{\partial g(Y)}{\partial Y}\,.$$

Dual theories (free p-forms)

A $p-$form and its dual

The Lagrangian is given in the form of ("Maxwell Lagrangian")

$$\mathcal{L} \sim F \wedge \star F \,, \quad F = dA \,.$$

Massless $p-$form and a $(d-2-p)$-form fields describe correspondingly particles of $p-$form and a $(d-2-p)$-form representations of the massless little group $ISO(d-2)$, dual to each other.

Attention!

Dual formulations do not admit the same interacting deformations!