FRACTONS

a new covariant gauge field theory

Erica Bertolini

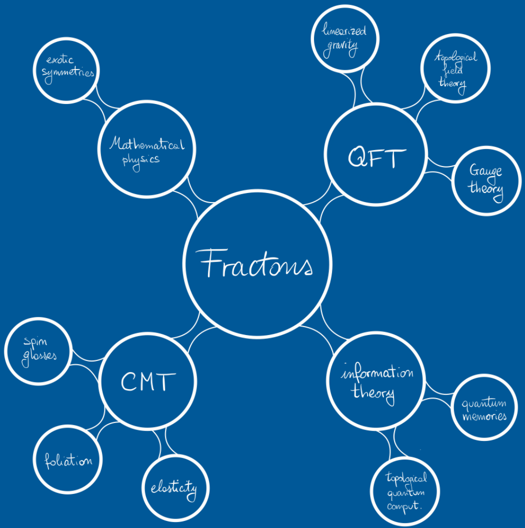
Dublin Institute for Advanced Studies

University of Hertfordshire, 15^{th} January 2025

Scheme of the seminar

- What are fractons?
  - General intro
  - Pretko's gauge theory
- Maxwell theory of fractons
- 3D fracton models
- Final remarks

What are fractons?


Fractons
Mathematical physics
exotic symmetries
QFT
linearized gravity
topological field theory
Gauge theory
CMT
spin glasses
foliation
elasticity
information theory
quantum memories
topological quantum comput.

It's a wonderfully rich world

C.Chamon, Phys.Rev.Lett. 94, 040402 (2005) *Quantum glassiness in strongly correlated clean systems: An example of topological overprotection*

J.Haah, Phys.Rev.A 83, 042330 (2011) *Local stabilizer codes in three dimensions without string logical operators*

B.Yoshida, Phys.Rev.B 88, 125122 (2013) *Exotic topological order in fractal spin liquids*

S.Vijay, J.Haah, L.Fu, Phys.Rev.B 92, 235136 (2015) *A new kind of topological quantum order: A dimensional hierarchy of quasiparticles built from stationary excitations*

S.Vijay, J.Haah, L.Fu, Phys.Rev.B 94, 235157 (2016) *Fracton topological order, generalized lattice gauge theory and duality*

H.Ma, E.Lake, X.Chen, M.Hermele, Phys.Rev.B 95, 245126 (2017) *Fracton topological order via coupled layers*

W.Shirley, K.Slagle, Z.Wang, X.Chen, Phys.Rev.X 8, 031051 (2018) *Fracton models on general 3-dimensional manifolds*

H.Ma, M.Hermele, X.Chen, Phys.Rev.B 98, 035111 (2018) *Fracton topological order from the Higgs and partial-confinement mechanisms of rank-two gauge theory*

D.Bulmash and M.Barkeshli, Phys.Rev.B 97, 235112 (2018) *Higgs mechanism in higher-rank symmetric $U(1)$ gauge theories*

N.Seiberg, SciPost Phys. 8, 050 (2020) *Field theories with a vector global symmetry*

N.Seiberg, S.-H.Shao, SciPost Phys. 9, 046 (2020) *Exotic $U(1)$ symmetries, duality and fractons in 3+1-dimensional QFT*

P.Gorantla, H.T.Lam, N.Seiberg, S.-H.Shao, SciPost Phys. 9, 073 (2020) *More exotic field theories in 3+1 dimensions*

N.Seiberg, S.-H.Shao, SciPost Phys. 10, 003 (2021) *Exotic $\mathbb{Z}_N$ symmetries, duality, and fractons in 3+1-dimensional QFT*

P.Gorantla, H.T.Lam, N.Seiberg, S.-H.Shao, Phys.Rev.B 106, 045112 (2022) *Global dipole symmetry, compact Lifshitz theory, tensor gauge theory, and fractons*

P.Gorantla, H.T.Lam, N.Seiberg, S.-H.Shao, Phys.Rev.B 107, 125121 (2023) *Gapped lineon and fracton models on graphs*

...

M. Pretko (et al.) **:**

Phys.Rev.B 95, 115139 (2017) *Subdimensional particle structure of higher rank $U(1)$ spin liquids*

Phys.Rev.B 96, 035119 (2017) *Generalized electromagnetism of subdimensional particles: A spin liquid story*

Phys.Rev.D 96, 024051 (2017) *Emergent gravity of fractons: Mach's principle revisited*

Phys.Rev.B 96, 125151 (2017) *Higher-spin Witten effect and two-dimensional fracton phases*

Phys.Rev.Lett. 120, 195301 (2018) *Fracton-Elasticity Duality*

Phys.Rev.B 98, 115134 (2018) *The fracton gauge principle*

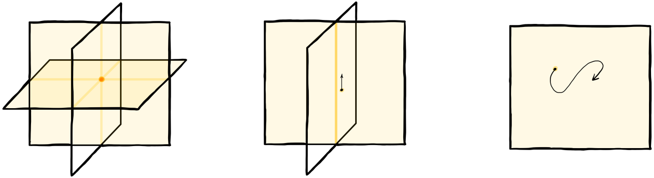
Phys.Rev.X 9, 021003 (2019) *Localization in Fractonic Random Circuits*

...

What are fractons? The defining property

Restricted mobility

- fracton (0D)
- lineon (1D)
- planon (2D)

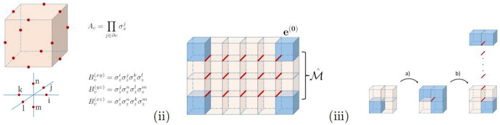


What does it mean?

- Dispersion ($\omega = 0$) ?
- Propagators ?
- Kinetic constraints ?
- lightcone limit ($c \to 0$) ?

Many Languages

- Lattice theory :
  - exactly solvable spin models
  - quantum error correcting codes
  - “gapped”
  - Type I : fractons, lineons and planons
    S.Vijay, J.Haah, L.Fu, Phys.Rev.B 94, 235157 (2016) *Fracton topological order, generalized lattice gauge theory and duality*
  - Type II : fractons only (*fractal* structure)
    J.Haah, Phys.Rev.A 83, 042330 (2011) *Local stabilizer codes in 3D without string logical operators*

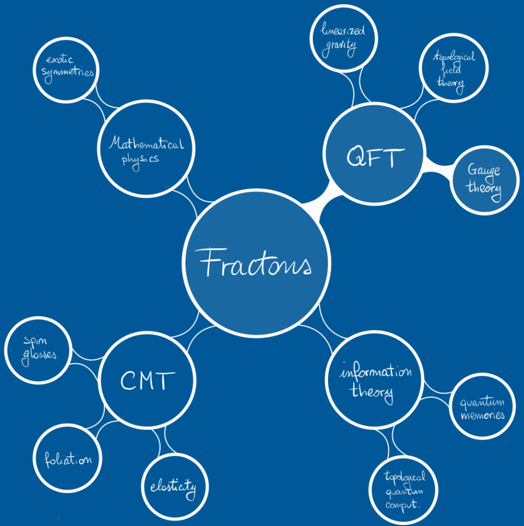


- Tensor gauge theory (Pretko) :
  - $A_{ij} = A_{ji}$
  - “gapless”
  - Maxwell-like ($H = E^2 + B^2$; Gauss $\rightarrow$ limited mobility)
  - Higher moment (dipole) conservations .

Origins and applications

- Spin Glasses : slow thermal equilibrium
  C. Chamon, Phys.Rev.Lett. 94, 040402 (2005) *Quantum glassiness in strongly correlated clean systems: An example of topological overprotection*
- Quantum Infomation : robust memories
  J. Haah, Phys.Rev.A 83, 042330 (2011) *Local stabilizer codes in 3D without string logical operators*
- Spin Liquids : higher spin generalization
  M. Pretko, Phys.Rev.B 95, 115139 (2017) *Subdimensional particle structure of higher rank $U(1)$ spin liquids*
- Topological Lattice Defects : "fracton-elasticity duality"
  M. Pretko and L. Radzihovsky, Phys.Rev.Lett. 120, 195301 (2018) *Fracton-Elasticity Duality*
- Systems with dipole/multipole moment conservations
  A. Gromov, Phys.Rev.X 9, 031035 (2019) *Towards classification of Fracton phases: the multipole algebra*

Pretko's gauge theories for fractons



The "Gauss" constraint and dipole conservation

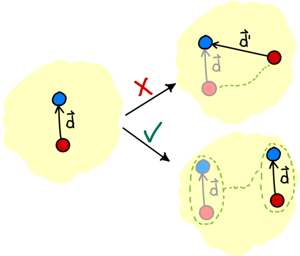
E^{ij} symmetric tensor "electric field"

Gauss : $\partial_i\partial_j E^{ij} = \rho \quad \Rightarrow \quad$ **dipole (D^k) conservation**

$$D^k = \int dV\, x^k \rho = \int dV\, x^k \partial_j\partial_i E^{ji} = -\int dV\, \partial_j E^{kj} = 0$$

single charges = **fractons**

dipoles = free



M. Pretko, Phys.Rev.B 95, 115139 (2017) *Subdimensional particle structure of higher rank $U(1)$ spin liquids*
M. Pretko, Phys.Rev.B 96, 035119 (2017) *Generalized electromagnetism of subdimensional particles*

The ingredients - 4D scalar charge theory

Symmetric rank-2 tensor and scalar Lagrange multiplier :

$$A_{ij} = A_{ji} \quad ; \quad A_0$$

Symmetry and Gauss constraint :

$$\delta_{fract} A_{ij} = \partial_i \partial_j \Lambda \quad ; \quad \partial_i \partial_j E^{ij} = \rho$$

Hamiltonian : Maxwell-like

$$H \sim E^2 + B^2$$

"Electric" and "magnetic" fields :

$$E_{ij} = \partial_i \partial_j A_0 - \partial_0 A_{ij} \quad ; \quad B_{ij} = \epsilon_{ikl} \partial^k A^l_j$$

M. Pretko, Phys.Rev.B 96, 035119 (2017) *Generalized electromagnetism of subdimensional particles*
M. Pretko, Phys.Rev.B 96, 125151 (2017) *Higher-spin Witten effect and two-dimensional fracton phases*

A generalized electromagnetism

Noncovariant Lagrangian

$$L \sim -A_0\partial_i\partial_j E^{ij} + \partial_t A_{ij}E^{ij} - E^{ij}E_{ij} - B^{ij}B_{ij} + A_0\rho + A_{ij}J^{ij}$$

Maxwell-like equations

$$\partial_i\partial_j E^{ij} = \rho \qquad \text{Gauss (by construction)}$$
$$\partial_i B^{ij} = 0 \qquad \text{null div of B (by def)}$$
$$\partial_t B^{ij} - \epsilon^{imn}\partial_m E_n^{\ j} = 0 \qquad \text{Faraday (postulated)}$$
$$\partial_t E^{ij} + \frac{1}{2}\left(\epsilon^{imn}\partial_m B_n^{\ j} + \epsilon^{jmn}\partial_m B_n^{\ i}\right) = J^{ij} \qquad \text{Ampère (EoM)}$$

Fractonic continuity equation

$$\partial_t\rho + \partial_i\partial_j J^{ij} = 0 \quad \Rightarrow \quad \partial_t D^i = \text{bd terms}$$

Traceless scalar theory

Constraints :

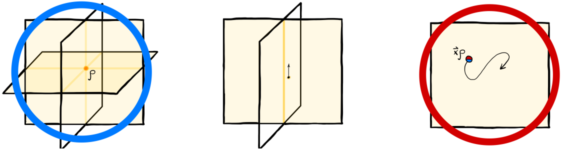
$$\partial_i \partial_j E^{ij} = \rho \quad \oplus \quad E_i^i = 0$$

Additional conservation

$$\int dV\, x^2 \rho = 0$$

a component of quadrupole moment

$\Rightarrow$ single charges ρ = **fractons** (0D particles)
dipoles $\vec{x}\rho$ = **planons** (2D particles)



Vector charge theory

Gauss constraint :

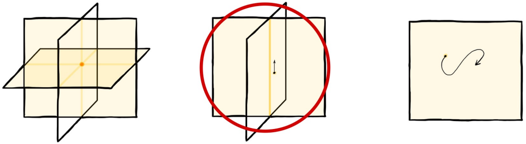
$$\partial_j E^{ij} = \rho^i$$

Conservations of

$$\int dV\, \vec{\rho} = 0 \qquad \int dV\, \vec{x} \times \vec{\rho} = 0$$

charge ("linear momentum") "angular momentum"

$\Rightarrow$ single charges $\vec{\rho}$ = **lineons** (1D particles)



Traceless vector theory

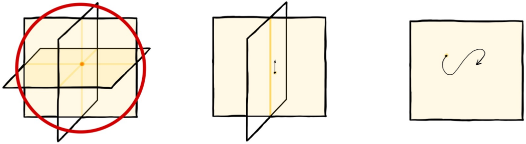
Constraints :

$$\partial_j E^{ij} = \rho^i \quad \text{and} \quad E^i_i = 0$$

Additional conservations

$$\int dV\, (\vec{\rho} \cdot \vec{x}) = 0 \quad ; \quad \int dV \left[(\vec{x} \cdot \vec{\rho})\, \vec{x} - \tfrac{1}{2} x^2 \vec{\rho} \right] = 0$$

$\Rightarrow$ single charges $\vec{\rho}$ = **fractons** (0D particles)

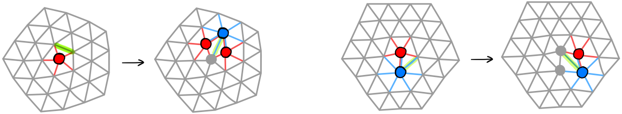


“Practical” example : elasticity duality (3D)

The dictionary :

Fractons	Elasticity
E^{ij} (Electric tensor field)	u^{ij} (Strain tensor)
B^i (Magnetic vector field)	π^i (Lattice momentum)
$\partial_t E^i_i + \epsilon^{ij}\partial_j B_i = J^i_i$ Trace of the “Ampère” equation	$\partial_t n_d + \partial_i J^i_d = J^i_i$ continuity equation for vacancy #

fracton $\leftrightarrow$ disclination ; dipole $\leftrightarrow$ dislocation



M. Pretko and L. Radzihovsky, Phys.Rev.Lett. 120, 195301 (2018) *Fracton-Elasticity Duality*

Weaknesses → Motivations

But (from a QFT point of view) :

- ad hoc definitions...
- A_0?
- non-covariant ;
- inhomogeneous # of ∂ ;
- bizarre mass dimensions.

Maxwell theory of fractons

Based on :

E.B. and N.Maggiore, Phys. Rev. D 106 (2022) no.12, 125008, *Maxwell theory of fractons*

A.Blasi and N.Maggiore, Phys. Lett. B 833 (2022), 137304, *The theory of symmetric tensor field: from fractons to gravitons and back*

E.B., A.Blasi, A.Damonte and N.Maggiore, Symmetry, 15 (2023), no.4, 945, *Gauging fractons and linearized gravity*

Approach : the symmetry

QFT covariant (symmetry-based) technique

$$\delta_{fract}A_{ij} = \partial_i\partial_j\Lambda \quad \rightarrow \quad \delta_{fract}A_{\mu\nu} = \partial_\mu\partial_\nu\Lambda \ ,$$

Symmetry	→	**Action**	→	**EoM**
$\delta_{gauge}A_\mu = \partial_\mu\Lambda(x)$	→	$\int d^4x\ F^{\mu\nu}F_{\mu\nu}$	→	$\partial_\mu F^{\mu\nu} = 0$
$\delta_{fract}A_{\mu\nu} = \partial_\mu\partial_\nu\Lambda(x)$	→	?	→	?

The symmetry is the only ingredient !

(together with covariance, locality and power-counting)

The building block: fracton field strength

New field strength $F \sim \partial A$

	Maxwell	**Fractons**
invariance	$\delta_{gauge} F_{\mu\nu} = 0$	$\delta_{fract} F_{\mu\nu\rho} = 0$
cyclicity	$F_{\mu\nu} + F_{\nu\mu} = 0$	$F_{\mu\nu\rho} + F_{\nu\rho\mu} + F_{\rho\mu\nu} = 0$
Bianchi	$\epsilon_{\mu\nu\rho\sigma}\partial^\nu F^{\rho\sigma} = 0$	$\epsilon_{\alpha\mu\nu\rho}\partial^\mu F^{\beta\nu\rho} = 0$

where

$$F_{\mu\nu\rho} = F_{\nu\mu\rho} = \partial_\mu A_{\nu\rho} + \partial_\nu A_{\mu\rho} - 2\partial_\rho A_{\mu\nu} \, .$$

The action

The most general invariant action ($\delta_{fract} S_{inv} = 0$) is

$$S_{inv} = g_1 S_{fract} + g_2 S_{LG} \ ,$$

where

$$S_{fract} \sim \int d^4x \ F^{\mu\nu\rho} F_{\mu\nu\rho} \sim \int F^2$$

$$S_{LG} = \int d^4x \left(\frac{1}{4} F^{\mu}{}_{\mu\nu} F_{\rho}{}^{\rho\nu} - \frac{1}{6} F^{\mu\nu\rho} F_{\mu\nu\rho} \right) .$$

diffs $\qquad\qquad$ **fract**

$$\delta A_{\mu\nu} = \partial_\mu \xi_\nu + \partial_\nu \xi_\mu \xrightarrow{\xi_\mu = \frac{1}{2}\partial_\mu \Lambda} \delta A_{\mu\nu} = \partial_\mu \partial_\nu \Lambda$$

M.Pretko, Phys.Rev.D 96, 024051 (2017) *Emergent gravity of fractons: Mach's principle revisited*

Remark : on the general theory

- From the canonical momentum $\Pi^{\mu\nu}$ and the EoM

$$\partial_i \partial_j \Pi^{ij} = 0$$

Gauss for the whole theory !

$\Rightarrow$ fractonic behaviour for the whole theory? (LG?)

- If $g_1 + 2g_2 = 0$

$$S_{inv}[A_{\mu\nu}] = S_{inv}[\bar{A}_{\mu\nu}] \quad ; \quad \eta^{\mu\nu}\bar{A}_{\mu\nu} = 0 \,,$$

Traceless theory $\Rightarrow$ Traceless fractons?

Maxwell theory for fractons ($g_2 = 0$)

Equations of Motion (EoM) :

$$\partial_\mu F^{\alpha\beta\mu} = 0 \xrightarrow{\alpha=\beta=0} \boxed{A_{\mu 0} \equiv \partial_\mu A_0}$$

EoM + Bianchi + A_0 :

Maxwell	Fractons
$\vec{\nabla} \cdot \vec{E} = 0$	$\partial_j E^{ij} = 0 \quad \left(\partial_i \partial_j E^{ij} = 0 \; ⚠\right)$
$\vec{\nabla} \cdot \vec{B} = 0$	$\partial^a B_a^{\ p} = 0$
$\vec{\nabla} \times \vec{E} - \partial_t \vec{B} = 0$	$\epsilon_{0lmj} \partial^m E^{ij} - \partial_0 B_l^{\ i} = 0$
$\vec{\nabla} \times \vec{B} + \partial_t \vec{E} = 0$	$\partial_0 E_{ij} + \frac{1}{2}\left(\epsilon_{0ikl} \partial^k B^l_{\ j} + \epsilon_{0jkl} \partial^k B^l_{\ i}\right) = 0$

with **"electric"** and **"magnetic"** tensor fields

$$E^{ij} \equiv -F^{ij0} = 2\left(\partial^0 A^{ij} - \partial^i \partial^j A^0\right) \quad ; \quad B_i^{\ j} \equiv \frac{1}{3} \epsilon_{0ikl} F^{jkl} = \epsilon_{0ilk} \partial^l A^{jk} .$$

The action

Using Electric and Magnetic tensors

$$S_{fract} = \frac{g_1}{6}\int d^4x\, F_{\mu\nu\rho}F^{\mu\nu\rho}$$
$$= -\frac{1}{4g_1}\int d^4x \left(E^{ij}E_{ij} - B^{ij}B_{ij}\right) ,$$

Maxwell-like !

Claim : fractons are embedded in the covariant theory

Energy-momentum tensor

$$T_{\alpha\beta} \equiv -\frac{2}{\sqrt{-g}} \frac{\delta S_{fract}}{\delta g^{\alpha\beta}}\bigg|_{g^{\alpha\beta}=\eta^{\alpha\beta}} = \frac{g_1}{6}\eta_{\alpha\beta}F^2 - \frac{g_1}{3}\left(2F_{\alpha\nu\rho}F_{\beta}{}^{\nu\rho} + F^{\mu\nu}{}_{\beta}F_{\mu\nu\alpha}\right)$$

whose components are :

Fractons	**Maxwell**
$T_{00} = -\frac{1}{4g_1}\left(E^{ij}E_{ij} + B^{ij}B_{ij}\right) \quad \Rightarrow g_1 < 0$	$u = \frac{1}{2}(E^iE_i + B^iB_i)$
$T_{0i} = \frac{1}{2}\epsilon_{0ijk}E^{ja}B^k{}_a$	$S_i = \epsilon_{0ijk}E^jB^k$
$T_{ij} = \eta_{ij}T_{00} - E_{ia}E_j{}^a - \frac{1}{2}\left(B_{ia}B_j{}^a - B_{aj}B^a{}_i\right)$	$\sigma_{ij} = \eta_{ij}u - E_iE_j - B_iB_j$

Continuity equation

EoM with matter coupling :

$$\partial_\mu F^{\alpha\beta\mu} = -J^{\alpha\beta} \ .$$

Continuity equation

$$\partial_\alpha\partial_\beta J^{\alpha\beta} = 2\partial_0\partial_i J^{0i} + \partial_i\partial_j J^{ij} = \boxed{\partial_0\rho + \partial_i\partial_j J^{ij} = 0} \ ,$$

with charge $\rho \sim \partial_i J^{0i}$ and Gauss now is

$$\partial_i\partial_j E^{ij} = \rho \ .$$

"Lorentz" force not *by intuition*

From the on-shell "conservation" of $T^{\mu\nu}$

$$\partial_\nu T^{\mu\nu} + f^\mu = 0 \,,$$

the 4D "Lorentz" force on a dipole is

$$\begin{aligned} \textbf{Fractons:} \quad f^0 &= E_{ab}J^{ab} \quad ; \quad f^i = 2J_{a0}E^{ia} - \tfrac{1}{2}\epsilon^{0imn}J_{am}B_n^{\ a} \\ \textbf{Maxwell:} \quad f^0 &= \vec{E}\cdot\vec{J} \quad ; \quad \vec{f} = q\vec{E} + q\vec{v}\times\vec{B} \,, \end{aligned}$$

where charge, dipole and (dipole) current are

$$\partial_i J^{0i} \sim \rho \quad ; \quad J^{0i} \sim p^i \equiv x^i\rho \quad ; \quad J^{ik} \sim v^i p^k + v^k p^i \,.$$

Finally a well defined theory!

- Covariant (Maxwell) theory for fractons :

  Symmetry → **Action** → **EoM**

  $$\delta A_{\mu\nu} = \partial_\mu \partial_\nu \Lambda(x) \quad \rightarrow \quad \int d^4x \; F^{\mu\nu\rho} F_{\mu\nu\rho} \quad \rightarrow \quad \partial_\mu F^{\alpha\beta\mu} = 0 \; ;$$

- Results of Literature from first principles of QFT :
  - ✓ electric and magnetic tensor fields ;
  - ✓ A_0 ;
  - ✓ "Maxwell" equations ;
  - ✓ energy density ;
  - ✓ "Lorentz" force on a dipole ;

- Energy-momentum tensor ;

- Strong relation with LG.

3D fracton models

Based on :

E.B., A.Blasi, N.Maggiore and D.Sacco Shaikh, JHEP 10 (2024) 232, *Hall-like behaviour of higher rank CS theory of fractons*

E.B., A.Blasi, N.Maggiore, to appear on EPJC *Quasi-topological fractons: a 3D dipolar gauge theory*

Motivations

- QFT perspective of 3D rank-2 (fracton?) theories
- Maxwell-like?
  M. Pretko, Phys.Rev.B 96, 035119 (2017) *Generalized electromagnetism of subdimensional particles*
- Chern-Simons-like?
  M. Pretko, Phys.Rev.B 96, 125151 (2017) *Higher-spin Witten effect and two-dimensional fracton phases*
- Topological?
- Elasticity?
  M. Pretko and L. Radzihovsky, Phys.Rev.Lett. 120, 195301 (2018) *Fracton-Elasticity Duality*
  A. Gromov, Phys. Rev. Lett. 122, 076403, *Chiral Topological Elasticity and Fracton Order*

1. Hall-like theory

Based on :

E.B., A.Blasi, N.Maggiore and D.Sacco Shaikh, JHEP 10 (2024) 232, *Hall-like behaviour of higher rank CS theory of fractons*

E.B., A.Blasi, N.Maggiore, to appear on EPJC *Quasi-topological fractons: a 3D dipolar gauge theory*

Symmetry principle

Symmetry		3D Action		3D EoM
$\delta_{fract} a_{\mu\nu} = \partial_\mu \partial_\nu \Lambda(x)$	$\rightarrow$	?	$\rightarrow$	?

But : the symmetry is **NOT** the only ingredient...

Mass dimensions :

- $[a_{\mu\nu}] = \frac{1}{2}$
- $[a_{\mu\nu}] = 1$

The invariant action

Most general invariant action

$$S_{inv} = \int d^3x\, \epsilon^{\mu\nu\rho} a_{\mu\lambda} \partial_\nu a^{\lambda}_{\rho} \,,$$

(traceless) Chern-Simons-like, but not topological

$$T_{\mu\nu} = -\epsilon^{\alpha\beta\gamma}(a_{\alpha\mu}\partial_\beta a_{\gamma\nu} + a_{\alpha\nu}\partial_\beta a_{\gamma\mu}) \neq 0 \,,$$

however, on-shell (fractonic embedding)

$$\int dV\, T_{00} = 0 \,.$$

Generalized (planar) electromagnetism

On-shell EoM$_{00}$

$$\frac{\delta S_{inv}}{\delta a_{00}} = 0 \quad \Rightarrow \quad a_{0a} = \partial_a a_0$$

Electromagnetic analogy

$$\frac{\delta S_{CS}}{\delta A_a} \propto \epsilon^{0ab} E_b \quad \rightarrow \quad \frac{\delta S_{inv}}{\delta a_{ab}} \equiv \frac{1}{2}\left(\epsilon^{0ak}\bar{E}^b{}_k + \epsilon^{0bk}\bar{E}^a{}_k\right)$$

$$\frac{\delta S_{CS}}{\delta A_0} \propto B \quad \rightarrow \quad \frac{\delta S_{inv}}{\delta a_{0a}} \equiv B^a \ ,$$

with

$$\bar{E}_{ab} = \bar{F}_{ab0} \quad ; \quad \bar{E}^a_{\ a} = 0 \quad ; \quad B^a = \frac{2}{3}\epsilon^{0mn}\bar{F}^a{}_{mn} \ ,$$

and $\bar{F}^{\mu\nu\rho}$ traceless invariant field strength.

Fractonic behaviour

Matter coupling

$$S = \int d^3x \left(\epsilon^{\mu\nu\rho} a_{\mu\lambda} \partial_\nu a^\lambda_\rho - a_{\mu\nu} J^{\mu\nu} \right) ,$$

fractonic continuity equation

$$\partial_0 \rho + \partial_m \partial_n J^{mn} = 0$$

with

$\rho \equiv 2\partial_m J^{0m}$;	$d^m \equiv -2J^{0m}$;	J^{mn}
charge	dipole	dipole current

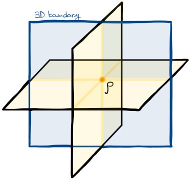
Physical interpretations

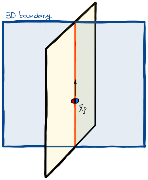
continuity equation → three conservation equations :

$$\partial_t \int \rho = const \quad ; \quad \partial_t \int \vec{x}\rho = const \quad ; \quad \partial_t \int x^2\rho = const \ ,$$

of charge, dipole and a component of the quadrupole.

Traceless scalar fracton theory : dipole = lineon (1D)





Equations of motion → Hall-like behaviour

std Chern-Simons	fractonic Chern-Simons	
$\rho = B$	$d^a = -2B^a$	flux attachment
$J^a = \sigma^{ab} E_b$	$J^{ab} = \sigma^{abmn} \bar{E}_{mn}$	Hall(-like) current

with (dipole-like) fractonic conductivity

$$\sigma^{abmn} \equiv \frac{1}{4}\left(\epsilon^{0am}\eta^{bn} + \epsilon^{0bm}\eta^{an} + \epsilon^{0an}\eta^{bm} + \epsilon^{0bn}\eta^{am}\right) .$$

M. Pretko, Phys.Rev.B 96, 125151 (2017) *Higher-spin Witten effect and two-dimensional fracton phases*

Summarizing

- Non-topological CS-like theory for fractons ;
- Generalized planar electromagnetism ;
- Fractonic interpretation ;
- Hall-like behaviour for dipoles.

2. Quasi-topological fractons

Based on :

E.B., A.Blasi, N.Maggiore and D.Sacco Shaikh, JHEP 10 (2024) 232, *Hall-like behaviour of higher rank CS theory of fractons*

E.B., A.Blasi, N.Maggiore, to appear on EPJC *Quasi-topological fractons: a 3D dipolar gauge theory*

Searching for a new 3D theory

field content	mass dim.	symmetry	theory
$a_{\mu\nu} \neq a_{\nu\mu}$	1	$\delta a_{\mu\nu} = \partial_\mu \xi_\nu$	?

Invariant action

$$S_{inv} = \int d^3x\, \epsilon^{\mu\nu\rho}\, a_\mu^{\ \lambda} \partial_\nu a_{\rho\lambda}\ .$$

On-shell vanishing energy momentum tensor

$$T_{\mu\nu}|_{EoM} = 0\ ,$$

$\Rightarrow$ **"quasi-topological" !**

Decomposition

$$a_{\mu\nu} = \underbrace{\boxed{h_{\mu\nu}}}_{\text{symm.}} + \underbrace{\boxed{\epsilon_{\mu\nu\rho}a^\rho}}_{\text{antisymm.}} \Rightarrow \begin{aligned} \delta h_{\mu\nu} &= \partial_\mu \xi_\nu + \partial_\nu \xi_\mu \\ \delta a^\rho &= -\epsilon^{\rho\alpha\beta}\partial_\alpha \xi_\beta \,. \end{aligned}$$

The invariant action becomes

$$S_{inv} = \int d^3x \left(\epsilon^{\mu\nu\rho} h_\mu^{\ \lambda} \partial_\nu h_{\rho\lambda} - \epsilon^{\mu\nu\rho} a_\mu \partial_\nu a_\rho + 2h_{\mu\nu}\partial^\mu a^\nu - 2h\partial_\mu a^\mu \right) .$$

T. L. Hughes, R. G. Leigh and E. Fradkin, Phys. Rev. Lett. 107 (2011), 075502, *torsional response and dissipationless viscosity in topological insulators*

Hidden fracton symmetry

$$\xrightarrow{\xi_\mu \propto \partial_\mu \phi} \quad \delta a_{\mu\nu} = \partial_\mu \partial_\nu \phi \quad \Rightarrow \quad \delta h_{\mu\nu} = \partial_\mu \partial_\nu \phi \quad ; \quad \delta a^\rho = 0 \,.$$

EoM

$$\frac{\delta S_{inv}}{\delta h_{ab}} = \frac{1}{2}\left(\epsilon^{0am}\tilde{E}^{b}{}_{m} + \epsilon^{0bm}\tilde{E}^{a}{}_{m}\right) \quad ; \quad \frac{\delta S_{inv}}{\delta h_{a0}} = B^a ,$$

with

$$\tilde{E}_{ab} = \tilde{G}_{ab0} \quad ; \quad \tilde{E}^{a}{}_{a} = 0 \quad ; \quad B^a = \tfrac{2}{3}\epsilon^{0mn}\tilde{G}^{a}{}_{mn} ,$$

where

$$\begin{aligned} G_{\mu\nu\rho} = G_{\nu\mu\rho} &\equiv \partial_\mu a_{\rho\nu} + \partial_\nu a_{\rho\mu} - \partial_\rho(a_{\mu\nu} + a_{\nu\mu}) \\ &= F_{\mu\nu\rho} + \epsilon_{\rho\mu\lambda}\partial_\nu a^\lambda + \epsilon_{\rho\nu\lambda}\partial_\mu a^\lambda , \end{aligned}$$

is the invariant field strength, and

$$\tilde{G}_{\mu\nu\rho} \equiv G_{\mu\nu\rho} - \tfrac{1}{4}\left(2\eta_{\mu\nu}G^{\lambda}{}_{\lambda\rho} - \eta_{\mu\rho}G^{\lambda}{}_{\lambda\nu} - \eta_{\nu\rho}G^{\lambda}{}_{\lambda\mu}\right) ,$$

its traceless part.

Matter source and fractons

$$S_J = -\int d^3x\, j^{\mu\nu}a_{\mu\nu} = -\int d^3x \left(J^{\mu\nu}h_{\mu\nu} + J^{\mu}a_{\mu}\right)\ ,$$

- $\partial_\mu\partial_\nu J^{\mu\nu} = 0$ as "Hall-like" theory

$$\rho \equiv 2\partial_m J^{0m} \quad ; \quad d^m \equiv -2J^{0m} \quad ; \quad J^{mn}$$

fracton charge — dipole — dipole current

- additional $\partial_\mu j^{\mu\nu} = 0$

$$\epsilon^{0ab}\partial_a J_b = -\rho \quad \Rightarrow \quad J_a = \partial_a f + \epsilon_{0ab}d^b$$

$$\partial_0\omega^n + \partial_m j^{mn} = 0\ ,$$

with new vector charge

$$\omega^n \equiv j^{0n} = -d^n - \tfrac{1}{2}\epsilon^{0na}\partial_a f\ .$$

Conserved quantities and physical intepretation

Charge and quadupole-like are conserved

$$\partial_t \int \omega^a = 0 \quad ; \quad \partial_t \int x_a \omega^a = 0 \,,$$

$\Rightarrow$ $\omega^a(x)$ =**lineon**.

If $J_0(x) = 0$ two additional conservations :

$$\partial_t \int \epsilon_{0ab} x^a \omega^b = \partial_t \int f = 0$$

$$\partial_t \int \left(x^a x_b \omega^b - \tfrac{1}{2} x^2 \omega^a \right) = 0 \,,$$

$\Rightarrow$ “traceless vector charge theory”: $\omega^a(x)$ = **fracton**.

M. Pretko, Phys.Rev.B 96, 035119 (2017) *Generalized electromagnetism of subdimensional particles*

Extended Hall-like behaviour

From the EoM

symmetric part	nonsymmetric	
$d^a = -2B^a$	$\omega^a = \mathcal{B}^a_+$	flux attachment
$J^{ab} = \tilde{\sigma}^{abmn}\tilde{E}_{mn}$	$j^{ab} = \sigma^{abmn}\mathcal{E}_{mn}$	Hall-like current

with

$$\mathcal{E}_{ab} \equiv G_{ab0} = \tilde{G}_{ab0} + \frac{1}{2}\eta_{ab}G^m{}_{m0} = \tilde{E}_{ab} + \frac{1}{2}\eta_{ab}\mathcal{E}^m{}_m$$

$$\mathcal{B}^a_\pm \equiv B^a \pm \frac{1}{2}\epsilon^{0ab}G^\mu{}_{\mu b}$$

$$\tilde{\sigma}^{abmn} \equiv \frac{1}{4}\left(\epsilon^{0am}\eta^{bn} + \epsilon^{0bm}\eta^{an} + \epsilon^{0an}\eta^{bm} + \epsilon^{0bn}\eta^{am}\right)$$

$$\sigma^{abmn} \equiv \tilde{\sigma}^{abmn} + \frac{1}{2}\epsilon^{0ab}\eta^{mn} \,,$$

Summarizing

- New 3D fracton theory :

field content	dim.	symmetry	theory
$a_{\mu\nu} \neq a_{\nu\mu}$	1	$\delta a_{\mu\nu} = \partial_\mu \xi_\nu$	quasi-topological fractons

- Generalized field strength and electromagnetic fields ;
- New quasiparticle content :
  - ρ =fracton ;
  - d^i =lineon ;
  - ω^i =lineon ($J_0 \neq 0$) or ω^i =fracton ($J_0 = 0$) ;
- Generalized Hall-like behaviour ;
- Relation with Cosserat elasticity?

Final Remarks :

Review of results

Review of results

- **New covariant** theory for fractons
  - Results of Literature from first principles of QFT
  - Energy-momentum tensor
  - Strong relation with LG.
- **3D Hall-like** theory for fractons
  - e.m. analogy
  - traceless fractons
  - Hall-like behaviour.
- **3D quasi-topological** fractons
  - extended e.m. analogy
  - scalar and vector charge th.
  - Hall-like behaviour for both charges.

Final Remarks :

Future

Future

- Fractons and elasticity (stay tuned!) ;
- Matter coupling (fracton QED) ;
- BF-like models ;
- Curved spacetime ;
- Dualities ;
- Relations with LG and gravity ;
- Higher dimensions ($d = 6$) ;
- ...$\longrightarrow \infty$

Thanks for your attention!