Exotic Spheres from Different Angles

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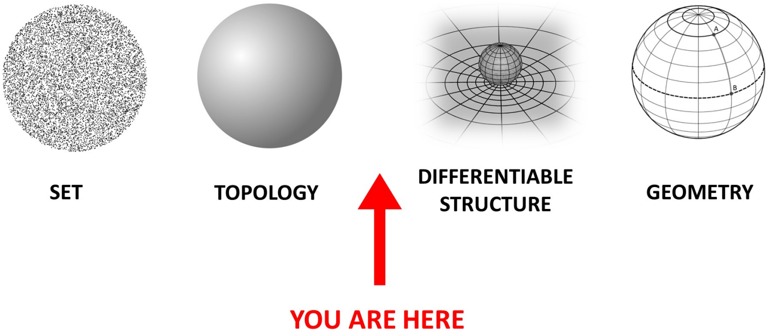
- Milnor's bundles were the first examples of exotic manifolds.
- There are no exotic manifolds for dimensions 1, 2 and 3.

Review - Illustration



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However, they can be described as non-principal S^3 bundles over S^4, which is Milnor's original construction. [1]

[1] J. Milnor, *On manifolds homeomorphic to the 7-sphere*, Ann. Math. 64 (1956) 399.

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- The transition function, in coordinates, is:

$$(\mathbb{H} \ni z\,, \quad y \in \mathbb{H}^*) \mapsto (\mathbb{H} \ni \frac{1}{z}\,, \quad \frac{(z^h y z^l)}{||z||^{h+l}} \in \mathbb{H}^*).$$

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- Hence, we have that $F = G/H$, for $H = \mathrm{SO}(3)$, since $S^3 = \mathrm{SO}(4)/\mathrm{SO}(3)$.

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- Milnor was able to prove that if $h + l = 1$, then the total space of the bundle is homeomorphic to the topological 7-sphere.
- He also showed that if $(h - l)^2 \neq 1(\mathrm{mod} 7)$, then the total space cannot be diffeomorphic to the standard 7-sphere.
- The simplest example of a pair (h, l) which defines an exotic sphere is $(2, -1)$, i.e. the **Gromoll-Meyer sphere**.

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- They have been shown to support numerous families of Einstein metrics (non-constructive!).

  $\implies$ even more suitable for M-theory compactifications (Freund-Rubin).

Geometry - Kaluza-Klein and Abelian Bundles

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K-K Ansatz and Bundle Interpretation

$$\underbrace{g_{\mu\nu}}_{\text{Base metric}} dx^\mu dx^\nu + \phi^2(\underbrace{A_\mu}_{\text{Connection}} dx^\mu + dx^5)^2 = \underbrace{g_{MN}}_{\text{Total space metric}} dx^M dx^N$$

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The (Riemannian) *Kaluza-Klein metric* on the total space obtained this way is well-defined, and it makes the horizontal and vertical subspaces orthogonal to each other.

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Kaluza-Klein metric on the total space [2]

$$g_{MN} = \begin{pmatrix} g_{\mu\nu}(x) + h_{ij}(y) A^i_\mu(x) A^j_\nu(x) & A^i_\mu(x) h_{ij}(y) \\ h_{ij}(y) A^j_\nu(x) & h_{ij}(y) \end{pmatrix}$$

[2] Y.M. Cho, *Higher-dimensional unifcations of gravitation and gauge theories*, J. Math. Phys. 16 (1975) 2029.

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- Fibre = $F = G$; metric on the fibre = $h_{ij}(y)$ - it is written in non-coordinate basis: $h_{ij}(y) = h(e_i, e_j)$. $= \delta_{ij}$
- Structure Group = G; connection on the bundle = $A^i_\mu(x)$ "$\in \mathrm{Lie}(G) \times \Omega^1(U_\alpha)$" $= \left(\frac{1}{x^2+1}\right) 2\eta^i_{\mu\nu} x^\nu$

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[3] R. Percacci and S. Randjbar-Daemi, *Kaluza-Klein theories on bundles with homogeneous fibers. 1*, J. Math. Phys. 24 (1983) 807.

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- Structure Group $= SO(4)$; we use $so(4) \simeq su(2) \oplus su(2)$:

$$A^i_\mu = \begin{cases} A^\alpha_\mu = -\bar{\eta}^\alpha_{\mu\nu} \frac{x_\nu}{x^2+1} \\ A^{\hat{i}}_\mu = -\frac{(x-a)^2 (x-b)^2\, \eta^{\hat{i}}_{\mu\nu}}{(x-a)^2(x-b)^2+\rho^2[(x-a)^2+(x-b)^2]} \left(\frac{\rho^2 (x-a)_\nu}{(x-a)^4} + \frac{\rho^2 (x-b)_\nu}{(x-b)^4} \right) \end{cases}$$

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- ${K_\gamma}^\alpha = \delta^\alpha_\gamma$; using $S^3 = \{(X,Y,Z,W)\ s.t.\ X^2+Y^2+Z^2+W^2 = 1\}$:

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- How does the instantons' moduli space enter the game?
- Does the metric solve Einstein's equations for some choice of moduli?
- Can the solution above be uplifted to a solution of supergravity with appropriate fluxes?

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[5] D. S. Berman, M. Cederwall and TSG, *Curvature of an Exotic 7-sphere*, arXiv:2410.01909.

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Vielbein with Quaternions

Let $ds^2 = E^a \otimes E^a + \varepsilon^i \otimes \varepsilon^i$. Treat S^3 as a quaternion $\boldsymbol{y}$ with $|\boldsymbol{y}| = 1$, and denote the two $su(2)$ gauge fields as $\boldsymbol{A}$, $\boldsymbol{B}$, then:

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Questions: what is the Ricci and what are its properties?

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$$E^a = \mathrm{d}x^m \underbrace{E_m{}^a}_{S^4}, \quad \boldsymbol{\varepsilon} = \underbrace{d\boldsymbol{y}\bar{\boldsymbol{y}}}_{S^3} + \boldsymbol{A} - \underbrace{\boldsymbol{y}\boldsymbol{B}\bar{\boldsymbol{y}}}_{\text{``}K_{\hat{i}}{}^\alpha A^{\hat{i}}_\mu\text{''}}$$

Questions: what is the Ricci and what are its properties?

What is the isometry of the Kaluza-Klein metric?

[5] D. S. Berman, M. Cederwall and TSG, *Curvature of an Exotic 7-sphere*, arXiv:2410.01909.

Geometry - Quaternions and Instantons

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$k = 1$ (regular)

$$\boldsymbol{B} = \frac{\mathrm{Im}((\bar{\boldsymbol{x}} - \bar{\boldsymbol{\xi}})\mathrm{d}\boldsymbol{x})}{(\lambda^2 + |\boldsymbol{x} - \boldsymbol{\xi}|^2)},$$

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$k = 2$ (singular)

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Question: What is the field strength and how can it be regularised?
Is there some symmetry enhancement for some choices of moduli?

Geometry - Curvature

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Ricci Tensor

$$R_{ab} = (R_{S^4})_{ab} - \frac{1}{2}\mathscr{F}_a{}^{ci}\mathscr{F}_{bc}{}^{i}, \quad R_{ij} = 2\delta_{ij} + \frac{1}{4}\mathscr{F}^{abi}\mathscr{F}_{ab}{}^{j},$$

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Ricci Scalar

$$R = R_{S^4} + R_{S^3} - \frac{1}{4}\mathscr{F}^{abi}\mathscr{F}_{ab}{}^{i}.$$

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Isometries of the Base

The $SO(5)$ isometries of the base act *non-trivially* on the gauge field and field strength.

Geometry - Field Strength Moduli

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Instanton moduli vs Kaluza-Klein Moduli

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$$\xi \mapsto \xi' = -\frac{(a-\xi c)^{-1}(b-\xi d)+\frac{\lambda^2 \bar{c} d}{|a-\xi c|^2}}{1+\frac{\lambda^2 |c|^2}{|a-\xi c|^2}} ,$$

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$k = 2$ Case

There is no fixed point. However, there is a natural choice: $\lambda_a = \lambda_b = \lambda$ and $b = -a$, $\mathrm{Im}(a) = 0$.

Geometry - Instantons Regularisation

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$k = 2$ Field Strength (singular)

Let $x + a = x_+$ and $x - a = x_-$. Then:

$$F = \frac{\lambda^2}{(|x_+|^2|x_-|^2 + \lambda^2|x_+|^2 + \lambda^2|x_-|^2)^2} \times \Bigg(|x_+|^2(\lambda^2 + |x_+|^2)\frac{\bar{x}_- \mathrm{d}x \wedge \mathrm{d}\bar{x} x_-}{|x_-|^2} +$$

$$+ |x_-|^2(\lambda^2 + |x_-|^2)\frac{\bar{x}_+ \mathrm{d}x \wedge \mathrm{d}\bar{x} x_+}{|x_+|^2} - \lambda^2(\bar{x}_+ \mathrm{d}x \wedge \mathrm{d}\bar{x} x_- + \bar{x}_- \mathrm{d}x \wedge \mathrm{d}\bar{x} x_+) \Bigg).$$

Geometry - Instantons Regularisation

$k = 2$ Field Strength (regular)

$$F = \frac{4/3}{((1+|x|^2)^2 + \frac{4}{3}|\operatorname{Im} x|^2)^2} \times \Bigg(Q_0 \mathrm{d}x \wedge \mathrm{d}\bar{x} + \\ + Q_1 \Big(\frac{\operatorname{Im} x}{|\operatorname{Im} x|} \mathrm{d}x \wedge \mathrm{d}\bar{x} - \mathrm{d}x \wedge \mathrm{d}\bar{x} \frac{\operatorname{Im} x}{|\operatorname{Im} x|} \Big) + Q_2 \frac{\operatorname{Im} x}{|\operatorname{Im} x|} \mathrm{d}x \wedge \mathrm{d}\bar{x} \frac{\operatorname{Im} x}{|\operatorname{Im} x|} \Bigg).$$

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Moreover, there is a $SO(2)$ subgroup of the $SO(5)$ isometries of the base, whose action is equivalent to a gauge transformation.

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- Some questions on the properties of these metrics are still to be answered (sectional curvature, bounds on the curvature, etc.).

1. Review
2. Geometry of Exotic Spheres
3. A New Numerical ML Method
4. Exotic Spaces and Shock Waves
5. Summary and Conclusions

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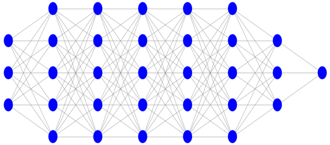

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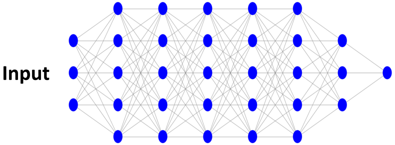

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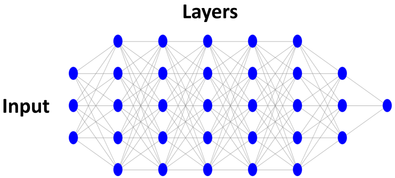

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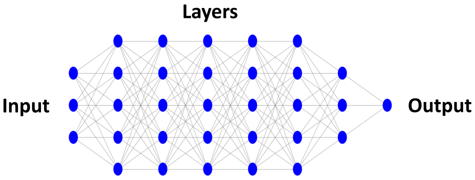

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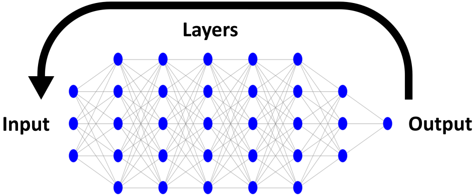

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- Input.
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- Training loop: adjust the weights and biases to minimise $|\text{predicted output} - \text{correct output}|$.

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Gross Structure

- Input: points on the manifold (subtle, see next slide).
- Layers - make sure that they are smooth.
- Output: metric g (subtle, see next slide).
- Loss function: $f_{loss}(g) = |R(g) - \lambda g|$, where $\lambda = 0, 1, -1$ (subtle, see next slide).

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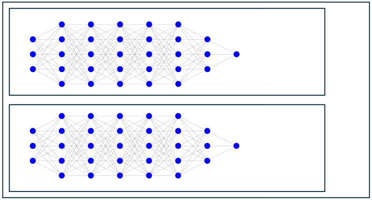

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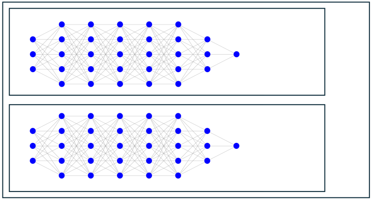

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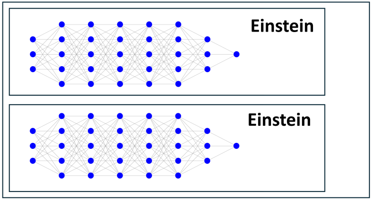

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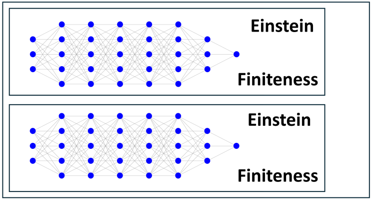

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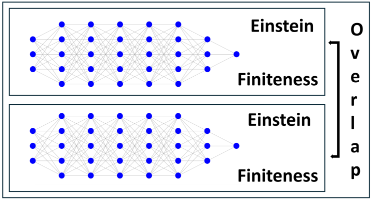
- Finiteness condition, to avoid convergence to "zero metric".

New ML Method - The Details

How do we sample points *globally* and feed them to the neural network appropriately?



Components of the loss function:

- Einstein condition, which uses autodifferentiation to calculate $R(g_1)$ and $R(g_2)$ and it is evaluated in the two patches separately.
- Finiteness condition, to avoid convergence to "zero metric".
- Overlap condition: $|g_1 - Jg_2J^T|$, where J is the Jacobian of the change of coordinates.

New ML Method - An Animation of S^2

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Analytic:

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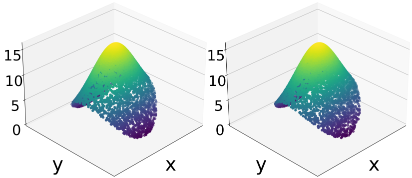

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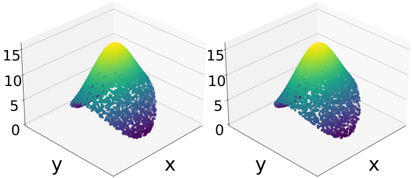

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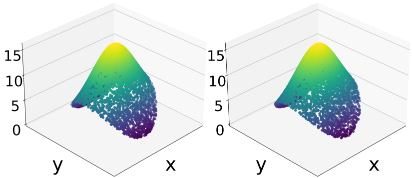

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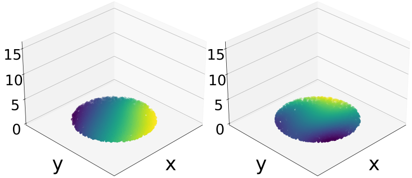

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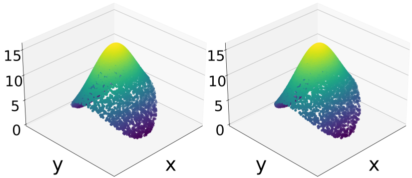

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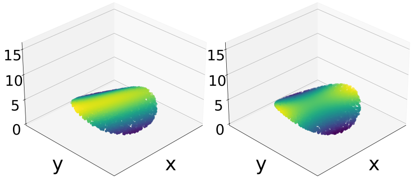

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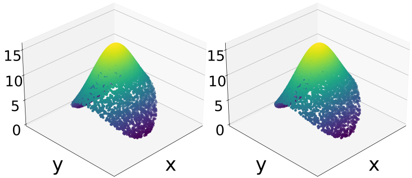

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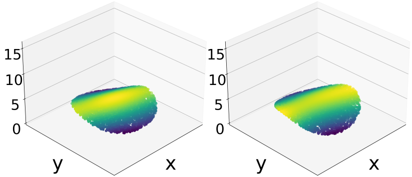

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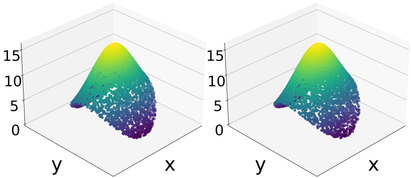

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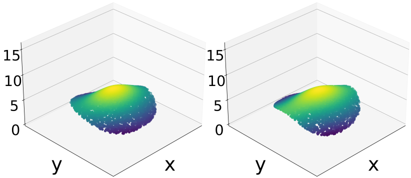

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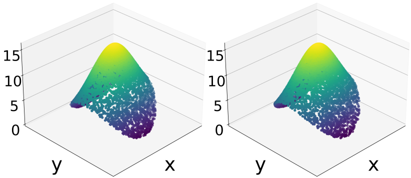

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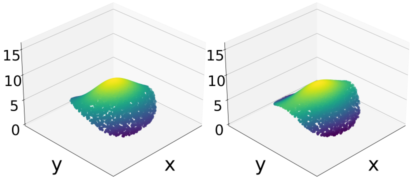

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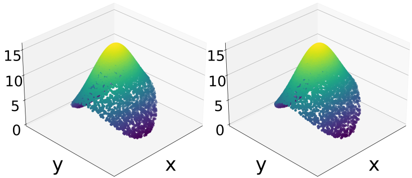

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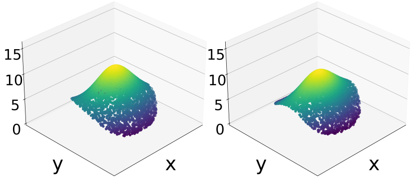

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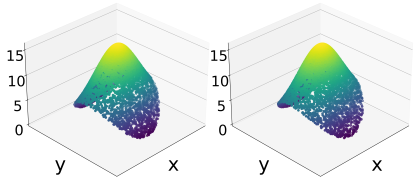

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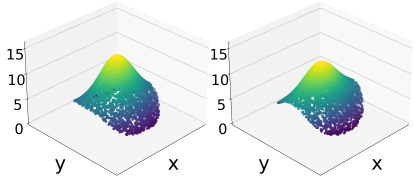

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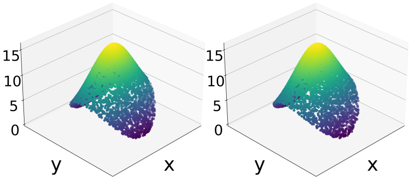

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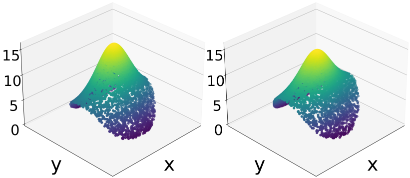

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- Change the number of patches, then look at $S^2 \times S^2$.
- Add parameters for moduli spaces, then look at T^2 to begin with.

1. Review
2. Geometry of Exotic Spheres
3. A New Numerical ML Method
4. Exotic Spaces and Shock Waves
5. Summary and Conclusions

"Exotic Shocks" - Motivation

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- Brans, Asselmeyer-Maluga and collaborators have worked extensively on exploring the consequences of exotic differentiable structures (including the ones on 7-spheres) in GR.

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Idea: to study this "difference", by explicitly looking at the map between the two exotic manifolds.
Similarly to the phenomenon of *topology change*, one can define an appropriate notion for a *change of differentiable structure* and study its implications.

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- By using the intuition about 1-dimensional foliation of spheres ([9]).

[9] I. Tamura, *Homeomorphy classification of total spaces of sphere bundles over spheres*, Jour. of Math. Soc. of Jap. 10 (1958).

“Exotic Shocks” - The Map of [9]

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$$g_m(l([a]_s, b), t) = (l([a]_s, b), t) \qquad (0 \leqq t \leqq 1/4),$$

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$$= \Big(l(a^{m+1}\, b_s\, b^{-1}_{s+4(1-s)(t-\frac{1}{2})}\, a^{-m},$$

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$$1 - \frac{3}{4}s\Big) \qquad (s \neq 0,\ 1/2 \leqq t \leqq 3/4),$$

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$$= \Big(l([a^{m+1}\, b_s\, a^{-m}]_{4(1-t)}, (a^{m+1}\, b_s\, a^{-m})^{-m}\, a^m\, b\, a^{-m} (a^{m+1}\, b_s\, a^{-m})^m),$$

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And, unless g' is Einstein, then g' is a *shock wave*.

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What is the interpretation?

- Shock waves arise from inequivalent differentiable structures.
- New (low regularity) metrics can be built on Σ^7 by pulling back those on S^7.

What about other exotic spaces?

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- The role of inequivalent differentiable structures in physics remains somewhat mysterious.

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- Some questions, however, are difficult to tackle analytically.
  In this direction, I introduced a numerical scheme which might help finding approximate solutions numerically.
- The role of inequivalent differentiable structures in physics remains somewhat mysterious.
  Exotic spheres might help providing a physical implication of the “change of differentiable structure”.

Wrapping Up

Exotic Spheres are a cornucopia of interesting open questions:

- Many open questions concern their geometry.
  In this regard, I presented some progress in understanding a set of geometries on exotic spheres analytically.
  Hopefully, this can help gauging their role in supegravity compactifications.
- Some questions, however, are difficult to tackle analytically.
  In this direction, I introduced a numerical scheme which might help finding approximate solutions numerically.
- The role of inequivalent differentiable structures in physics remains somewhat mysterious.
  Exotic spheres might help providing a physical implication of the "change of differentiable structure".
  I presented a possible manifestation of such a phenomenon, which consists of shock waves.

Thank you!