YANGIAN SYMMETRY, GKZ EQUATIONS AND INTEGRABLE FEYNMAN GRAPHS

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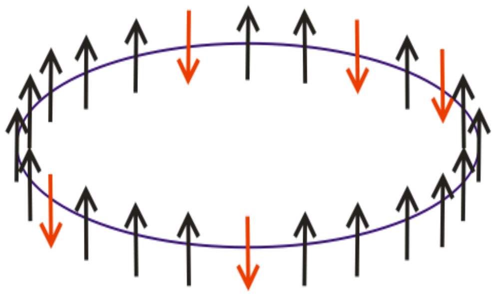
based on

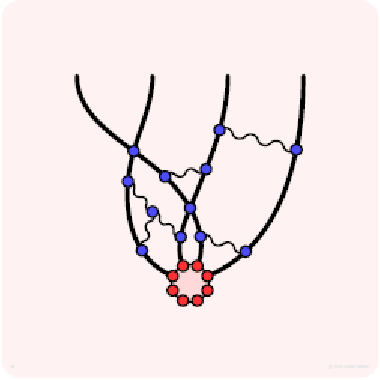
2304.04654 with Kazakov, Mishnyakov

2412.19296 with Mishnyakov

Introduction

Usually integrability comes up in 1d or 2d
(spin chains, IQFT in 2d, lattice models, ...)





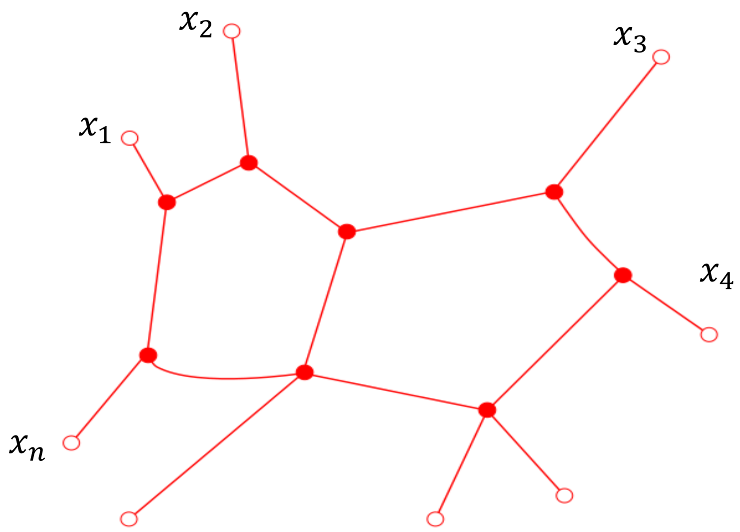
More recently – powerful examples in d>2,
such as N=4 super Yang-Mills in 4d

Goal: understand higher-dim integrability

Solution of N=4 SYM draws much from AdS/CFT duality with string theory

We will instead explore integrability directly in field theory

We will show how integrability helps to compute individual Feynman graphs



Rich structures: geometry, special functions/numbers, differential equations, etc

- We find a new large class of integrable Feynman graphs
- They satisfy PDEs based on Yangian symmetry

⇒ often fix the result completely!

Greatly extend known results
[Kazakov, Loebbert, Zhong 17-18]
[Loebbert, …]

famously used for other observables in N=4 SYM

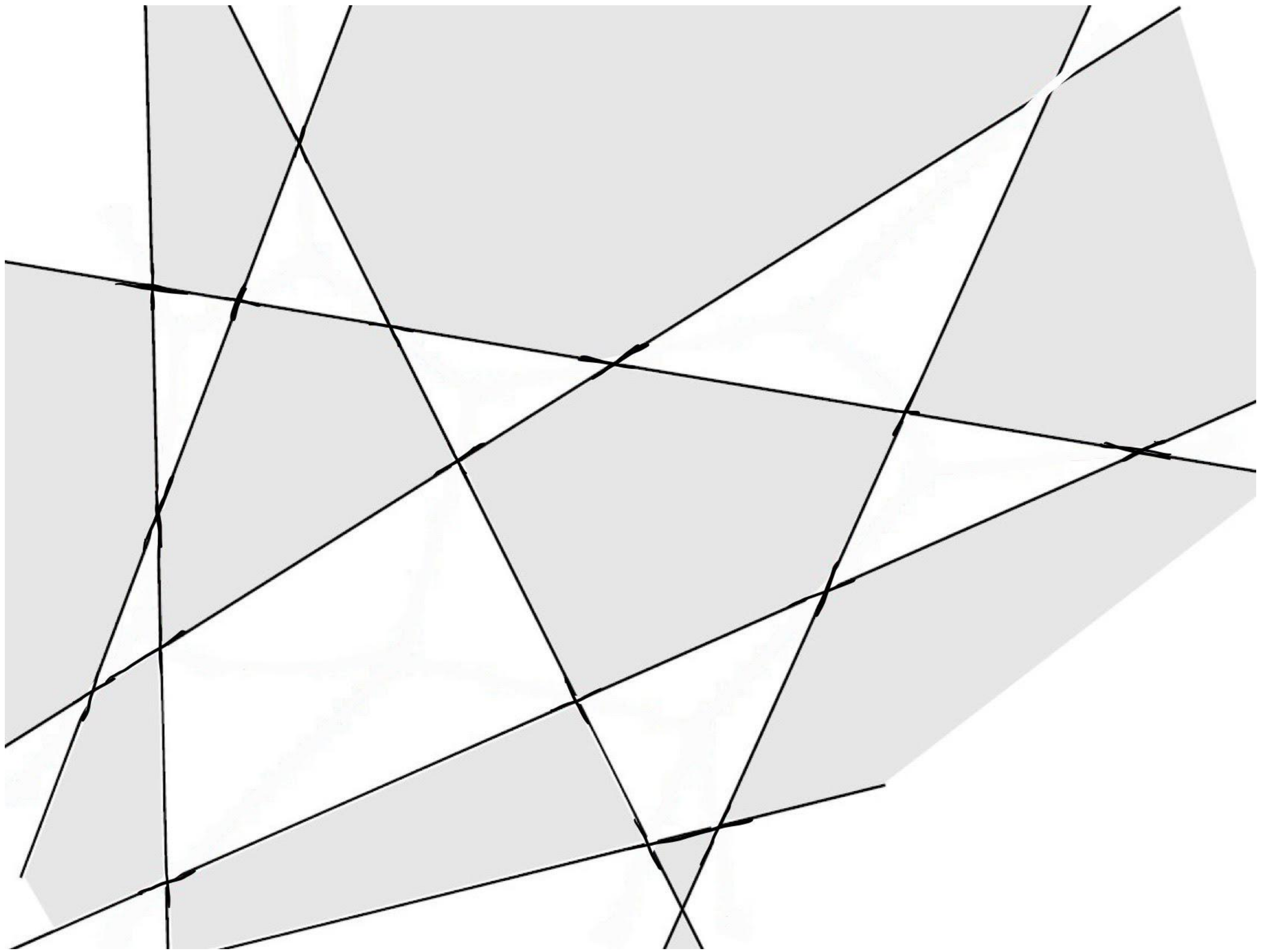
[Lukowski, Ferro, Frassek, Loebbert, Staudacher, …] [Arkani-Hamed, …]

Feynman graphs from the "Loom"

Study conformal Feynman integrals in any D arising from geometric "loom" construction

[Zamolodchikov 80]
[Kazakov, Olivucci 22]

Start from 'Baxter lattice' (finite set of lines)



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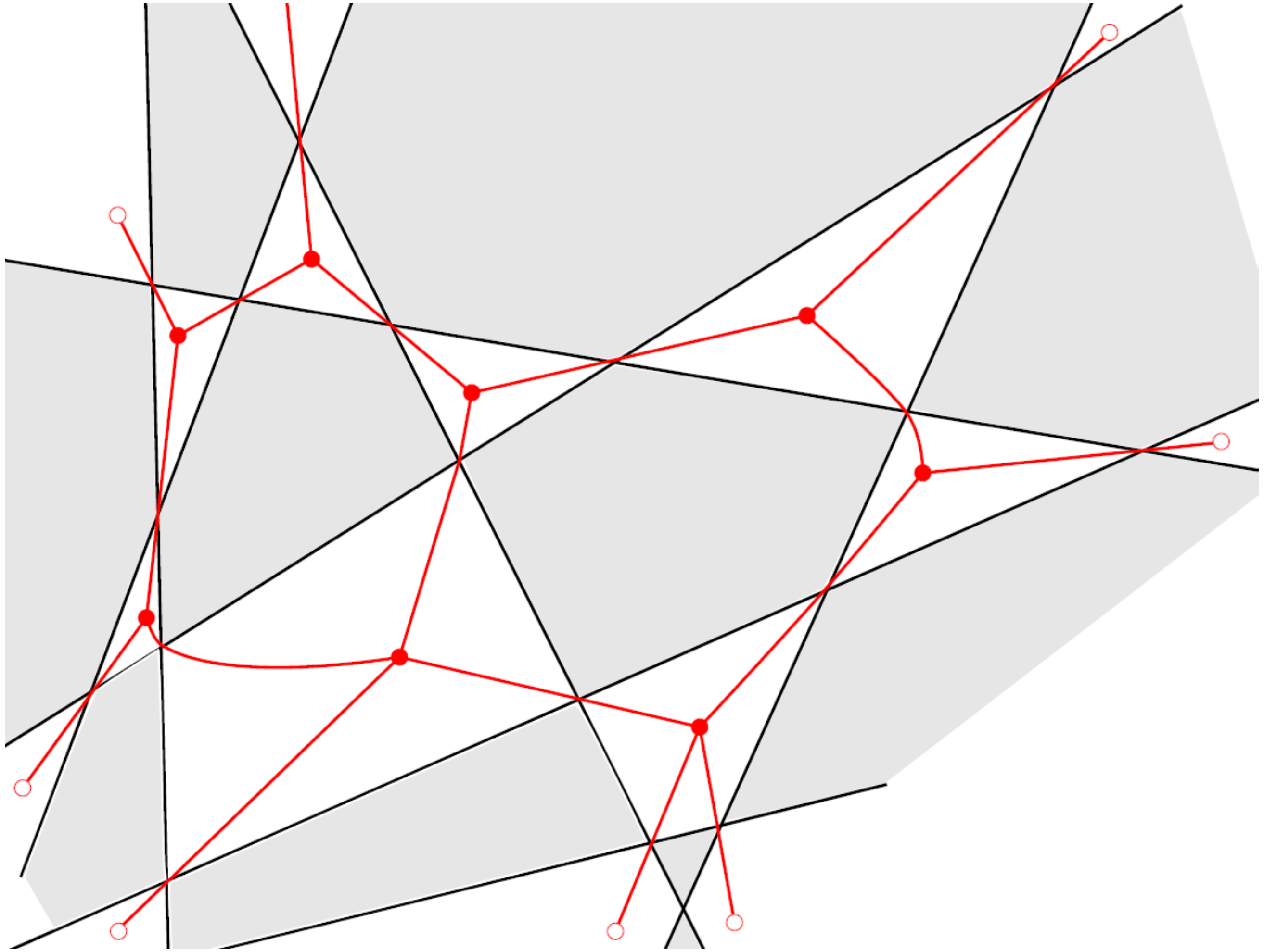

Start from 'Baxter lattice' (finite set of lines)

Put vertices inside white faces, connect neighbours via propagators

↓

get Feynman graph

Study conformal Feynman integrals in any D arising from geometric "loom" construction

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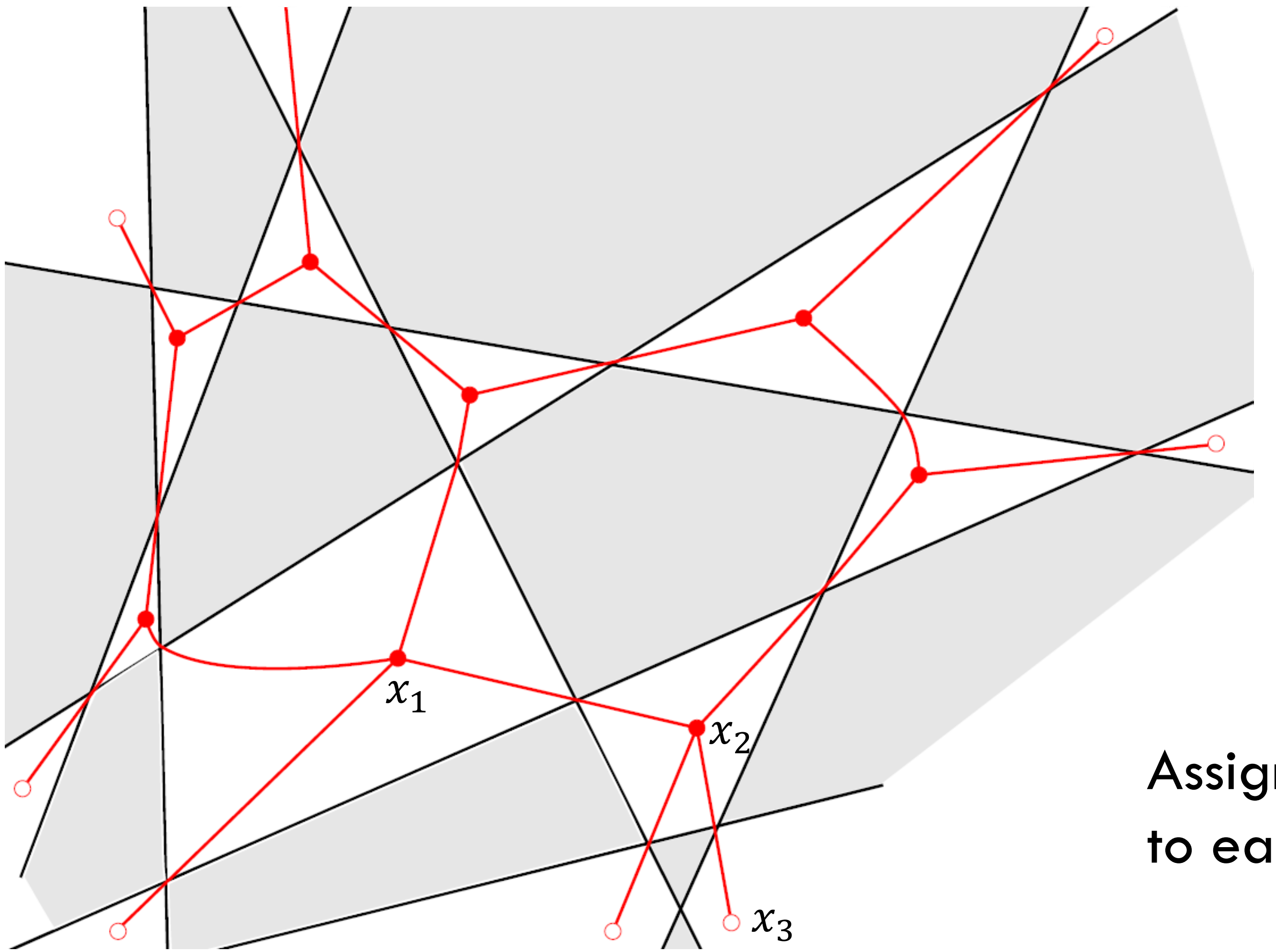


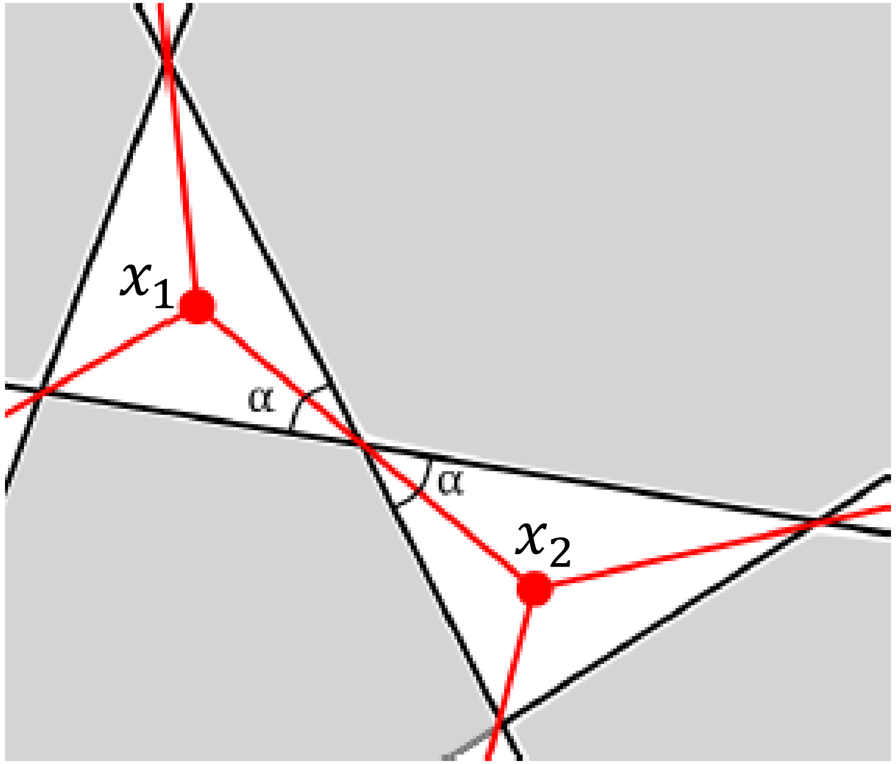
Start from 'Baxter lattice' (finite set of lines)

Put vertices inside white faces, connect neighbours via propagators

↓

get Feynman graph

Assign D-dim coordinates x_k to each vertex, integrate over internal



$$\text{propagator} = \frac{1}{|x_1 - x_2|^{2\Delta}} \qquad \Delta = D\frac{\pi - \alpha}{2\pi}$$

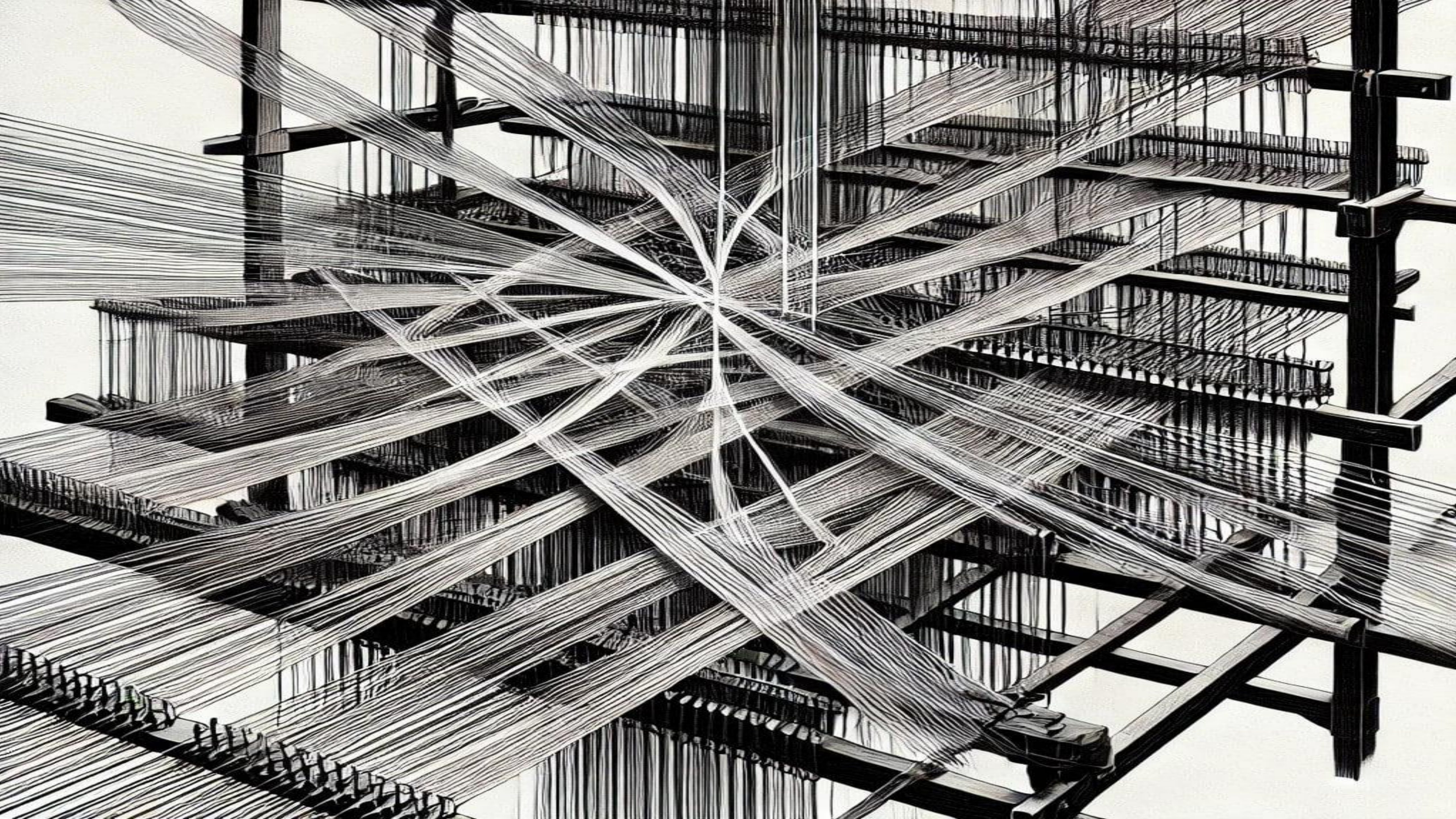
[Zamolodchikov 80]

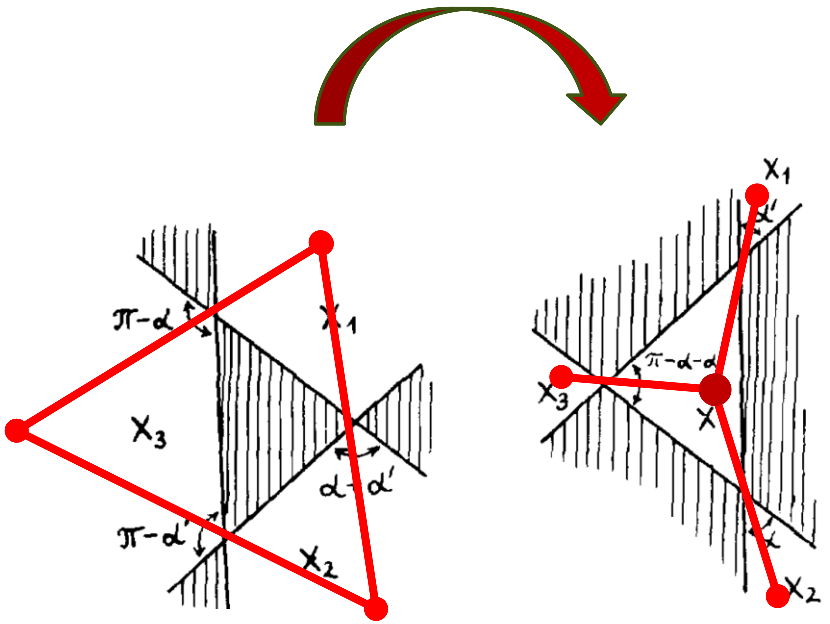
At each vertex the sum of Δ's is D !
So we have conformal symmetry

After integration we get a conformally invariant function of external coordinates

Can be viewed as position-space correlation function

$$\langle \mathrm{Tr}\,[\Phi_1(x_1)\Phi_2(x_2)\ldots\Phi_n(x_n)]\rangle$$

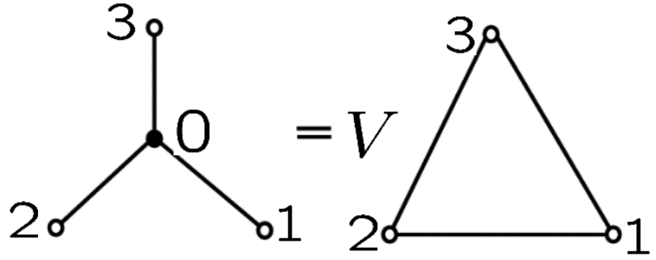




These Feynman graphs should be **integrable** in any D

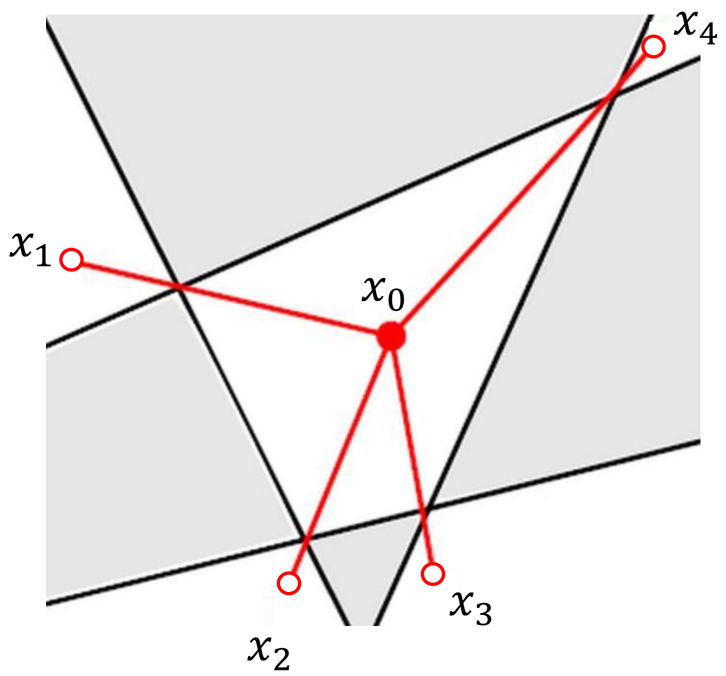
Moving lines of the Baxter lattice gives **star-triangle** transformation for the graph

$$\int \frac{d^D x_0}{|x_{10}|^{2a}|x_{20}|^{2b}|x_{30}|^{2c}} = \frac{V(a,b,c)}{|x_{12}|^{D-2c}|x_{23}|^{D-2a}|x_{31}|^{D-2b}}, \qquad (a+b+c = D\,, \quad x_{ij} := x_i - x_j)$$



$$V(a,b,c) = \pi^{D/2}\frac{\Gamma(\frac{D}{2}-a)\Gamma(\frac{D}{2}-b)\Gamma(\frac{D}{2}-c)}{\Gamma(a)\Gamma(a)\Gamma(c)}$$

Example: general cross integral



$$I = \int \frac{d^D x_0}{(x_{10})^{2\Delta_1}(x_{20})^{2\Delta_2}(x_{30})^{2\Delta_3}(x_{40})^{2\Delta_4}}$$

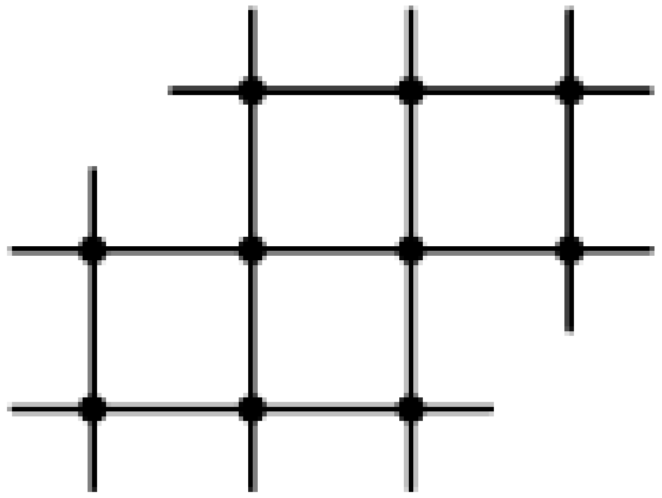
with $\sum_{k=1}^{4} \Delta_k = D$

Notice in general Delta's are not integer

One can still write down a Lagrangian that gives these graphs

[Kazakov Olivucci 22]

Example: square lattice in D=4



$\Delta = D\left(2 - \frac{\alpha}{\pi}\right)$ =1 (usual massless scalar field)

propagator = $\frac{1}{|x_1 - x_2|^2}$

$$S = \frac{N}{2}\int d^4x \, \mathrm{tr} \, \left(\partial^\mu \phi_1^\dagger \partial_\mu \phi_1 + \partial^\mu \phi_2^\dagger \partial_\mu \phi_2 + 2\xi^2 \, \phi_1^\dagger \phi_2^\dagger \phi_1 \phi_2\right)$$

[Gurdogan, Kazakov 15]

Fishnet CFT, actually an extreme deformation of N=4 SYM
(inherits integrability from it)

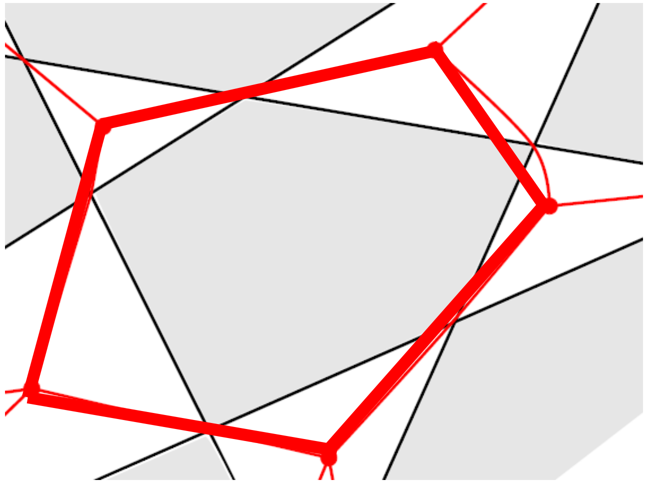
Constraints

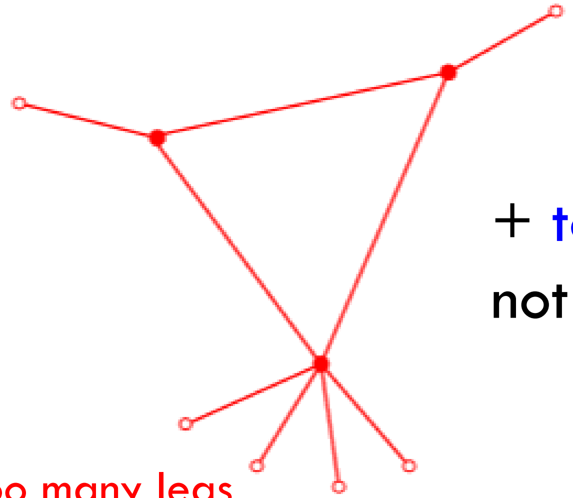
$$\Delta = D\frac{\pi - \alpha}{2\pi}$$

Two types of constraints on Δ 's

- Local: sum=D at each vertex $\Longleftrightarrow$ conformality
- Non-local: dual conformal

  sum around an n-gon = (n-2)D/2





too many legs

+ topological constraints:

not every graph can be drawn on a loom at all

Yangian symmetry and integrability

Conformal symmetry

Any loom graph $I(x_1, \ldots, x_n)$ is conformally invariant

Each leg carries a principal series rep of *so(D,2)* labelled by Δ_j

$$P_j^\mu = -i\partial_{x_j^\mu}, \quad D_j = x_j^\mu \partial_{x_j^\mu} - i\Delta_j, \quad L_j^{\mu\nu} = \ldots, \quad K_j^\mu = \ldots$$

Summing over all legs we get a symmetry:

$$\left(\sum_{j=1}^{n} P_j^\mu\right) I(x_1, \ldots, x_n) = 0 \qquad \text{etc}$$

But there is also a powerful hidden symmetry

Yangian symmetry

$$J^a = \sum_i J_i^a$$ level-0 (conformal)

$$\hat{J}^a = \sum_{i>j} f^a{}_{bc} J_i^b J_j^c + \sum_i s_i J_i^a$$ level-1 (dual conformal)

+ inf many more

Yangian symm was used to great effect in N=4 SYM

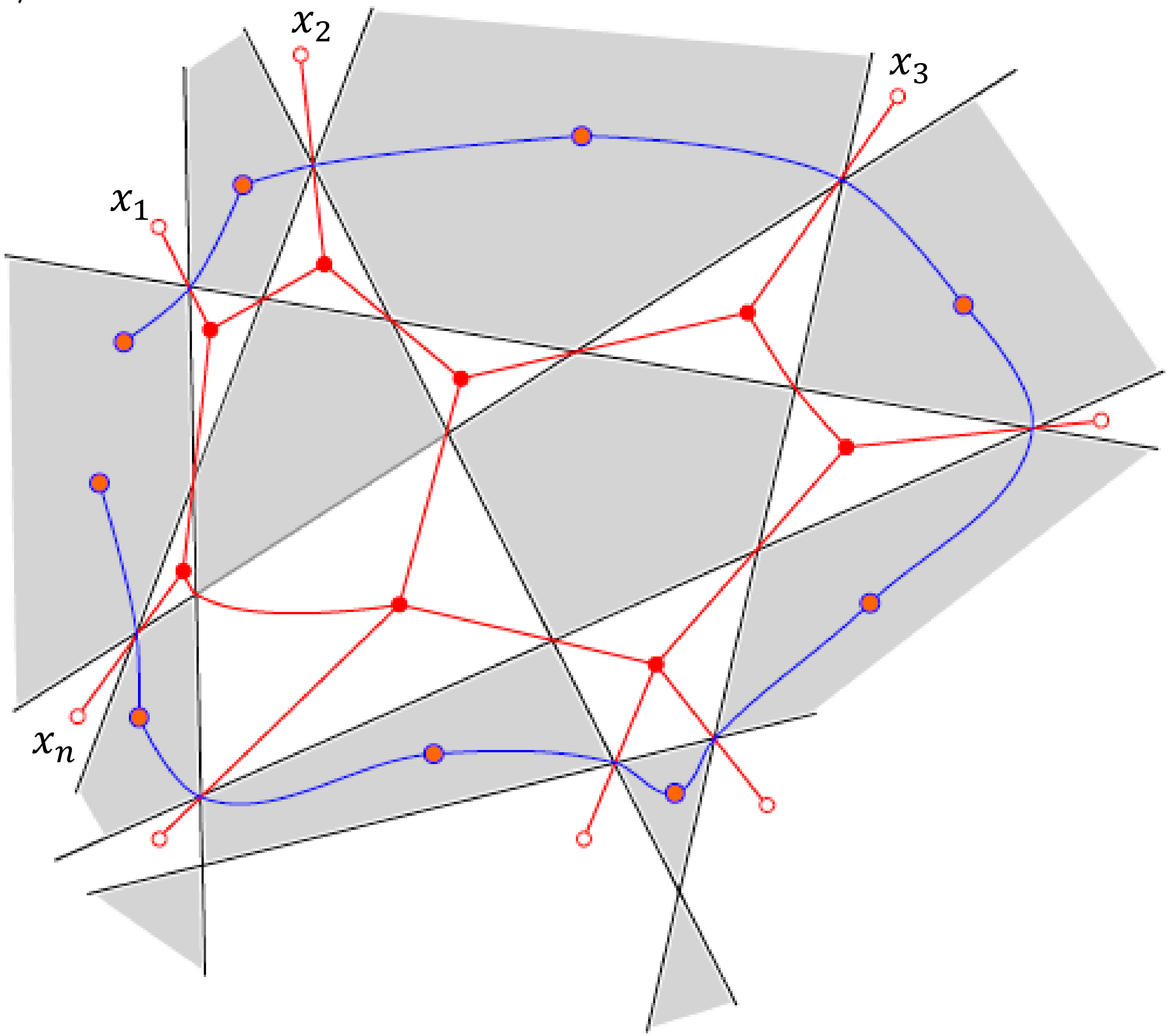
[Lukowski, Ferro, Frassek, Loebbert, Staudacher, …]
[Arkani-Hamed, …]

But may have problems with regularisation
We avoid them by working with correlators not amplitudes

We find that any "loom" graph is an eigenstate of an integrable SO(D,2) spin chain

$$(L_1 L_2 \ldots L_n)_{\alpha\beta}\,|\text{graph}\rangle = \lambda(u)\delta_{\alpha\beta}|\text{graph}\rangle$$

[Kazakov, **FLM**, Mishnyakov 23]



Laxes act on external legs

$$L(u_+, u_-) = \begin{pmatrix} u_+ - \mathbf{px} & \mathbf{p} \\ \mathbf{x}(u_+ - u_-) - \mathbf{xpx} & \mathbf{xp} + u_- \end{pmatrix}$$

[Chicherin, Derkachov, Isaev 12]

$$\mathbf{x} = -i\bar{\sigma}^\mu x_\mu \ , \quad \mathbf{p} = -\frac{i}{2}\sigma^\mu \partial_{x_\mu}$$

Generalize results known for mostly square lattice [Chicherin, Kazakov, Loebbert, Muller, Zhong 17] [..]

L acts in $V_{phys} \otimes V_{aux}$

principal series rep labelled by Delta

4d spinor rep

$$L(u_+, u_-) = \begin{pmatrix} u_+ - \mathbf{px} & \mathbf{p} \\ \mathbf{x}(u_+ - u_-) - \mathbf{xpx} & \mathbf{xp} + u_- \end{pmatrix}$$

encode rep label and spectral parameter

$$\mathbf{x} = -i\bar{\sigma}^\mu x_\mu \ , \quad \mathbf{p} = -\frac{i}{2}\sigma^\mu \partial_{x_\mu}$$

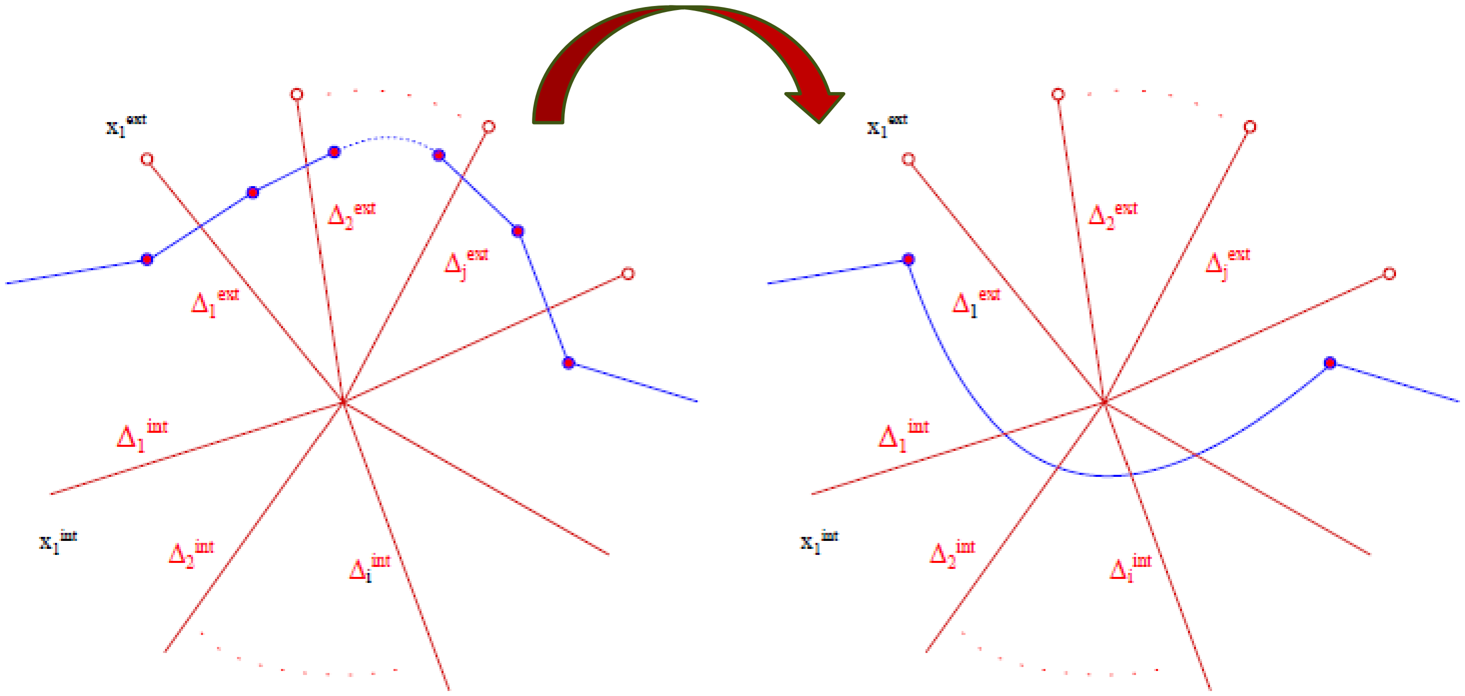
$$(L_1 L_2 \dots L_n)_{\alpha\beta}\, |\text{graph}\rangle = \lambda(u)\delta_{\alpha\beta}|\text{graph}\rangle$$

choose shifts of u according
to graph geometry

Prove using the "Lasso"
based on intertwining relation
& full derivative tricks

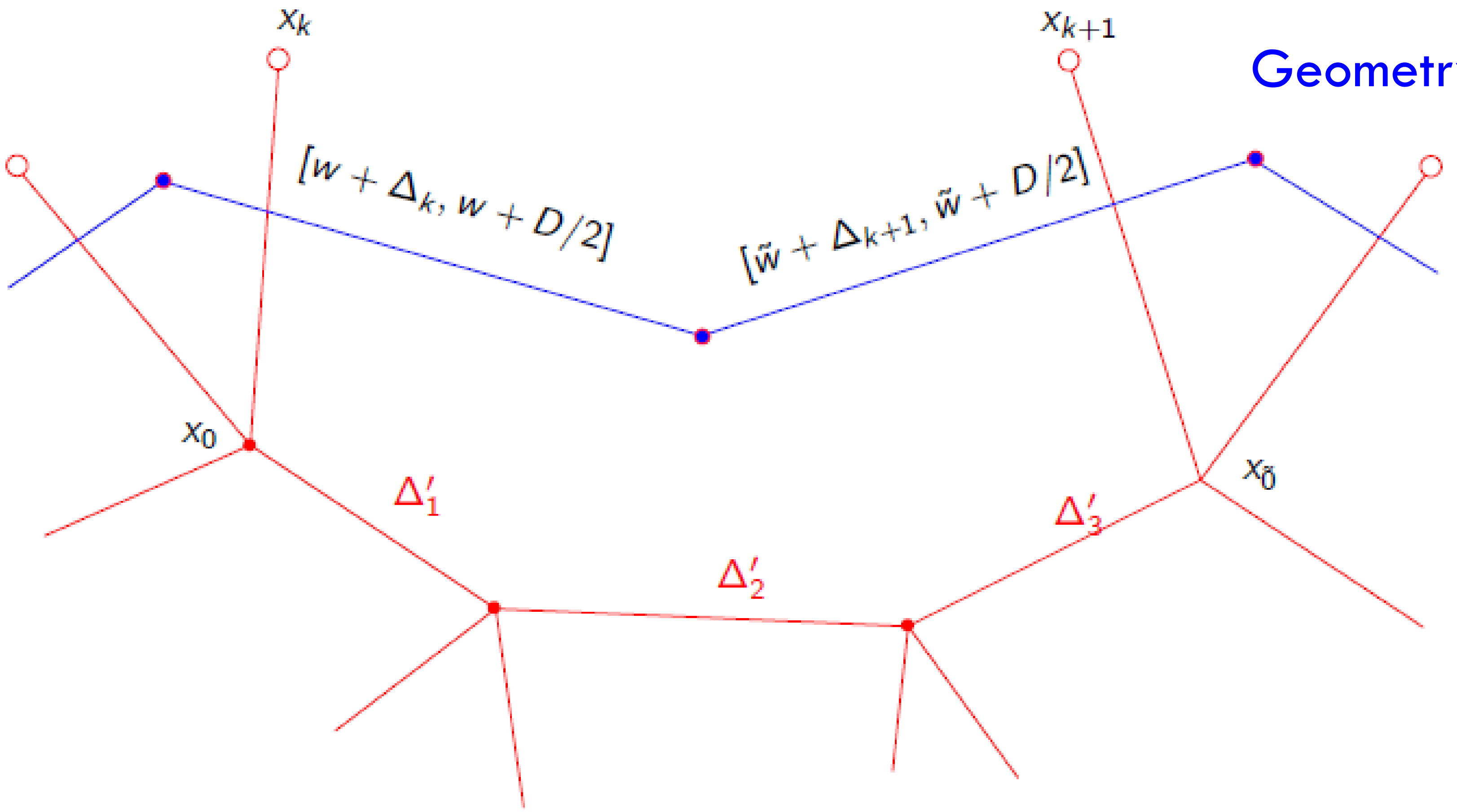
$$L_1(u+\Delta, u')L_2(v,u)\frac{1}{x_{12}^{2\Delta}} = \frac{1}{x_{12}^{2\Delta}}L_1(u,u')L_2(v,u+\Delta)$$

[Chicherin, Kazakov, Loebbert, Muller, Zhong 17]
[Kazakov, **FLM**, Mishnyakov 23]



In the end all constraints on Delta's are used nontrivially

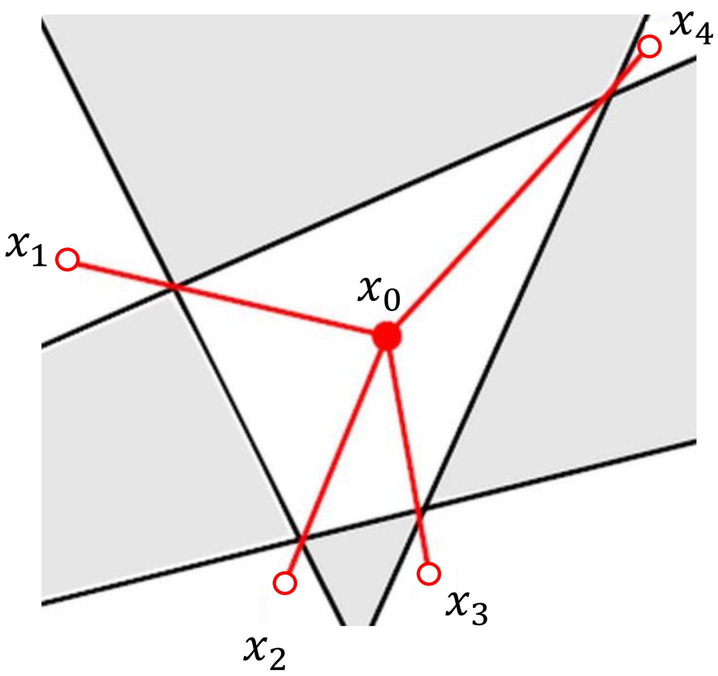
Geometry can be quite involved



We found a prescription for labels
for (almost) arbitrary graphs

[Kazakov, **FLM**, Mishnyakov 23]

Example: cross integral



The shifts are:

$$L_4[D, \Delta_1 + \Delta_2 + \Delta_3 + D/2] L_3[\Delta_1 + \Delta_2 + \Delta_3, \Delta_1 + \Delta_2 + D/2]$$
$$L_2[\Delta_1 + \Delta_2, \Delta_1 + D/2] L_1[\Delta_1, D/2]$$

Yangian symmetry

$$\underbrace{(L_1 L_2 \dots L_n)}_{T(u)}{}_{\alpha\beta}\,|\text{graph}\rangle = \lambda(u)\delta_{\alpha\beta}|\text{graph}\rangle$$

Expanding in powers of u we get Yangian algebra

$$T(u) \sim u^n\left(\mathbb{1} + \frac{1}{u}J + \frac{1}{u^2}\hat{J} + \dots\right)$$

level-0
conformal symmetry

level-1
Yangian symmetry

Enough to impose

$$\widehat{P}^{\mu} = -\frac{i}{2}\sum_{j<k}[(L_j^{\mu\nu} + g^{\mu\nu}D_j)P_{k,\nu} - (j \leftrightarrow k)] + \sum_j v_j(\Gamma)P_j^{\mu}$$

bilocal part

graph-dependent
evaluation parameters

$$\widehat{P}^{\mu}\,|\text{graph}\rangle = 0$$

New PDEs!

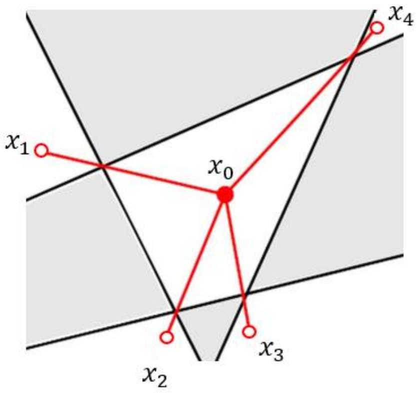
Equations in cross-ratios

The graph actually depends on cross-ratios $I_n = V_n\,\phi(u_1, u_2, \dots)$

$$0 = \widehat{\mathrm{P}}^\mu I_n = V_n \sum_{j<k=1}^{n} \frac{x_{jk}^\mu}{x_{jk}^2} D_{jk}\phi \longrightarrow D_{jk}\,\phi = 0, \qquad 1 \leq j < k \leq n.$$

Highly nontrivial coupled 2nd order diff eqs!

E.g. for cross:



$$z\bar{z} = \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2} = u, \qquad (1-z)(1-\bar{z}) = \frac{x_{14}^2 x_{23}^2}{x_{13}^2 x_{24}^2} = v.$$

$$0 = \left(\alpha\beta + (\alpha+\beta)(u\partial_u + v\partial_v) + (u\partial_u + v\partial_v)^2 - u\partial_u^2 - \gamma\partial_u\right) I_+(u,v)$$
$$0 = \left(\alpha\beta + (\alpha+\beta)(u\partial_u + v\partial_v) + (u\partial_u + v\partial_v)^2 - v\partial_v^2 - \gamma'\partial_v\right) I_+(u,v)$$

[Loebbert, Muller, Munkler 19]

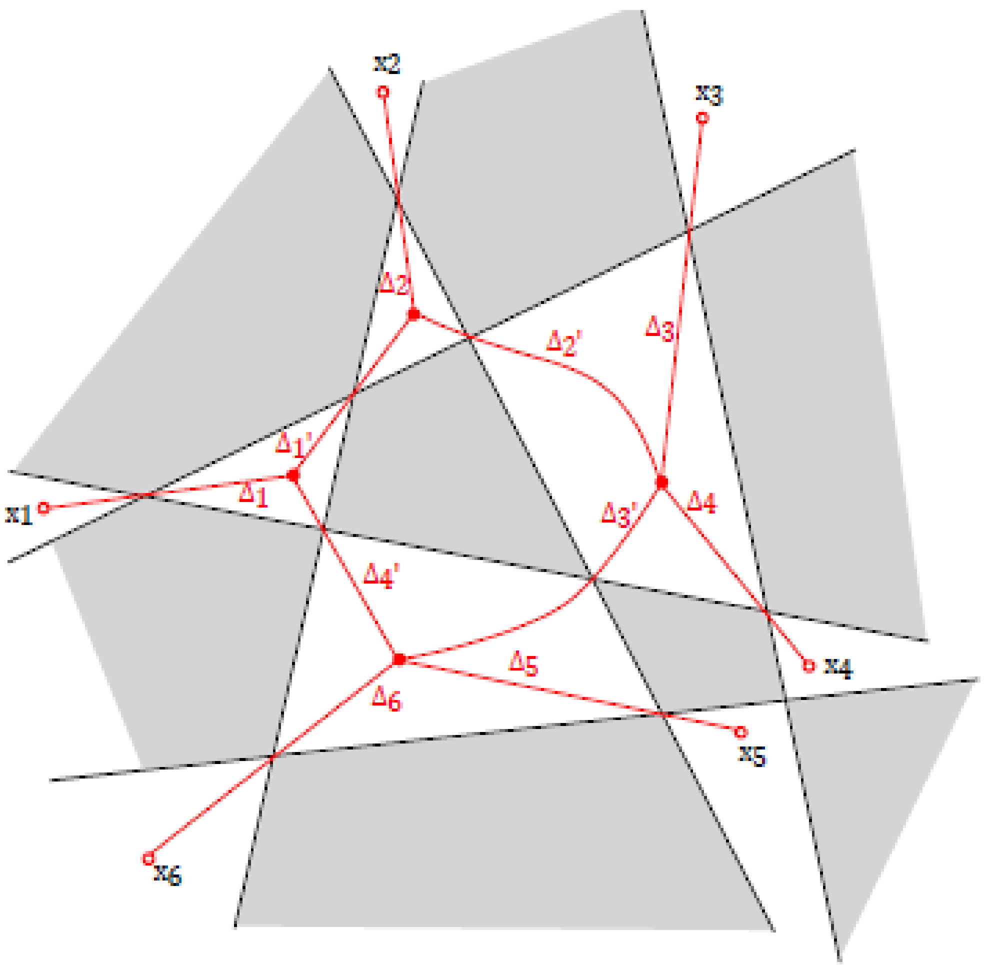
Hypergeometric system! Solved by Appell F4

Example: square with 6 legs [Kazakov, **FLM**, Mishnyakov 23]

$$\widehat{P}^{\mu} = -\frac{i}{2}\sum_{j<k}[(L_j^{\mu\nu} + g^{\mu\nu}D_j)P_{k,\nu} - (j \leftrightarrow k)] + \sum_j v_j(\Gamma)P_j^{\mu}$$

bilocal part

graph-dependent evaluation parameters



We find

$$v_j = \Big\{0, -\Delta_1' - \frac{\Delta_1}{2} - \frac{\Delta_2}{2} + D/2, -\frac{\Delta_1}{2} - \frac{\Delta_3}{2}, -\frac{\Delta_3}{2} - D/2,$$
$$-\Delta_1' - \frac{\Delta_1}{2} - \Delta_2 - \frac{\Delta_5}{2} + D/2, -\Delta_1' - \frac{\Delta_1}{2} - \frac{\Delta_2}{2} - \frac{\Delta_5}{2}\Big\}$$

Diff equations provide concise structure

… especially for 2d square lattice – linked to Calabi-Yau geometry
(Yangian eqs = Picard-Fuchs eqs for periods)

[Duhr, Klemm, Loebbert, Nega, Porkert 22, 23]

Should have finite-dim solution space!
So fix the graph up to boundary conditions

Need to classify & explore them

Equations in cross ratios

[**FLM,** Mishnyakov 24]

How to write Yangian in cross ratios?

Was known only for a few special cases

$$\widehat{P}^\mu = -\frac{i}{2}\sum_{j<k}[(L_j^{\mu\nu} + g^{\mu\nu}D_j)P_{k,\nu} - (j \leftrightarrow k)] + \sum_i s_i P_i^\mu \quad \longrightarrow \quad \widehat{P}^\mu = -2\sum_{i<k}\frac{x_{ik}^\mu}{x_{ik}^2}\mathrm{PDE}_{ik}(\xi)$$

We found the general answer! [**FLM,** Mishnyakov 24]

$$\mathrm{PDE}_{ik} = 2\left(\sum_{l>j>i} - \sum_{l<j<i} + \sum_{l<k<i;j} - \sum_{l>k>i;j}\right)\chi_{iklj}\theta_{il}\theta_{jk} + \sum_{j\neq i}(\delta_{j>i} - \delta_{j<i})\theta_{ik}\theta_{ij}$$
$$+ \left(\delta_{i<k}\left(\Delta_k - D\right) - \delta_{i>k}\left(\Delta_i - D\right)\right)\theta_{ik} + 2(s_i - s_k)\theta_{ik}$$

$$\theta_{ij} = \sum_A \alpha_{ij}^A \xi^A \frac{\partial}{\partial \xi^A} + \beta_{ij} \qquad \chi_{ijkl} = \frac{x_{ij}^2 x_{kl}^2}{x_{ik}^2 x_{jl}^2}$$

$$\xi^A = \prod_{i<j} x_{ij}^{2\alpha_{ij}^A} \qquad W_N(\mathbf{x}|\boldsymbol{\beta}) = \prod_{i<j} x_{ij}^{2\beta_{ij}}$$

we assume high enough D
but should be possible to relax [in progress]

Relation to GKZ systems

The most involved part of our eqs are
the famous Gelfand-Kapranov-Zelevinsky operators

[Gelfand Kapranov Zelevinsky 89, 92,93]

$$\hat{L}_{iklj} = \frac{\partial^2}{\partial x_{ik}^2 \partial x_{lj}^2} - \frac{\partial^2}{\partial x_{il}^2 \partial x_{kj}^2}$$

Gelfand-Kapranov-Zelevinsky systems
[1989]
Good & well-known systems of d.e.:
finite-dim space of solutions, constructed algorithmically

$$\mathrm{PDE}_{ik} = \mathcal{L}^{\kappa}_{ik} + R^{\kappa}_{ik}$$

[**FLM**, Mishnyakov 24]

sum of GKZ ops

1 st order part (remainder)

[see also Pal 2023
for N-cross graphs]

$$\mathcal{L}^{\kappa}_{ik} = 2\left(\sum_{l>j>i} - \sum_{l<j<i} + \sum_{l<k<i,j} - \sum_{l>k>i,j}\right)\hat{\mathcal{L}}^{\kappa}_{iklj}$$

$$L^{\kappa}_{iklj} = \left(\prod_{m<n} x_{mn}^{2\kappa_{mn}}\right)^{-1} L_{iklj} \left(\prod_{m<n} x_{mn}^{2\kappa_{mn}}\right)$$

trivial conjugation by propagators

GKZ systems are defined by the following data:

- A $m \times n$ matrix $\mathcal{A}$, also called the toric matrix, with integer entries, such that the vector $\{1, \ldots, 1\}$ lies in its column span.
- A vector $b \in \mathbb{R}^m$.

The GKZ system then is the following family of differential equations in variables z_i, $i = 1 \ldots n$ for a function $\Phi(z_1, \ldots, z_n)$:

- For all $\ell \in \mathbb{Z}^n$ such that $\ell \in \ker(\mathcal{A})$,

$$\mathcal{A}\ell = 0 ,$$

  one has an equation:

$$\left(\prod_{\ell_i > 0} \partial_{z_i}^{\ell_i} - \prod_{\ell_i < 0} \partial_{z_i}^{-\ell_i} \right) \Phi = 0$$

- The scaling equations

$$\left(\sum_j \mathcal{A}_{ij} z_j \frac{\partial}{\partial z_j} - b_i \right) \Phi = 0$$

Powerful solution theory exists for GKZ eqs

Gives A-hypergeometric functions
= basis of functions for Feynman graph

with integral representations. We only note that among the Euler type integrals associated with systems of the form (0.2) there are the integrals $\int \Pi P_i (t_1, \ldots, t_n)^{\alpha_i} t_1^{\beta_1} \ldots t_n^{\beta_n} dt_1 \ldots dt_n$, where P_i are polynomials, i.e., practically all integrals which arise in quantum field theory. A separate paper will be devoted to these integrals.

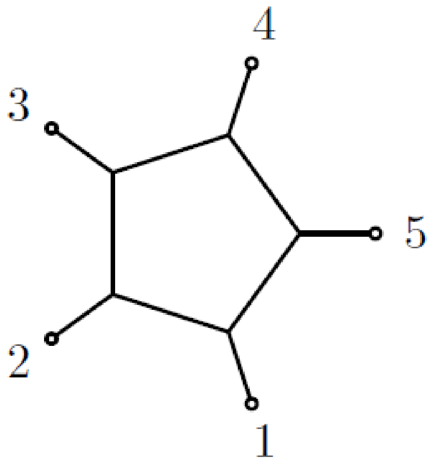
$$\mathrm{PDE}_{ik} = \mathcal{L}^{\kappa}_{ik} + R^{\kappa}_{ik}$$

sum of GKZ ops

1^{st} order part (remainder)

If remainder vanishes we can
use powerful GKZ solution methods

- N-cross graphs
- New examples:



$$2D = \Delta_1 + \Delta_2 + \Delta_3 + \Delta_4 + \Delta_5$$

Note GKZ eqs are much studied
for Feynman graphs

[Grimm Hoefnagel 24]
[de la Cruz 19]
…

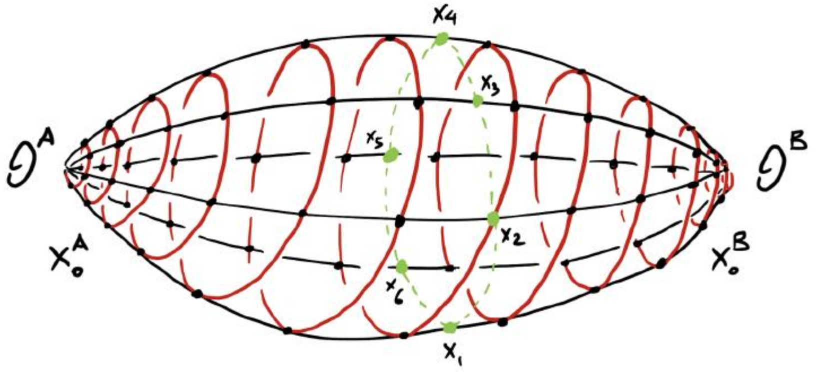
But typically in Lee-Pomeransky rep

Here we see them directly in coord space

Much to be explored

Other realisations of integrability

In 4d fishnets one can compute some graphs using separation of variables for conformal spin chain, should explore links [Cavaglia, Gromov, FLM 21]

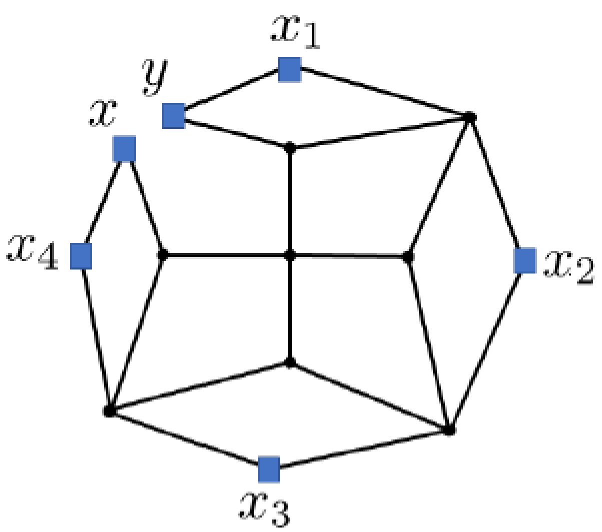


So far we took spinor rep in auxiliary space

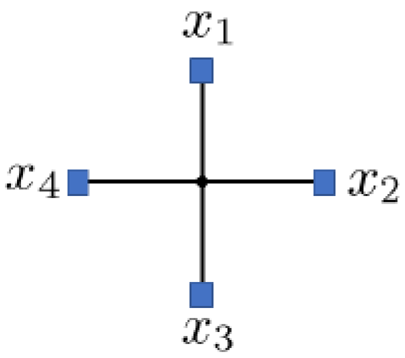
Another natural choice – principal series like in phys space [Kazakov, **FLM**, Mishnyakov 23]

Then instead of Lax use R-matrix – an integral operator [Chicherin, Derkachov, Isaev 12]

⇨ new integral equations for the diagrams



$$= \lambda(u)\,\delta^{(D)}(x-y)\times$$



Implications
to be explored

OUTLOOK

Long road to integrable Feynman graphs:

AdS/CFT and N=4 SYM → fishnet limit → Yangian symmetry

- Need full understanding and general form of PDEs, solution theory
  Link with powerful math methods (GKZ, D-module theory)

- Beyond scalar graphs – fermions, gauge fields, massive
  [Ferrando, Sever] [Loebbert et al]

- Calabi-Yau geometry appears in several different ways; beyond 2d?

- Connection with separation of variables (SoV)