BV Analysis of N=1, D=4 Sugra

Filippo Fila-Robattino

March 5, 2025

University of Zurich

Outline of the talk

Outline of the talk

- Introduction to the BV formalism in classical field theory: Motivation and a paradigmatic example.

Outline of the talk

- Introduction to the BV formalism in classical field theory: Motivation and a paradigmatic example.
- The BV formulation of Palatini–Cartan gravity.

Outline of the talk

- Introduction to the BV formalism in classical field theory: Motivation and a paradigmatic example.
- The BV formulation of Palatini–Cartan gravity.
- The BV formulation of $N = 1, D = 4$ Supergravity

Outline of the talk

- Introduction to the BV formalism in classical field theory: Motivation and a paradigmatic example.
- The BV formulation of Palatini–Cartan gravity.
- The BV formulation of $N = 1, D = 4$ Supergravity
- Extra: How to obtain the infinitesimal symmetry transformations from the boundary structure of a field theory.

The BV formalism in field theory

The BV formalism in field theory

Definition

Let M be a smooth manifold. A **field theory** on M is given by a space of fields F_M and an action functional $S_M \in \mathcal{C}^\infty(F_M)$ on it.

The BV formalism in field theory

Definition

Let M be a smooth manifold. A **field theory** on M is given by a space of fields F_M and an action functional $S_M \in \mathcal{C}^\infty(F_M)$ on it.

- we are interested in the symmetries of the systems, i.e. vector fields $X \in \mathfrak{X}(F_M)$ such that

$$\mathrm{L}_X(S_M) = 0.$$

The BV formalism in field theory

Definition

Let M be a smooth manifold. A **field theory** on M is given by a space of fields F_M and an action functional $S_M \in \mathcal{C}^\infty(F_M)$ on it.

- we are interested in the symmetries of the systems, i.e. vector fields $X \in \mathfrak{X}(F_M)$ such that
$$\mathrm{L}_X(S_M) = 0.$$

- in a gauge theory, symmetries are generated by elements of Lie(G), G Lie group

The BV formalism in field theory

Definition

Let M be a smooth manifold. A **field theory** on M is given by a space of fields F_M and an action functional $S_M \in \mathcal{C}^\infty(F_M)$ on it.

- we are interested in the symmetries of the systems, i.e. vector fields $X \in \mathfrak{X}(F_M)$ such that
$$\mathrm{L}_X(S_M) = 0.$$
- in a gauge theory, symmetries are generated by elements of Lie(G), G Lie group
- Here we consider an involutive distribution $\mathfrak{D} \subset \mathfrak{X}(F_M)$ s.t. $\mathrm{L}_X(S_M) = 0$ for all $X \in \mathfrak{D}$.

Definition

A BV manifold on M is the assignment of data $(\mathcal{F}_M, \mathcal{S}_M, Q, \varpi_M)$, where $(\mathcal{F}_M, \varpi_M)$ is a $\mathbb{Z}$-graded supermanifold endowed with a -1-symplectic form ϖ_M, and $\mathcal{S}_M$ and Q are respectively a degree 0 funcional (called BV action) and a degree 1 vector field on $\mathcal{F}_M$ such that

- $\iota_Q \varpi_M = \delta \mathcal{S}_M$, i.e. Q is the Hamiltonian vector field of $\mathcal{S}_M$;
- $Q^2 = \frac{1}{2}[Q, Q] = 0$, i.e. Q is cohomological.

Definition

A BV manifold on M is the assignment of data $(\mathcal{F}_M, \mathcal{S}_M, Q, \varpi_M)$, where $(\mathcal{F}_M, \varpi_M)$ is a $\mathbb{Z}$-graded supermanifold endowed with a -1-symplectic form ϖ_M, and $\mathcal{S}_M$ and Q are respectively a degree 0 funcional (called BV action) and a degree 1 vector field on $\mathcal{F}_M$ such that

- $\iota_Q \varpi_M = \delta \mathcal{S}_M$, i.e. Q is the Hamiltonian vector field of $\mathcal{S}_M$;
- $Q^2 = \frac{1}{2}[Q, Q] = 0$, i.e. Q is cohomological.

Remark

As a consequence of Q being cohomological, the BV action satisfies the classical master equation

$$(S, S) = 0.$$

Why is it useful?

Why is it useful?

- BV formalism used to obtain cohomological description of spaces of observables

Why is it useful?

- BV formalism used to obtain cohomological description of spaces of observables
- generalizes BRST when symmetries close only "on-shell"

Why is it useful?

- BV formalism used to obtain cohomological description of spaces of observables
- generalizes BRST when symmetries close only "on-shell"
- compatible with cutting and gluing, TQFT-like theories and axiomatic definitions of QFT's

Why is it useful?

- BV formalism used to obtain cohomological description of spaces of observables
- generalizes BRST when symmetries close only "on-shell"
- compatible with cutting and gluing, TQFT-like theories and axiomatic definitions of QFT's
- provides consistent formalism for quantization of gauge theories

A paradigmatic example: gauge field theory

A paradigmatic example: gauge field theory

- Let G be a gauge group on M, with Lie algebra $\mathfrak{g}$, and F_M be the space of fields (section of associated vector bundle, space of connections, etc.). There exists an action of the Lie algebra

$$\rho\colon \Gamma(M,\mathfrak{g}) \longrightarrow \mathfrak{X}(F_M)$$

forming an involutive distribution $\mathfrak{D} := \rho(\Gamma(M,\mathfrak{g})) \subset \mathfrak{X}(F_M)$

A paradigmatic example: gauge field theory

- Let G be a gauge group on M, with Lie algebra $\mathfrak{g}$, and F_M be the space of fields (section of associated vector bundle, space of connections, etc.). There exists an action of the Lie algebra

$$\rho\colon \Gamma(M, \mathfrak{g}) \longrightarrow \mathfrak{X}(F_M)$$

  forming an involutive distribution $\mathfrak{D} := \rho(\Gamma(M, \mathfrak{g})) \subset \mathfrak{X}(F_M)$

- Consider the bundle

$$\mathfrak{D}[1] \to F_M,$$

  definining the multiplet $\Phi^\alpha(x) = (\varphi^i(x), c^a(x)) \in \mathfrak{D}[1]$, with fields $\phi^i(x) \in F_M$ and ghosts $c^a(x) \in \Gamma(M, \mathfrak{g}[1])$

A paradigmatic example: gauge field theory

- The space of BV fields is given by

$$\mathcal{F}_M := T^*[-1]\mathfrak{D}[1]$$

with fiber given by the multiplet of anti–fields $\Phi^\dagger_\alpha(x) = (\varphi^\dagger_i(x), c^\dagger_\alpha(x))$ containing the field momenta $\varphi^\dagger_i(x) \in T^*[-1]F_M$ and the ghost momenta $c^\dagger_a \in \Gamma(M, \mathfrak{g}^*[-2])$

A paradigmatic example: gauge field theory

- The space of BV fields is given by

$$\mathcal{F}_M := T^*[-1]\mathfrak{D}[1]$$

  with fiber given by the multiplet of anti–fields $\Phi^\dagger_\alpha(x) = (\varphi^\dagger_i(x), c^\dagger_\alpha(x))$ containing the field momenta $\varphi^\dagger_i(x) \in T^*[-1]F_M$ and the ghost momenta $c^\dagger_a \in \Gamma(M, \mathfrak{g}^*[-2])$
- there exists a canonical -1 symplectic form

$$\varpi_M = \int_M \delta\Phi^\dagger_\alpha \wedge \delta\Phi^\alpha$$

A paradigmatic example: gauge field theory

- The space of BV fields is given by

$$\mathcal{F}_M := T^*[-1]\mathfrak{D}[1]$$

  with fiber given by the multiplet of anti–fields $\Phi^\dagger_\alpha(x) = (\varphi^\dagger_i(x), c^\dagger_\alpha(x))$ containing the field momenta $\varphi^\dagger_i(x) \in T^*[-1]F_M$ and the ghost momenta $c^\dagger_a \in \Gamma(M, \mathfrak{g}^*[-2])$
- there exists a canonical -1 symplectic form

$$\varpi_M = \int_M \delta\Phi^\dagger_\alpha \wedge \delta\Phi^\alpha$$

- Assume $\mathbb{Q}_0 \in \mathfrak{D}[1]$ is the vector field encoding the gauge symmetry, i.e. $\mathbb{Q}_0(S_M) = 0$

A paradigmatic example: gauge field theory

- In the simplest cases, the BV action is given by

$$\mathcal{S}_M = S_M + \int_M \Phi^\dagger_\alpha \wedge \mathbb{Q}_0(\phi^\alpha)$$

A paradigmatic example: gauge field theory

- In the simplest cases, the BV action is given by

$$\mathcal{S}_M = S_M + \int_M \Phi^\dagger_\alpha \wedge \mathbb{Q}_0(\phi^\alpha)$$

obtaining

$$Q(\Phi^\alpha) = \mathbb{Q}_0(\Phi^\alpha) \quad \text{and} \quad Q(\Phi^\dagger_\alpha) = \frac{\delta L_M}{\delta \Phi^\alpha} - (-1)^\beta \Phi^\dagger_\beta \frac{\delta(\mathbb{Q}_0 \phi^\beta)}{\delta \Phi^\alpha},$$

A paradigmatic example: gauge field theory

- In the simplest cases, the BV action is given by

$$\mathcal{S}_M = S_M + \int_M \Phi^\dagger_\alpha \wedge \mathbb{Q}_0(\phi^\alpha)$$

obtaining

$$Q(\Phi^\alpha) = \mathbb{Q}_0(\Phi^\alpha) \quad \text{and} \quad Q(\Phi^\dagger_\alpha) = \frac{\delta L_M}{\delta \Phi^\alpha} - (-1)^\beta \Phi^\dagger_\beta \frac{\delta(\mathbb{Q}_0 \phi^\beta)}{\delta \Phi^\alpha},$$

- Letting C^k be the functions of ghost degree k on $\mathcal{F}_M$, one can intuitively see

$$H^0(Q) = \frac{\ker Q : C^0 \to C^1}{\operatorname{Im} Q : C^{-1} \to C^0} \simeq C^\infty \left\{ \Phi^\alpha \;\middle|\; \frac{\delta L_M}{\delta \Phi^\alpha} = 0 \right\} / \mathcal{G},$$

A paradigmatic example: gauge field theory

- It is often the case that the symmetries close only on shell, i.e.

$$\mathbb{Q}_0^2 \Phi^\alpha = \frac{\delta L_M}{\delta \Phi^\beta} M^{\alpha\beta}(\Phi)$$

A paradigmatic example: gauge field theory

- It is often the case that the symmetries close only on shell, i.e.

$$\mathbb{Q}_0^2 \Phi^\alpha = \frac{\delta L_M}{\delta \Phi^\beta} M^{\alpha\beta}(\Phi)$$

- the BV action then needs to be modified by adding a quadratic term in the anti–fields

$$\mathcal{S}_M = S_M + \int_M \Phi^\dagger_\alpha \mathbb{Q}_0(\phi^\alpha) + \frac{1}{2} \Phi^\dagger_\alpha \Phi^\dagger_\beta M^{\alpha\beta}(\Phi),$$

A paradigmatic example: gauge field theory

- It is often the case that the symmetries close only on shell, i.e.

$$\mathbb{Q}_0^2\Phi^\alpha = \frac{\delta L_M}{\delta\Phi^\beta} M^{\alpha\beta}(\Phi)$$

- the BV action then needs to be modified by adding a quadratic term in the anti–fields

$$\mathcal{S}_M = S_M + \int_M \Phi^\dagger_\alpha \mathbb{Q}_0(\phi^\alpha) + \frac{1}{2}\Phi^\dagger_\alpha\Phi^\dagger_\beta M^{\alpha\beta}(\Phi),$$

- obtaining

$$Q(\Phi^\alpha) = \mathbb{Q}_0(\Phi^\alpha) + \Phi^\dagger_\beta M^{\alpha\beta},$$
$$Q(\Phi^\dagger_\alpha) = \frac{\delta L_M}{\delta\Phi^\alpha} - (-1)^\beta \Phi^\dagger_\beta \frac{\delta(\mathbb{Q}_0\phi^\beta)}{\delta\Phi^\alpha} + \frac{(-1)^{\beta+\gamma}}{2}\Phi^\dagger_\beta\Phi^\dagger_\gamma \frac{\delta M^{\beta\gamma}}{\delta\Phi^\alpha}.$$

The BV formulation of Palatini–Cartan gravity

- Let $P_{SO} \to M$ be a principal $\mathrm{SO}(3,1)$–bundle and V be a 4–dimensional vector space equipped with the Minkowski metric

The BV formulation of Palatini–Cartan gravity

- Let $P_{SO} \to M$ be a principal $\mathrm{SO}(3,1)$–bundle and V be a 4–dimensional vector space equipped with the Minkowski metric
- The Minkowski bundle is the associated vector bundle

$$\mathcal{V} := P_{SO} \times_\lambda V,$$

where λ is the vector (spin 1) representation of $\mathrm{SO}(3,1)$

The BV formulation of Palatini–Cartan gravity

- Let $P_{SO} \to M$ be a principal SO(3,1)–bundle and V be a 4–dimensional vector space equipped with the Minkowski metric
- The Minkowski bundle is the associated vector bundle

$$\mathcal{V} := P_{SO} \times_\lambda V,$$

where λ is the vector (spin 1) representation of SO(3,1)

Definition

The Palatini–Cartan theory of gravity is given by the following data

$$F_{PC} := \Omega^1_{\mathrm{n.d.}}(M, \mathcal{V}) \times \mathcal{A}_M \ni (e, \omega) \quad \text{and} \quad S_{PC} = \int_M \frac{e^2}{2} F_\omega,$$

where

- e is the vielbein, seen as a linear isomorphism $TM \to \mathcal{V}$;
- ω is a local connection, locally modeled by elements in $\Omega^1(M, \wedge^2\mathcal{V})$, after noticing $\mathfrak{so}(3,1) \simeq \wedge^2 V$.

The BV formulation of Palatini–Cartan gravity: Symmetries

The BV formulation of Palatini–Cartan gravity: Symmetries

- Let $c \in \Gamma[1](\wedge^2\mathcal{V})$ be the gauge parameter related to internal Lorentz symmetry, the infinitesimal transformation is

$$\delta_c e = [c, e] \qquad \text{and} \qquad \delta_c \omega = d_\omega c.$$

The BV formulation of Palatini–Cartan gravity: Symmetries

- Let $c \in \Gamma[1](\wedge^2\mathcal{V})$ be the gauge parameter related to internal Lorentz symmetry, the infinitesimal transformation is

$$\delta_c e = [c, e] \qquad \text{and} \qquad \delta_c \omega = d_\omega c.$$

- Let $\xi \in \mathfrak{X}[1](M)$ be the gauge parameter of diffeomorphisms, then

$$\delta_\xi e = \mathrm{L}_\xi^\omega e := \iota_\xi d_\omega e - d_\omega \iota_\xi e \qquad \text{and} \qquad \delta_\xi \omega = \iota_\xi F_\omega$$

The BV formulation of Palatini–Cartan gravity: Symmetries

- Let $c \in \Gamma[1](\wedge^2\mathcal{V})$ be the gauge parameter related to internal Lorentz symmetry, the infinitesimal transformation is

$$\delta_c e = [c, e] \qquad \text{and} \qquad \delta_c \omega = d_\omega c.$$

- Let $\xi \in \mathfrak{X}[1](M)$ be the gauge parameter of diffeomorphisms, then

$$\delta_\xi e = \mathrm{L}_\xi^\omega e := \iota_\xi d_\omega e - d_\omega \iota_\xi e \qquad \text{and} \qquad \delta_\xi \omega = \iota_\xi F_\omega$$

- We obtain the distribution $\mathfrak{D}[1] \subset \mathfrak{X}[1](F_{PC})$ given by elements $\mathbb{X} \in \mathfrak{D}[1]$ of the kind

$$\mathbb{X}(e,\omega) = \int_M (\mathrm{L}_\xi^\omega e - [c, e])\frac{\delta}{\delta e} + (\iota_\xi F_\omega - d_\omega c)\frac{\delta}{\delta \omega} \in \mathfrak{X}[1](F_{PC}),$$

being parametrized by the space $\Gamma[1](\wedge^2\mathcal{V}) \times \mathfrak{X}[1](M) \ni (c, \xi)$.

The BV formulation of Palatini–Cartan gravity

- The BV space of fields is given by

$$\mathcal{F}_{PC} := T^*[-1]\big(F_{PC} \times \Gamma[1](\wedge^2\mathcal{V}) \times \mathfrak{X}[1](M)\big),$$

The BV formulation of Palatini–Cartan gravity

- The BV space of fields is given by

$$\mathcal{F}_{PC} := T^*[-1]\big(F_{PC} \times \Gamma[1](\wedge^2\mathcal{V}) \times \mathfrak{X}[1](M)\big),$$

- With the canonical -1 symplectic form

$$\varpi_{PC} = \int_M \delta e \delta e^\dagger + \delta\omega\delta\omega^\dagger + \delta c \delta c^\dagger + \iota_{\delta\xi}\delta\xi^\dagger$$

The BV formulation of Palatini–Cartan gravity

- The BV space of fields is given by

$$\mathcal{F}_{PC} := T^*[-1]\big(F_{PC} \times \Gamma[1](\wedge^2\mathcal{V}) \times \mathfrak{X}[1](M)\big),$$

- With the canonical -1 symplectic form

$$\varpi_{PC} = \int_M \delta e \delta e^\dagger + \delta\omega\delta\omega^\dagger + \delta c \delta c^\dagger + \iota_{\delta\xi}\delta\xi^\dagger$$

- The symmetries are encoded in the vector field $\mathbb{Q}_{PC}$, with

$$\mathbb{Q}_{PC} e = L_\xi^\omega e - [c, e] \qquad \mathbb{Q}_{PC}\omega = \iota_\xi F_\omega - d_\omega c$$

$$\mathbb{Q}_{PC} c = \frac{1}{2}(\iota_\xi \iota_\xi F_\omega - [c.c]) \qquad \mathbb{Q}_{PC}\xi = \frac{1}{2}[\xi, \xi].$$

The BV formulation of Palatini–Cartan gravity

With all the above information, we have

Theorem

The collection $(\mathcal{F}_{PC}, \varpi_{PC}, \mathbb{Q}_{PC}, \mathcal{S}_{PC})$ defines a BV structure, where

$$\mathcal{S}_{PC} = \int_M \frac{e^2}{2} F_\omega - (L_\xi^\omega e - [c, e])e^\dagger + (\iota_\xi F_\omega - d_\omega c)\omega^\dagger + \frac{1}{2}(\iota_\xi \iota_\xi F_\omega - [c, c])c^\dagger + \frac{1}{2}\iota_{[\xi,\xi]}\xi^\dagger.$$

Remark

Notice that in the presence of a boundary, the vector field $\mathbb{Q}_{PC}$ is not the Hamiltonian vector field of $\mathcal{S}_{PC}$, but instead

$$\delta \mathcal{S}_{PC} = \iota_{\mathbb{Q}_{PC}} \varpi_{PC} + \vartheta_{PC}^{\partial},$$

where ϑ_{PC} is a boundary term.

The $N = 1, D = 4$ Supergravity in the first order formalism

The $N = 1, D = 4$ Supergravity in the first order formalism

- Let $P \to M$ be a principal Spin$(3, 1)$–bundle and V be a 4–dimensional vector space equipped with the Minkowski metric

The $N = 1, D = 4$ Supergravity in the first order formalism

- Let $P \to M$ be a principal Spin$(3,1)$–bundle and V be a 4–dimensional vector space equipped with the Minkowski metric
- The Minkowski bundle is the associated vector bundle

$$\mathcal{V} := P \times_\lambda V,$$

where λ is the vector (spin 1) representation of Spin$(3,1)$

The $N = 1, D = 4$ Supergravity in the first order formalism

- Let $P \to M$ be a principal Spin$(3, 1)$–bundle and V be a 4–dimensional vector space equipped with the Minkowski metric
- The Minkowski bundle is the associated vector bundle

$$\mathcal{V} := P \times_\lambda V,$$

where λ is the vector (spin 1) representation of Spin$(3, 1)$

- The Dirac spinor bundle is the associated bundle

$$\mathbb{S}_D := P \times_\Gamma \mathbb{C}^4,$$

where Γ is the gamma representation of the Clifford algebra $\mathcal{C}(V)$ restricted to the spin subgroup Spin(V) $\simeq$Spin$(3, 1)$.

The $N = 1, D = 4$ Supergravity in the first order formalism

- Let $P \to M$ be a principal $\mathrm{Spin}(3,1)$–bundle and V be a 4–dimensional vector space equipped with the Minkowski metric
- The Minkowski bundle is the associated vector bundle

$$\mathcal{V} := P \times_\lambda V,$$

  where λ is the vector (spin 1) representation of $\mathrm{Spin}(3,1)$
- The Dirac spinor bundle is the associated bundle

$$\mathbb{S}_D := P \times_\Gamma \mathbb{C}^4,$$

  where Γ is the gamma representation of the Clifford algebra $\mathcal{C}(V)$ restricted to the spin subgroup $\mathrm{Spin}(V) \simeq \mathrm{Spin}(3,1)$.

The $N = 1, D = 4$ Supergravity in the first order formalism

Definition

The first–order formulation of $N = 1, D = 4$ Supergravity is given by the following data

$$F_{SG} := \Omega^1_{\mathrm{n.d.}}(M, \mathcal{V}) \times \mathcal{A}_M \times \Omega^1(M, \Pi\mathbb{S}_M) \ni (e, \omega, \psi),$$
$$S_{PC} = \int_M \frac{e^2}{2} F_\omega + \frac{1}{3!}\bar{\psi}\gamma^3 d_\omega \psi,$$

where

- e is the vielbein, seen as a linear isomorphism $TM \to \mathcal{V}$;
- ω is a local connection, locally modeled by elements in $\Omega^1(M, \wedge^2\mathcal{V})$;
- ψ is the gravitino, seen as 1–form with values in the Majorana spinor bundle

$$\mathbb{S}_M := \{\chi \in \mathbb{S}_D \mid \bar{\chi} := \chi^\dagger\gamma_0 = \chi^t C\}$$

The $N = 1, D = 4$ Supergravity in the first order formalism

The equations of motion are

The $N = 1, D = 4$ Supergravity in the first order formalism

The equations of motion are

- The Einstein equation

$$eF_{\omega} + \frac{1}{3!}\bar{\psi}\gamma^{3}d_{\omega}\psi = 0,$$

The $N = 1, D = 4$ Supergravity in the first order formalism

The equations of motion are

- The Einstein equation

$$eF_\omega + \frac{1}{3!}\bar{\psi}\gamma^3 d_\omega \psi = 0,$$

- The torsion equation

$$d_\omega e - \frac{1}{2}\bar{\psi}\gamma\psi = 0,$$

The $N = 1, D = 4$ Supergravity in the first order formalism

The equations of motion are

- The Einstein equation

$$eF_\omega + \frac{1}{3!}\bar{\psi}\gamma^3 d_\omega \psi = 0,$$

- The torsion equation

$$d_\omega e - \frac{1}{2}\bar{\psi}\gamma\psi = 0,$$

- The Rarita-Schwinger equation

$$e d_\omega \bar{\psi}\gamma^3 + \frac{1}{2} d_\omega e \bar{\psi}\gamma^3 = 0.$$

The $N = 1, D = 4$ (Super)symmetry transformations

The $N = 1, D = 4$ (Super)symmetry transformations

- Let $\chi \in \Gamma[1](\Pi S_M)$ be the spinorial gauge parameter. Postulate the following supersymmetric transformations for e and ψ

$$\delta_\chi e = -\bar{\chi}\gamma\psi, \qquad \delta_\chi \psi = d_\omega \chi,$$

The $N = 1, D = 4$ (Super)symmetry transformations

- Let $\chi \in \Gamma[1](\Pi S_M)$ be the spinorial gauge parameter. Postulate the following supersymmetric transformations for e and ψ

$$\delta_\chi e = -\bar{\chi}\gamma\psi, \qquad \delta_\chi \psi = d_\omega \chi,$$

- Imposing the invariance of S_{SG}, one obtains

$$\delta\chi S_{SG} = \int_M \left(d_\omega e - \frac{1}{2}\bar{\psi}\gamma\psi \right) \left(e\delta_\chi \omega - \frac{1}{3!}\bar{\chi}\gamma^3 d_\omega \psi \right),$$

$$\text{hence} \quad e\delta_\chi \omega = \frac{1}{3!}\bar{\chi}\gamma^3 d_\omega \psi.$$

The $N = 1, D = 4$ (Super)symmetry transformations

- Let $\chi \in \Gamma[1](\Pi S_M)$ be the spinorial gauge parameter. Postulate the following supersymmetric transformations for e and ψ

$$\delta_\chi e = -\bar{\chi}\gamma\psi, \qquad \delta_\chi \psi = d_\omega \chi,$$

- Imposing the invariance of S_{SG}, one obtains

$$\delta\chi S_{SG} = \int_M \left(d_\omega e - \frac{1}{2}\bar{\psi}\gamma\psi \right) \left(e\delta_\chi \omega - \frac{1}{3!}\bar{\chi}\gamma^3 d_\omega \psi \right),$$

$$\text{hence} \quad e\delta_\chi \omega = \frac{1}{3!}\bar{\chi}\gamma^3 d_\omega \psi.$$

- The other gauge parameters are $c \in \Gamma[1](\wedge^2 \mathcal{V})$ and $\xi \in \mathfrak{X}[1](M)$, respectively generating the internal Lorentz symmetry and the diffeomorphism symmetry as

$$\delta_\xi e = \mathrm{L}_\xi^\omega e \qquad \delta_\xi \omega = \iota_\xi F_\omega \qquad \delta_\xi \psi = \mathrm{L}_\xi^\omega \psi$$
$$\delta_c e = [c, e] \qquad \delta_c \omega = d_\omega c \qquad \delta_c \psi = [c, \psi].$$

Squaring the supersymmetry

An interesting exercise is to compute the expression of δ_χ^2.

Squaring the supersymmetry

An interesting exercise is to compute the expression of δ_χ^2. We obtain

$$\delta_\chi^2 e = -\frac{1}{2}\mathrm{L}_\varphi^\omega e + \frac{1}{2}\iota_\varphi\left(d_\omega e - \frac{1}{2}\bar{\psi}\gamma\psi\right)$$

$$\delta_\chi^2 \psi = -\frac{1}{2}\mathrm{L}_\varphi^\omega \psi + \frac{1}{2}\iota_\varphi d_\omega\psi - \left(\bar{\chi}\kappa(< \bar{e}, \underline{\gamma} d_\omega \psi >) + \frac{1}{8}\bar{\chi}\iota_{\hat{\gamma}}\iota_{\hat{\gamma}}(\underline{\gamma} d_\omega\psi)\right)\chi$$

$$e\delta_\chi^2 \omega = -\frac{1}{2}e\iota_\varphi F_\omega + \frac{1}{2}\iota_\varphi\left(eF_\omega + \frac{1}{3!}\bar{\psi}\gamma^3 d_\omega\psi\right) - \frac{1}{2\cdot 3!}\bar{\psi}\iota_\varphi(\gamma^3 d_\omega\psi)$$
$$- \frac{1}{3!}\bar{\psi}\gamma^3\chi\left(\bar{\chi}\kappa(< \bar{e}, \underline{\gamma} d_\omega \psi >) + \frac{1}{8}\bar{\chi}\iota_{\hat{\gamma}}\iota_{\hat{\gamma}}(\underline{\gamma} d_\omega\psi)\right),$$

where $\varphi = \bar{\chi}\hat{\gamma}\chi$, $\hat{\gamma} := \gamma^\mu\partial_\mu$ and $\underline{\gamma} = \gamma_\mu dx^\mu$.

Remark

Notice that $\delta_\chi^2\Phi = -\frac{1}{2}\delta_\varphi\Phi + f(\mathrm{Eom_s})$.

The first attempt to a BV action

- The main construction tells us space of BV fields is

$$\mathcal{F}_{SG} = T^*[-1]\big(F_{SG} \times \Gamma[1](\wedge^2\mathcal{V}) \times \mathfrak{X}[1](M) \times \Gamma[1](M, \Pi\mathbb{S}_M)\big),$$

The first attempt to a BV action

- The main construction tells us space of BV fields is

$$\mathcal{F}_{SG} = T^*[-1]\big(F_{SG} \times \Gamma[1](\wedge^2 \mathcal{V}) \times \mathfrak{X}[1](M) \times \Gamma[1](M, \Pi\mathbb{S}_M)\big),$$

- The -1–symplectic forms reads

$$\varpi_{SG} = \int_M \delta e \delta e^\dagger + \delta\omega\delta\omega^\dagger + i\delta\bar{\psi}\delta\psi_\dagger + \delta c \delta c^\dagger + \iota_{\delta\xi}\delta\xi^\dagger + i\delta\bar{\chi}\delta\chi_\dagger.$$

The first attempt to a BV action

- The main construction tells us space of BV fields is

$$\mathcal{F}_{SG} = T^*[-1]\big(F_{SG} \times \Gamma[1](\wedge^2\mathcal{V}) \times \mathfrak{X}[1](M) \times \Gamma[1](M, \Pi\mathbb{S}_M)\big),$$

- The -1–symplectic forms reads

$$\varpi_{SG} = \int_M \delta e\delta e^\dagger + \delta\omega\delta\omega^\dagger + i\delta\bar{\psi}\delta\psi_\dagger + \delta c\delta c^\dagger + \iota_{\delta\xi}\delta\xi^\dagger + i\delta\bar{\chi}\delta\chi_\dagger.$$

- The vector field $\mathbb{Q}_0$ encoding the symmetries is given by the sum of the infinitesimal gauge transformations, i.e.

$$\mathbb{Q}_0 e = L_\xi^\omega e - [c, e] + \bar{\chi}\gamma\psi \qquad \mathbb{Q}_0\omega = \iota_\xi F_\omega - d_\omega c + \delta_\chi\omega$$
$$\mathbb{Q}_0\psi = L_\xi^\omega\psi - [c, \psi] - d_\omega\chi \qquad \mathbb{Q}_0\xi = \frac{1}{2}[\xi, \xi] + \frac{1}{2}\varphi$$
$$\mathbb{Q}_0 c = \frac{1}{2}(\iota_\xi\iota_\xi F_\omega - [c.c]) + \iota_\xi\delta_\chi\omega \quad \mathbb{Q}_0\chi = L_\xi^\omega\chi - [c, \chi] - \frac{1}{2}\iota_\varphi\psi,$$

The first attempt to a BV action

- However, the Classical Master Equation fails, indeed

$$
\begin{aligned}
\mathbb{Q}_0^2 e &= \frac{1}{2}\iota_\varphi\left(d_\omega e - \frac{1}{2}\bar{\psi}\gamma\psi\right)\\
\mathbb{Q}_0^2 \psi &= \frac{1}{2}\iota_\varphi d_\omega \psi - \left(\bar{\chi}\kappa(< \bar{e}, \underline{\gamma} d_\omega \psi >) + \frac{1}{8}\bar{\chi}\iota_{\hat{\gamma}}\iota_{\hat{\gamma}}(\underline{\gamma} d_\omega \psi)\right)\chi\\
e\mathbb{Q}_0^2 \omega &= \frac{1}{2}\iota_\varphi\left(eF_\omega + \frac{1}{3!}\bar{\psi}\gamma^3 d_\omega\psi\right) + \frac{1}{2\cdot 3!}\bar{\psi}\gamma^3\iota_\varphi d_\omega\psi\\
&\quad - \frac{1}{3!}\bar{\psi}\gamma^3\chi\left(\bar{\chi}\kappa(< \bar{e}, \underline{\gamma} d_\omega \psi >) + \frac{1}{8}\bar{\chi}\iota_{\hat{\gamma}}\iota_{\hat{\gamma}}(\underline{\gamma} d_\omega \psi)\right)\\
\mathbb{Q}_0^2 c &= \frac{1}{2}\iota_\varphi\delta_\chi\omega + \iota_\xi\mathbb{Q}_0^2\omega \qquad \mathbb{Q}_0^2\chi = 0 \qquad \mathbb{Q}_0^2\xi = 0.
\end{aligned}
$$

The first attempt to a BV action

- However, the Classical Master Equation fails, indeed

$$\mathbb{Q}_0^2 e = \frac{1}{2}\iota_\varphi\left(d_\omega e - \frac{1}{2}\bar{\psi}\gamma\psi\right)$$

$$\mathbb{Q}_0^2 \psi = \frac{1}{2}\iota_\varphi d_\omega\psi - \left(\bar{\chi}\kappa(< \bar{e}, \underline{\gamma} d_\omega\psi >) + \frac{1}{8}\bar{\chi}\iota_{\hat{\gamma}}\iota_{\hat{\gamma}}(\underline{\gamma} d_\omega\psi)\right)\chi$$

$$e\mathbb{Q}_0^2\omega = \frac{1}{2}\iota_\varphi\left(eF_\omega + \frac{1}{3!}\bar{\psi}\gamma^3 d_\omega\psi\right) + \frac{1}{2\cdot 3!}\bar{\psi}\gamma^3\iota_\varphi d_\omega\psi$$
$$- \frac{1}{3!}\bar{\psi}\gamma^3\chi\left(\bar{\chi}\kappa(< \bar{e}, \underline{\gamma} d_\omega\psi >) + \frac{1}{8}\bar{\chi}\iota_{\hat{\gamma}}\iota_{\hat{\gamma}}(\underline{\gamma} d_\omega\psi)\right)$$

$$\mathbb{Q}_0^2 c = \frac{1}{2}\iota_\varphi\delta_\chi\omega + \iota_\xi\mathbb{Q}_0^2\omega \qquad \mathbb{Q}_0^2\chi = 0 \qquad \mathbb{Q}_0^2\xi = 0.$$

- The BV action needs to be complemented with a rank–2 contribution

The Full BV action of $N=1, D=4$ SuGra

- To make the computations easier, we can uniquely define $\check{c} \in \Omega^{(2,0)}[-1]$, $\check{\omega} \in \Omega^{(2,1)}[-1]$ and $\psi_\dagger^0 \in \Omega^1[-1](M, \Pi\mathbb{S}_M)$ such that

$$c^\dagger = \frac{e^2}{2}\check{c}, \qquad \omega^\dagger = e\check{\omega} \quad \text{and} \quad \psi_\dagger := \frac{1}{3!}e\gamma^3\underline{\gamma}\psi_\dagger^0$$

The Full BV action of $N = 1, D = 4$ SuGra

- To make the computations easier, we can uniquely define $\check{c} \in \Omega^{(2,0)}[-1]$, $\check{\omega} \in \Omega^{(2,1)}[-1]$ and $\psi_\dagger^0 \in \Omega^1[-1](M, \Pi\mathbb{S}_M)$ such that

$$c^\dagger = \frac{e^2}{2}\check{c}, \qquad \omega^\dagger = e\check{\omega} \quad \text{and} \quad \psi_\dagger := \frac{1}{3!} e\gamma^3 \underline{\gamma} \psi_\dagger^0$$

- Furthermore, we define $k^\dagger := \omega^\dagger - \iota_\xi c^\dagger$ and $\check{k} \in \Omega^{(2,1)}[-1]$ such that $k^\dagger = e\check{k}$

The Full BV action of $N = 1, D = 4$ SuGra

- To make the computations easier, we can uniquely define $\check{c} \in \Omega^{(2,0)}[-1]$, $\check{\omega} \in \Omega^{(2,1)}[-1]$ and $\psi_{\dagger}^{0} \in \Omega^{1}[-1](M, \Pi\mathbb{S}_M)$ such that

$$c^{\dagger} = \frac{e^2}{2}\check{c}, \qquad \omega^{\dagger} = e\check{\omega} \quad \text{and} \quad \psi_{\dagger} := \frac{1}{3!} e\gamma^3 \underline{\gamma}\psi_{\dagger}^{0}$$

- Furthermore, we define $k^{\dagger} := \omega^{\dagger} - \iota_{\xi} c^{\dagger}$ and $\check{k} \in \Omega^{(2,1)}[-1]$ such that $k^{\dagger} = e\check{k}$

Then we have

Theorem

The collection $(\mathcal{F}_{SG}, \varpi_{PC}, \mathbb{Q}_{SG}, \mathcal{S}_{SG})$ defines a BV structure, where

$$\mathcal{S}_{SG} = S_{SG}^{0} + \int_M \Phi_{\alpha}^{\dagger} \mathbb{Q}_{SG}(\Phi^{\alpha}) + s_2$$

The Full BV action of $N = 1, D = 4$ SuGra

where

$$
\begin{aligned}
s_2 := \int_M \frac{1}{2}\check{k}\iota_\varphi e^\dagger &+ \frac{1}{4}\left(\frac{1}{2}\bar{\psi}^0_\dagger\underline{\gamma} + \alpha(\check{k}\bar{\psi})\underline{\gamma} - \frac{i}{2}\check{c}\bar{\chi}\right)\iota_\varphi\psi_\dagger \\
&+ \frac{i}{4\cdot 3!}\left(\alpha(\check{k}\bar{\psi})\underline{\gamma} - \frac{i}{2}\check{c}\bar{\chi}\right)\gamma^3\iota_\varphi(\breve{\omega}\psi) \\
&- \frac{i}{2\cdot 3!}\left(\frac{1}{2}\bar{\psi}^0_\dagger\underline{\gamma} + \frac{1}{2}\alpha(\check{k}\bar{\psi})\underline{\gamma}\right)\gamma^3\chi < e, \bar{\chi}[\breve{\omega},\gamma]\psi > \\
&+ \frac{1}{2\cdot 3!}\left(\frac{1}{4}\bar{\psi}^0_\dagger\underline{\gamma} + \frac{1}{2}\alpha(\check{k}\bar{\psi})\underline{\gamma}\right)\gamma^3\chi < e, \bar{\chi}\underline{\gamma}^2\psi^0_\dagger > \\
&- \frac{1}{32}\left(i\bar{\psi}_\dagger\chi + \frac{1}{3!}\check{k}\bar{\psi}\gamma^3\chi\right)\bar{\chi}\iota_{\hat{\gamma}}\iota_{\hat{\gamma}}([\breve{\omega},\gamma]\psi) \\
&- \frac{i}{32}\left(i\bar{\psi}_\dagger\chi + \frac{1}{3!}\check{k}\bar{\psi}\gamma^3\chi\right)\bar{\chi}\iota_{\hat{\gamma}}\iota_{\hat{\gamma}}(\underline{\gamma}^2\psi^0_\dagger).
\end{aligned}
$$

Extra: How to compute infinitesimal gauge transformations

Extra: How to compute infinitesimal gauge transformations

- Assume $\Sigma = \partial M$. There is a boundary term in the variation of the action
$$\delta S_{SG} = \int_M \mathsf{EL}_M - \int_\Sigma \frac{e^2}{2}\delta\omega + \frac{1}{3!} e\bar{\psi}\gamma^3\delta\psi,$$
defined on the space of (pre–)boundary fields $\widetilde{F}^{\partial}_{SG} = \Omega^1_{\mathrm{n.d.}}(\Sigma, \mathcal{V}) \times \mathcal{A}(\Sigma) \times \Omega^1(\Sigma, \Pi\mathbb{S}_M)$

Extra: How to compute infinitesimal gauge transformations

- Assume $\Sigma = \partial M$. There is a boundary term in the variation of the action
$$\delta S_{SG} = \int_M \mathsf{EL}_M - \int_\Sigma \frac{e^2}{2}\delta\omega + \frac{1}{3!}e\bar{\psi}\gamma^3\delta\psi,$$
defined on the space of (pre–)boundary fields
$\widetilde{F}^{\partial}_{SG} = \Omega^1_{\mathrm{n.d.}}(\Sigma, \mathcal{V}) \times \mathcal{A}(\Sigma) \times \Omega^1(\Sigma, \Pi\mathbb{S}_M)$
- the variation of the boundary term yields a closed 2–form
$$\tilde{\varpi}_\Sigma = \int_\Sigma e\delta e\delta\omega + \frac{1}{3!}\bar{\psi}\gamma^3\delta\psi\delta e + \frac{1}{3!}e\delta\bar{\psi}\gamma^3\delta\psi$$
which is degenerate

Extra: How to compute infinitesimal gauge transformations

- Assume $\Sigma = \partial M$. There is a boundary term in the variation of the action
$$\delta S_{SG} = \int_M \mathrm{EL}_M - \int_\Sigma \frac{e^2}{2}\delta\omega + \frac{1}{3!} e\bar{\psi}\gamma^3\delta\psi,$$
defined on the space of (pre–)boundary fields
$\widetilde{F}^{\partial}_{SG} = \Omega^1_{\mathrm{n.d.}}(\Sigma, \mathcal{V}) \times \mathcal{A}(\Sigma) \times \Omega^1(\Sigma, \Pi\mathbb{S}_M)$
- the variation of the boundary term yields a closed 2–form
$$\tilde{\varpi}_\Sigma = \int_\Sigma e\delta e\delta\omega + \frac{1}{3!}\bar{\psi}\gamma^3\delta\psi\delta e + \frac{1}{3!} e\delta\bar{\psi}\gamma^3\delta\psi$$
which is degenerate
- For any $\mathbb{X} = \int_\Sigma \mathbb{X}_e \frac{\delta}{\delta e} + \mathbb{X}_\omega \frac{\delta}{\delta\omega} + \mathbb{X}_\psi \frac{\delta}{\delta\psi}$
$$\mathrm{Ker}(\tilde{\varpi}_\Sigma) = \left\{ \int_\Sigma \mathbb{X}_\omega \frac{\delta}{\delta\omega} \in \mathfrak{X}(\tilde{F}_\Sigma) \mid e\mathbb{X}_\omega = 0,\ \mathbb{X}_\omega \in \Omega^1(\Sigma, \wedge^2\mathcal{V}) \right\}$$

Extra: How to compute infinitesimal gauge transformations

- The space of boundary fields is the symplectic manifold $(F^{\partial}_{SG}, \varpi_{\Sigma})$, where

$$F^{\partial}_{SG} = \Omega^1_{\mathrm{n.d.}}(\Sigma, \mathcal{V}) \times \mathcal{A}_{\mathrm{red}}(\Sigma) \times \Omega^1(\Sigma, \Pi\mathbb{S}_M),$$
$$\mathcal{A}_{red}(\Sigma) = \{[\omega] \mid \omega' \sim \omega \text{ iff } \omega' = \omega + v,\ ev = 0\}$$

Extra: How to compute infinitesimal gauge transformations

- The space of boundary fields is the symplectic manifold $(F^{\partial}_{SG}, \varpi_{\Sigma})$, where

$$F^{\partial}_{SG} = \Omega^1_{\mathrm{n.d.}}(\Sigma, \mathcal{V}) \times \mathcal{A}_{\mathrm{red}}(\Sigma) \times \Omega^1(\Sigma, \Pi\mathbb{S}_M),$$
$$\mathcal{A}_{red}(\Sigma) = \{[\omega] \mid \omega' \sim \omega \text{ iff } \omega' = \omega + v,\ ev = 0\}$$

- Interested in space of Cauchy data C_{Σ}

$$C_{\Sigma} := \{\Phi \in F^{\partial}_{SG} \mid \text{s.t. } (\mathrm{EL})|_{\Sigma}(\Phi) = 0\}.$$

Extra: How to compute infinitesimal gauge transformations

- The space of boundary fields is the symplectic manifold $(F^{\partial}_{SG}, \varpi_{\Sigma})$, where

$$F^{\partial}_{SG} = \Omega^1_{\mathrm{n.d.}}(\Sigma, \mathcal{V}) \times \mathcal{A}_{\mathrm{red}}(\Sigma) \times \Omega^1(\Sigma, \Pi\mathbb{S}_M),$$
$$\mathcal{A}_{red}(\Sigma) = \{[\omega] \mid \omega' \sim \omega \text{ iff } \omega' = \omega + v,\ ev = 0\}$$

- Interested in space of Cauchy data C_{Σ}

$$C_{\Sigma} := \{\Phi \in F^{\partial}_{SG} \mid \text{s.t. } (\mathrm{EL})|_{\Sigma}(\Phi) = 0\}.$$

- The restriction of the EL equations to the boundary give rise to constraints,

$$eF_{\omega} + \frac{1}{3!}\bar{\psi}\gamma^3 d_{\omega}\psi = 0, \quad d_{\omega}e - \frac{1}{2}\bar{\psi}\gamma\psi = 0,$$
$$\frac{e}{3!}\gamma^3 d_{\omega}\psi - \frac{1}{2}d_{\omega}e\gamma^3\psi = 0$$

Extra: How to compute infinitesimal gauge transformations

Theorem

The constraints are invariant under the v–translation except the torsion equation, which splits into the invariant constraint

$$e\left(d_\omega e - \frac{1}{2}\bar{\psi}\gamma\psi\right) = 0,$$

and the structural constraint

$$\epsilon_n\left(d_\omega e - \frac{1}{2}\bar{\psi}\gamma\psi\right) = e\sigma$$

where $\{e_i, \epsilon_n\}$ is a basis of $\mathcal{V}$.

Furthermore, for all $\omega' \in \mathcal{A}(\Sigma)$ there exists a unique decomposition $\omega' = \omega + v$, where ω satisfies the structural constraint and $ev = 0$, fixing uniquely a representative $\omega \in [\omega']$.

Extra: How to compute infinitesimal gauge transformations

- The constraint are casted into functionals over F_{SG}^{∂} by using Lagrange multipliers

$$J_\mu = \int_\Sigma \mu \left(eF_\omega + \frac{1}{3!}\bar{\psi}\gamma^3 d_\omega \psi \right)$$

$$L_c = \int_\Sigma c \left(ed_\omega e - \frac{1}{2} e\bar{\psi}\gamma\psi \right)$$

$$M_\chi = \int_\Sigma \frac{1}{3}\bar{\chi} \left(e\gamma^3 d_\omega \psi - \frac{1}{2} d_\omega e\gamma^3 \psi \right),$$

$\mu \in \Gamma[1](\Sigma, \mathcal{V})$, $c \in \Gamma[1](\Sigma, \wedge^2\mathcal{V})$, $\chi \in \Gamma[1](\Sigma, \Pi\mathbb{S}_M)$

Extra: How to compute infinitesimal gauge transformations

- The constraint are casted into functionals over F^{∂}_{SG} by using Lagrange multipliers

$$J_\mu = \int_\Sigma \mu \left(eF_\omega + \frac{1}{3!}\bar{\psi}\gamma^3 d_\omega \psi \right)$$
$$L_c = \int_\Sigma c \left(ed_\omega e - \frac{1}{2} e\bar{\psi}\gamma\psi \right)$$
$$M_\chi = \int_\Sigma \frac{1}{3}\bar{\chi} \left(e\gamma^3 d_\omega \psi - \frac{1}{2} d_\omega e \gamma^3 \psi \right),$$

$\mu \in \Gamma[1](\Sigma, \mathcal{V})$, $c \in \Gamma[1](\Sigma, \wedge^2\mathcal{V})$, $\chi \in \Gamma[1](\Sigma, \Pi\mathbb{S}_M)$

- can use the basis $\{e_i, \epsilon_n\}$ of $\mathcal{V}$ to split $\mu = \iota_\xi e + \lambda\epsilon_n$. obtaining

$$P_\xi = \int_\Sigma \frac{1}{2}\iota_\xi(e^2)F_\omega + \frac{1}{3!}\iota_\xi e\bar{\psi}\gamma^3 d_\omega\psi,$$
$$H_\lambda = \int_\Sigma \lambda\epsilon_n \left(eF_\omega + \frac{1}{3!}\bar{\psi}\gamma^3 d_\omega\psi \right).$$

Extra: How to compute infinitesimal gauge transformations

- can preserve the boundary set by introducing transformation $P_\xi \mapsto P_\xi - L_{\iota_\xi(\omega-\omega_0)} - M_{\iota_\xi \psi}$, then

$$P_\xi = \int_\Sigma \frac{1}{2}\iota_\xi e^2 F_\omega + \iota_\xi(\omega - \omega_0) e d_\omega e - \frac{1}{3!} e\bar{\psi}\gamma^3 \mathrm{L}_\xi^{\omega_0}\psi.$$

Extra: How to compute infinitesimal gauge transformations

- can preserve the boundary set by introducing transformation $P_\xi \mapsto P_\xi - L_{\iota_\xi(\omega-\omega_0)} - M_{\iota_\xi \psi}$, then

$$P_\xi = \int_\Sigma \frac{1}{2}\iota_\xi e^2 F_\omega + \iota_\xi(\omega - \omega_0) e d_\omega e - \frac{1}{3!} e\bar{\psi}\gamma^3 \mathrm{L}_\xi^{\omega_0}\psi.$$

Proposition

The Hamiltonian vector fields of the constraints are the infinitesimal gauge transformations, i.e. vector fields $\mathbb{X}_I = \{\mathbb{P}_\xi, \mathbb{L}_c, \mathbb{H}_\lambda, \mathbb{M}_\chi\}$ such that $\iota_{\mathbb{X}_I}\varpi_\Sigma = \delta f_I$, where $f_I = \{P_\xi, L_c, H_\lambda, M_\chi\}$. Explicitly

$$\begin{aligned}
&\mathbb{L}_e = [c, e] && \mathbb{L}_\omega = d_\omega c && \mathbb{L}_\psi = [c, \psi] \\
&\mathbb{P}_e = -\mathrm{L}_\xi^{\omega_0} e && \mathbb{P}_\omega = -\iota_\xi F_{\omega_0} - \mathrm{L}_\xi^{\omega_0}(\omega - \omega_0) && \mathbb{P}_\psi = -\mathrm{L}_\xi^{\omega_0}\psi \\
&\mathbb{M}_e = -\bar{\chi}\gamma\psi && e\mathbb{M}_\omega = \frac{1}{3!}\bar{\chi}\gamma^3 d_\omega \psi && \mathbb{M}_\psi = d_\omega \chi
\end{aligned} \qquad (1)$$