Syzygy Modules and the D(2)-problem

In the Case of Dihedral Groups

John Evans
j.evans8@herts.ac.uk

University of Hertfordshire UH

Plan for today

Wall's D(2)-problem

Syzygy Modules

Syzygies of Cyclic Groups

Syzygies of Dihedral Groups D_{2p}

Bringing this back to Wall's D(2)-problem

Wall's D(2)-Problem

- Arose from an attempt during the 1960s to classify compact manifolds by means of 'surgery'.
- Key Figures: C.T.C. Wall, W. Browder, S.P. Novikov
- Earlier key figures: Thom, Wallace, Milnor, Smale.

Wall's D(2)-Problem

- Arose from an attempt during the 1960s to classify compact manifolds by means of 'surgery'.
- Key Figures: C.T.C. Wall, W. Browder, S.P. Novikov
- Earlier key figures: Thom, Wallace, Milnor, Smale.
- Two key papers by Wall:
  - Poincaré Complexes 1
  - Finiteness Conditions for CW Complexes.

  Foundational Question: What conditions are necessary to impose before a space can be homotopy equivalent to one of dimension $\leq n$.

How do we answer this question?

- Obvious first condition: homology groups should vanish in dimensions $> n$. Clearly a necessary condition.

How do we answer this question?

- Obvious first condition: homology groups should vanish in dimensions $> n$. Clearly a necessary condition.
- Problem: homology is a notoriously bad indicator of dimension, e.g. 'Moore Space'.

Let m be a positive integer. The Moore space $M(m, n)$ is formed from the n-sphere by attaching an $(n+1)$-cell via an attaching map $S^n \to S^n$ of degree m. Then $M(m, n)$ has dimension $n+1$.

How do we answer this question?

- Obvious first condition: homology groups should vanish in dimensions $> n$. Clearly a necessary condition.
- Problem: homology is a notoriously bad indicator of dimension, e.g. 'Moore Space'.

Let m be a positive integer. The Moore space $M(m, n)$ is formed from the n-sphere by attaching an $(n+1)$-cell via an attaching map $S^n \to S^n$ of degree m. Then $M(m, n)$ has dimension $n+1$.
However, computing integral homology gives,

$$H_k(M(m, n); \mathbb{Z}) = \begin{cases} \mathbb{Z}/m\mathbb{Z}, & k = n; \\ 0, & k > n. \end{cases}$$

This seems to indicate the dimesion is $dim = n$.

Introducing cohomology

A better indicator of dimension is cohomology:

$$H^k(M(m,\, n); \mathbb{Z}) = \begin{cases} \mathbb{Z}/m\mathbb{Z}, & k = n+1; \\ 0, & k > n+1. \end{cases}$$

This now gives the correct answer that $dim = n + 1$.

Stating the D(n)-problem

- If $\tilde{X}$ denotes the universal covering of X, the assumption that $H_k(\tilde{X}, \mathbb{Z}) = 0$ for all $k > n$ is enough to guarantee that X is equivalent to a space of dimension $\leq n + 1$, but not necessarily of dimension $\leq n$.

Stating the D(n)-problem

- If $\tilde{X}$ denotes the universal covering of X, the assumption that $H_k(\tilde{X}, \mathbb{Z}) = 0$ for all $k > n$ is enough to guarantee that X is equivalent to a space of dimension $\leq n + 1$, but not necessarily of dimension $\leq n$.
- Given this, we pose the problem as follows:

Wall's D(n)-problem: Let X be a complex of geometrical dimension $n + 1$. What further conditions are necessary and sufficient for X to be homotopy equivalent to a complex of dimension n?

Current state of the D(n)-problem

- *Simply connected case:* Milnor showed that the condition $H^{n+1}(X, \mathbb{Z}) = 0$ is sufficient.
- *Non-simply connected case:* We still require $H_{n+1}(\tilde{X}, \mathbb{Z}) = 0$.
  - Wall showed that for $n \geq 3$, the additional condition, both necessary and sufficient, is that $H^{n+1}(X, B) = 0$ for all coefficient bundles $\mathcal{B}$.
  - The case $n = 1$ resolved following the Stallings-Swan proof that groups of cohomological dimension one are free.

This leaves the case $n = 2$

Wall's D(2)-problem: Let X be a finite connected cell complex of geometrical dimension 3, and suppose that

$$H_3(\tilde{X}, \mathbb{Z}) = H^3(X, \mathcal{B}) = 0$$

for all coeficient systems $\mathcal{B}$ on X. Is it true that X is homotopy equivalent to a finite complex of dimension 2?

- The D(2)-problem is parametrized by the fundamental group, i.e. each finitely presented group G has its own D(2)-problem.
- We say G has the D(2)-property when the above question is answered in the affirmative.
- We say the D(2)-property fails for G if there is a finite 3-complex X_G with $\pi_1(X_G) \cong G$ which answers the above question in the negative.

Relationship with an older problem

- If K is a finite 2-complex with $\pi_1(K) = G$, we obtain an exact sequence of $\mathbb{Z}[G]$-modules,

$$0 \to \pi_2(K) \to C_2(K) \xrightarrow{\partial_2} C_1(K) \xrightarrow{\partial_1} C_0(K) \xrightarrow{\epsilon} \mathbb{Z} \to 0.$$

where $C_n(K) = H_n(\tilde{K}^{(n)}, \tilde{K}^{(n-1)})$ is the group of cellular n-chains in the universal cover of K.

Relationship with an older problem

- If K is a finite 2-complex with $\pi_1(K) = G$, we obtain an exact sequence of $\mathbb{Z}[G]$-modules,

$$0 \to \pi_2(K) \to C_2(K) \xrightarrow{\partial_2} C_1(K) \xrightarrow{\partial_1} C_0(K) \xrightarrow{\epsilon} \mathbb{Z} \to 0.$$

where $C_n(K) = H_n(\tilde{K}^{(n)}, \tilde{K}^{(n-1)})$ is the group of cellular n-chains in the universal cover of K.
- Each $C_n(K)$ is a free module over $\mathbb{Z}[G]$. This suggests that we take, as algebraic models for geometric 2-complexes, arbitrary exact sequences of the form,

$$0 \to J \to F_2 \xrightarrow{\partial_2} F_1 \xrightarrow{\partial_1} F_0 \xrightarrow{\epsilon} \mathbb{Z} \to 0,$$

where F_i is finitely generated and free over $\mathbb{Z}[G]$.
- We call such objects algebraic 2-complexes over G.

Relationship with an older problem

Realization Problem: Let G be a finitely presented group. Is every algebraic 2-complex geometrically realizable; that is, homotopy equivalent in the algebraic sense, to a complex of the form $C_*(K)$, where K is a finite 2-complex?

Key point: The D(2)-property holds for G if and only if each algebraic 2-complex over G is geometrically realizable.

What are syzygy modules?

- The basic model in module theory is that of a vector space over a field.
- Linear algebra is rendered tractable by the fact that every module over a field is free.

What are syzygy modules?

- The basic model in module theory is that of a vector space over a field.
- Linear algebra is rendered tractable by the fact that every module over a field is free.
- Over a general ring, this freeness need not be the norm.
- Nevertheless, when a module M is not free, it is natural to make a first approximation to it being free. We do this by taking a surjective homomorphism $\varphi : F_0 \to M$, where F_0 is free. We obtain an exact sequence,

$$0 \to K_1 \to F_0 \xrightarrow{\varphi} M \to 0.$$

- We sometimes regard the kernel K_1 as a *first derivative* of M.

What are syzygy modules?

- The basic model in module theory is that of a vector space over a field.
- Linear algebra is rendered tractable by the fact that every module over a field is free.
- Over a general ring, this freeness need not be the norm.
- Nevertheless, when a module M is not free, it is natural to make a first approximation to it being free. We do this by taking a surjective homomorphism $\varphi : F_0 \to M$, where F_0 is free. We obtain an exact sequence,

$$0 \to K_1 \to F_0 \xrightarrow{\varphi} M \to 0.$$

- We sometimes regard the kernel K_1 as a *first derivative* of M.
- We can now do the same thing and again approximate K_1 by a free module:

$$0 \to K_2 \to F_1 \to K_1 \to 0.$$

Free resolutions

Iterating, we obtain a long exact sequence[1]:

$$\begin{tikzcd}[column sep=small]
 & & & K_5 \arrow[dr] & & & & K_3 \arrow[dr] & & & & K_1 \arrow[dr] & & & \\
\cdots \arrow[r] & F_6 \arrow[rr] \arrow[dr] & & F_5 \arrow[r] \arrow[u] & F_4 \arrow[rr] \arrow[dr] & & F_3 \arrow[r] \arrow[u] & F_2 \arrow[rr] \arrow[dr] & & F_1 \arrow[r] \arrow[u] & F_0 \arrow[r] & M \arrow[r] & 0 \\
 & & K_6 \arrow[ur] & & & K_4 \arrow[ur] & & & K_2 \arrow[ur] & & & &
\end{tikzcd}$$

The intermediate models K_n are called the syzygies of M, with K_n being the n^{th} syzygy module of M.

[1]Free resolutions were made famous by Hilbert and his work on Invariant Theory.

Uniqueness of syzygies

- It is immediately obvious that syzygy module cannot be unique.
- Return to $0 \to K_1 \to F_0 \to M \to 0$. If F is a free module then we can stabilize the middle term to give another exact sequence,

$$0 \to K_1 \oplus F \to F_0 \oplus F \to M \to 0.$$

If K_1 is a first syzygy of M, then so too is $K_1 \oplus F$.

Uniqueness of syzygies

- It is immediately obvious that syzygy module cannot be unique.
- Return to $0 \to K_1 \to F_0 \to M \to 0$. If F is a free module then we can stabilize the middle term to give another exact sequence,

$$0 \to K_1 \oplus F \to F_0 \oplus F \to M \to 0.$$

  If K_1 is a first syzygy of M, then so too is $K_1 \oplus F$.
- In the original context of Invariant Theory, the idea of uniqueness was introduced by trying to make the resolution minimal in some sense.
- This fails quite badly in our context and so we instead turn to stable syzygies.

Schanuel's Lemma and Stability

Suppose we have two exact sequences of modules over a ring Λ:

$$0 \to K \to \Lambda^n \to M \to 0 \quad \text{and} \quad 0 \to K' \to \Lambda^m \to M \to 0.$$

Schanuel: Then $K \oplus \Lambda^m \cong K' \oplus \Lambda^n$.

Schanuel's Lemma and Stability

Suppose we have two exact sequences of modules over a ring Λ:

$$0 \to K \to \Lambda^n \to M \to 0 \quad \text{and} \quad 0 \to K' \to \Lambda^m \to M \to 0.$$

Schanuel: Then $K \oplus \Lambda^m \cong K' \oplus \Lambda^n$.

Given the above, we say two Λ-modules $K,\ K'$ are *stably equivalent* (written $K \sim K'$) if there exist some positive integers m, n such that $K \oplus \Lambda^m \cong K' \oplus \Lambda^n$.

Schanuel's Lemma and Stability

Suppose we have two exact sequences of modules over a ring Λ:

$$0 \to K \to \Lambda^n \to M \to 0 \quad \text{and} \quad 0 \to K' \to \Lambda^m \to M \to 0.$$

Schanuel: Then $K \oplus \Lambda^m \cong K' \oplus \Lambda^n$.

Given the above, we say two Λ-modules K, K' are *stably equivalent* (written $K \sim K'$) if there exist some positive integers m, n such that $K \oplus \Lambda^m \cong K' \oplus \Lambda^n$.

- $\sim$ is an equivalence relation.
- If we look at *stable syzygies* then we have uniqueness.
- We will write $\Omega_n = [K_n]$ for the stable class of the n^{th} syzygy.
- Stability does not need to equal isomorphism, i.e. $K \sim K' \not\Rightarrow K \cong K'$, in general.

How many (stable) syzygies are there?

Let $M = \mathbb{Z}$. For a finite group G, there are *a priori* two possibilities:

- The stable modules $\Omega_n(\mathbb{Z})$ are isomorphically distinct; or
- $\Omega_n(\mathbb{Z}) \cong \Omega_m(\mathbb{Z})$ for some $m,\ n \in \mathbb{Z}$ with $m \neq n$.

How many (stable) syzygies are there?

Let $M = \mathbb{Z}$. For a finite group G, there are *a priori* two possibilities:

- The stable modules $\Omega_n(\mathbb{Z})$ are isomorphically distinct; or
- $\Omega_n(\mathbb{Z}) \cong \Omega_m(\mathbb{Z})$ for some $m, n \in \mathbb{Z}$ with $m \neq n$.

We say that $n > 0$ is a *free cohomological period* of G if $\Omega_{n+k}(\mathbb{Z}) \cong \Omega_k(\mathbb{Z})$ for all $k \in \mathbb{Z}$. We note that this is equivalent to the existence of an exact sequence of the form,

$$0 \to \mathbb{Z} \to F_{n-1} \to \cdots \to F_0 \to \mathbb{Z} \to 0,$$

in which each F_i is finitely generated and free over $\mathbb{Z}$.

- A cohomological free period is necessarily even.

Cohomological free period - Examples

- Cyclic of order n, C_n: This has cohomological period 2.
- Dihedral of order $4n+2$, D_{4n+2}: This has cohomological period 4.
- Groups of order pq, where p, q are distinct primes such that $q|p-1$, $C_p \rtimes C_q$: This has cohomological period $2q$
- Quaternionic groups of order $4n$, where $n \geq 2$, $Q(4n)$: This has cohomological period 4.
- For an odd prime p, $C_p \times C_p$ is *not* periodic.

A group structure on syzygies

- Suppose G has cohomological period $n > 0$.
- If we consider the tensor product $- \otimes_{\mathbb{Z}} -$ over $\mathbb{Z}$, then we can form and abelian group of order n on $\Omega_r(\mathbb{Z})$.
- Commutativity and associativity follow directly from the tensor product. It therefore remains to show closure, identity and inverse.
- We will show this for two examples.

Some preliminaries

- Throughout, we work with $\mathbb{Z}[G]$-lattices, i.e. $\mathbb{Z}[G]$-modules whose underlying abelian group is finitely generated and free.
- When M, N are $\mathbb{Z}[G]$-lattices of ranks m, n, respectively, the tensor product $M \otimes N$ is a $\mathbb{Z}[G]$-lattice of rank mn with G-action given by $(\nu \otimes \omega)g = \nu g \otimes \omega g$.
- Working with lattices confers several advantages, notably that the dual of a short exact sequence of $\mathbb{Z}[G]$-lattices is again a short exact sequence of $\mathbb{Z}[G]$-lattices (in which the arrows are reversed). This property extends to exact sequences of finite length.
- We denote the category of finitely generated $\mathbb{Z}[G]$-lattices by $\mathcal{F}(\mathbb{Z}[G])$.

Syzygy modules for cyclic groups

- Set $\Lambda_0 := \mathbb{Z}[C_p]$, where $C_n = \langle x \mid x^p = 1 \rangle$.

Syzygy modules for cyclic groups

- Set $\Lambda_0 := \mathbb{Z}[C_p]$, where $C_n = \langle x \mid x^p = 1 \rangle$.
- There is a free resolution of period two given by:

$$0 \to \mathbb{Z} \xrightarrow{\epsilon^*} \Lambda_0 \xrightarrow{x-1} \Lambda_0 \xrightarrow{\epsilon} \mathbb{Z} \to 0,$$

  where ϵ is the augmentation map, and ϵ^* is its dual.
- We denote the augmentation ideal of Λ_0 by $I_C = ker(\epsilon)$.

Syzygy modules for cyclic groups

- Set $\Lambda_0 := \mathbb{Z}[C_p]$, where $C_n = \langle x \mid x^p = 1 \rangle$.
- There is a free resolution of period two given by:

$$0 \to \mathbb{Z} \xrightarrow{\epsilon^*} \Lambda_0 \xrightarrow{x-1} \Lambda_0 \xrightarrow{\epsilon} \mathbb{Z} \to 0,$$

  where ϵ is the augmentation map, and ϵ^* is its dual.
- We denote the augmentation ideal of Λ_0 by $I_C = ker(\epsilon)$.
- We can now read off the syzygies from the above as follows:

$$\Omega_r(\mathbb{Z}) = \begin{cases} [\mathbb{Z}], & r \equiv 0 (mod\, 2); \\ [I_C], & r \equiv 1 (mod\, 2). \end{cases}$$

The group operation

We will show the following for any prime p:

$$I_C \otimes I_C \cong \mathbb{Z} \oplus \Lambda_0^{p-2}. \tag{1}$$

The group operation

We will show the following for any prime p:

$$I_C \otimes I_C \cong \mathbb{Z} \oplus \Lambda_0^{p-2}. \tag{1}$$

- I_C can be written, $I_C = span_{\mathbb{Z}}\{x-1,\ x^2-1, \ldots,\ x^{p-1}-1\}$, with $rk_{\mathbb{Z}}(I_C) = p-1$.

The group operation

We will show the following for any prime p:

$$I_C \otimes I_C \cong \mathbb{Z} \oplus \Lambda_0^{p-2}. \tag{1}$$

- I_C can be written, $I_C = span_{\mathbb{Z}}\{x-1,\ x^2-1, \ldots,\ x^{p-1}-1\}$, with $rk_{\mathbb{Z}}(I_C) = p-1$.
- Consider the exact sequence, $0 \to I_C \xrightarrow{i} \Lambda_0 \xrightarrow{\epsilon} \mathbb{Z} \to 0$ and dualise,

$$0 \to \mathbb{Z} \xrightarrow{\epsilon^*} \Lambda_0 \xrightarrow{i^*} I_C^* \to 0$$

where $\epsilon^*(1) = \Sigma = \sum_{r=0}^{p-1} x^r$ is central.

The group operation

We will show the following for any prime p:

$$I_C \otimes I_C \cong \mathbb{Z} \oplus \Lambda_0^{p-2}. \tag{1}$$

- I_C can be written, $I_C = span_{\mathbb{Z}}\{x-1,\ x^2-1, \ldots,\ x^{p-1}-1\}$, with $rk_{\mathbb{Z}}(I_C) = p-1$.
- Consider the exact sequence, $0 \to I_C \xrightarrow{i} \Lambda_0 \xrightarrow{\epsilon} \mathbb{Z} \to 0$ and dualise,

  $$0 \to \mathbb{Z} \xrightarrow{\epsilon^*} \Lambda_0 \xrightarrow{i^*} I_C^* \to 0$$

  where $\epsilon^*(1) = \Sigma = \sum_{r=0}^{p-1} x^r$ is central.
- Therefore, $Im(\epsilon^*)$ is the two-sided ideal of Λ_0, generated by Σ.
- Consequently, we identify I_C^* with $\Lambda_0/(\Sigma)$, which is naturally a ring.

A basis for I_C^*

- Next, we put $\nu_r = i^*(x^r)$ where $\nu_0 = 1$, and $\nu_r = (\nu_1)^r = \nu^r$.
- If we think of I_C^* as a Λ_0-module, then I_C^* has a $\mathbb{Z}$-basis $\{1, \nu, \nu^2, \ldots, \nu^{p-2}\}$, in which
  - $\nu^p = 1$,
  - $\nu^{p-1} = -1 - \nu - \cdots - \nu^{p-2}$,
  - the action of x is to multiply by ν.
- It is well-known that $I_C \cong_{\Lambda_0} I_C^*$.

A useful description

If $p = 2$ then $I_C \otimes I_C \cong \mathbb{Z}$ is trivial.
We therefore let $n \geq 3$ and define the following for $1 \leq r \leq p - 2$:

$$V(r) = span_{\mathbb{Z}}\{\nu^{r+k} \otimes \nu^k \mid 0 \leq k \leq p - 1\} \subset I_C^* \otimes I_C^*.$$

A useful description

If $p = 2$ then $I_C \otimes I_C \cong \mathbb{Z}$ is trivial.
We therefore let $n \geq 3$ and define the following for $1 \leq r \leq p-2$:

$$V(r) = span_{\mathbb{Z}}\{\nu^{r+k} \otimes \nu^k \mid 0 \leq k \leq p-1\} \subset I_C^* \otimes I_C^*.$$

- For each $1 \leq r \leq p-2$, we have $V(r) \cong \Lambda_0$.
- For any $1 \leq r \leq p-2$,

$$V(r) \cap (V(1) + \cdots + V(r-1) + V(r+1) + \cdots + V(p-2)) = \{0\}.$$

- Set $V := V(1) \oplus \cdots \oplus V(p-2)$

‘Proof’ that $I_C \otimes I_C \cong \mathbb{Z} \oplus \Lambda_0^{p-2}$

- Observe $rk_{\mathbb{Z}}(V) = p(p-2)$ and so $rk_{\mathbb{Z}}((I_C^* \otimes I_C^*)/V) = 1$.

'Proof' that $I_C \otimes I_C \cong \mathbb{Z} \oplus \Lambda_0^{p-2}$

- Observe $rk_{\mathbb{Z}}(V) = p(p-2)$ and so $rk_{\mathbb{Z}}((I_C^* \otimes I_C^*)/V) = 1$.
- By first showing $(I_C^* \otimes I_C^*)/V$ is torsion free, we have $(I_C^* \otimes I_C^*)/V \cong \mathbb{Z}$.

‘Proof’ that $I_C \otimes I_C \cong \mathbb{Z} \oplus \Lambda_0^{p-2}$

- Observe $rk_{\mathbb{Z}}(V) = p(p-2)$ and so $rk_{\mathbb{Z}}((I_C^* \otimes I_C^*)/V) = 1$.
- By first showing $(I_C^* \otimes I_C^*)/V$ is torsion free, we have $(I_C^* \otimes I_C^*)/V \cong \mathbb{Z}$.
- In particular, we can construct a useful basis.
- Consider the basis $\{\nu^i \otimes \nu^j \mid 0 \leq i, j \leq p-2\}$ of $I_C^* \otimes I_C^*$.

‘Proof’ that $I_C \otimes I_C \cong \mathbb{Z} \oplus \Lambda_0^{p-2}$

- Observe $rk_{\mathbb{Z}}(V) = p(p-2)$ and so $rk_{\mathbb{Z}}((I_C^* \otimes I_C^*)/V) = 1$.
- By first showing $(I_C^* \otimes I_C^*)/V$ is torsion free, we have $(I_C^* \otimes I_C^*)/V \cong \mathbb{Z}$.
- In particular, we can construct a useful basis.
- Consider the basis $\{\nu^i \otimes \nu^j \mid 0 \leq i, j \leq p-2\}$ of $I_C^* \otimes I_C^*$. By performing elementary basis transformations, this can be replaced by the following basis:

$$\{\nu^{r+k} \otimes \nu^k \mid 1 \leq r \leq p-2,\ 0 \leq k \leq p-1\} \cup \{T\},$$

where

$$\begin{array}{rcccccccccc} T & = & 1 \otimes 1 & + & 1 \otimes \nu & + & 1 \otimes \nu^2 & + & \cdots & + & 1 \otimes \nu^{p-2} \\ & & & + & \nu \otimes \nu & + & \nu \otimes \nu^2 & + & \cdots & + & \nu \otimes \nu^{p-2} \\ & & & & & + & \nu^2 \otimes \nu^2 & + & \cdots & + & \nu^2 \otimes \nu^{p-2} \\ & & & & & & & & \ddots & & \vdots \\ & & & & & & & & & + & \nu^{p-2} \otimes \nu^{p-2}. \end{array}$$

‘Proof’ that $I_C \otimes I_C \cong \mathbb{Z} \oplus \Lambda_0^{p-2}$

- So $(I_C^* \otimes I_C^*)/V$ is generated by $\natural(T)$, where $\natural : I_C^* \otimes I_C^* \to (I_C^* \otimes I_C^*)/V$ is the natural surjection.

‘Proof’ that $I_C \otimes I_C \cong \mathbb{Z} \oplus \Lambda_0^{p-2}$

- So $(I_C^* \otimes I_C^*)/V$ is generated by $\natural(T)$, where $\natural : I_C^* \otimes I_C^* \to (I_C^* \otimes I_C^*)/V$ is the natural surjection.
- Clear that $T \in I_C^* \otimes I_C^*$ but $T \notin V$, and that $Tx = T$, thereby showing that x acts trivially on $(I_C^* \otimes I_C^*)/V$.
- Since x clearly acts trivially on $\mathbb{Z}$, our isomorphism extends to one over Λ_0, i.e. $(I_C^* \otimes I_C^*)/V \cong_{\Lambda_0} \mathbb{Z}$.
- This gives the following short exact sequence,

$$0 \to V \to I_C^* \otimes I_C^* \to [\natural(T)) \to 0.$$

- By dualising and using the self-duality of $V \cong \Lambda_0^{p-2}$ and $[\natural(T)) \cong \mathbb{Z}$, the proof is complete.

Moving to dihedral groups

- Set $\Lambda := \mathbb{Z}[D_{2p}]$, where $p = 2n + 1$ is an odd prime.
- Associated to Λ_0 is the canonical injection $i : \Lambda_0 \hookrightarrow \Lambda$.
- From this we can induce two maps on the categories of finitely generated lattices,
  - $i^* : \mathcal{F}(\Lambda) \to \mathcal{F}(\Lambda_0)$, given by restricting scalars to Λ_0,
  - $i_* : \mathcal{F}(\Lambda_0) \to \mathcal{F}(\Lambda)$, given by extending scalars; that is, $i_*(M) = M \otimes_{\Lambda_0} \Lambda$.

Moving to dihedral groups

- Set $\Lambda := \mathbb{Z}[D_{2p}]$, where $p = 2n + 1$ is an odd prime.
- Associated to Λ_0 is the canonical injection $i : \Lambda_0 \hookrightarrow \Lambda$.
- From this we can induce two maps on the categories of finitely generated lattices,
  - $i^* : \mathcal{F}(\Lambda) \to \mathcal{F}(\Lambda_0)$, given by restricting scalars to Λ_0,
  - $i_* : \mathcal{F}(\Lambda_0) \to \mathcal{F}(\Lambda)$, given by extending scalars; that is, $i_*(M) = M \otimes_{\Lambda_0} \Lambda$.
- Similarly, we have the canonical injection $j : \mathbb{Z}[C_2] \hookrightarrow \Lambda$ and this too induces two maps on the categories of finitely generated lattices.

Moving to dihedral groups

- Set $\Lambda := \mathbb{Z}[D_{2p}]$, where $p = 2n + 1$ is an odd prime.
- Associated to Λ_0 is the canonical injection $i : \Lambda_0 \hookrightarrow \Lambda$.
- From this we can induce two maps on the categories of finitely generated lattices,
  - $i^* : \mathcal{F}(\Lambda) \to \mathcal{F}(\Lambda_0)$, given by restricting scalars to Λ_0,
  - $i_* : \mathcal{F}(\Lambda_0) \to \mathcal{F}(\Lambda)$, given by extending scalars; that is, $i_*(M) = M \otimes_{\Lambda_0} \Lambda$.
- Similarly, we have the canonical injection $j : \mathbb{Z}[C_2] \hookrightarrow \Lambda$ and this too induces two maps on the categories of finitely generated lattices.
- Both the restriction and extension of scalars functors have easily verified properties. They are,
  - additive,
  - exact, and
  - take free modules to free modules.

Moving to dihedral groups

- The augmentation ideal of Λ will be denoted by I_G, and the augmentation ideals of Λ_0 and $\mathbb{Z}[C_2]$ will be denoted by I_C and I_2, respectively.

Moving to dihedral groups

- The augmentation ideal of Λ will be denoted by I_G, and the augmentation ideals of Λ_0 and $\mathbb{Z}[C_2]$ will be denoted by I_C and I_2, respectively.
- By $[\alpha)$ we mean the right ideal generated by α; that is $[\alpha) = \{\alpha\lambda \mid \lambda \in \Lambda\}$.
- In particular, any ideal in Λ is a Λ-lattice.

Moving to dihedral groups

- The augmentation ideal of Λ will be denoted by I_G, and the augmentation ideals of Λ_0 and $\mathbb{Z}[C_2]$ will be denoted by I_C and I_2, respectively.
- By $[\alpha)$ we mean the right ideal generated by α; that is $[\alpha) = \{\alpha\lambda \mid \lambda \in \Lambda\}$.
- In particular, any ideal in Λ is a Λ-lattice.
- Set $\pi := (x^n - 1)(y - 1)$ and $\tilde{\rho} = (y - 1)(x - 1)$. We then define

$$P = [\pi), \quad R = [\tilde{\rho}). \tag{2}$$

- Write $\Sigma_x = 1 + x + \cdots + x^{2n}$, which we observe is central in Λ.

Moving to dihedral groups

- The augmentation ideal of Λ will be denoted by I_G, and the augmentation ideals of Λ_0 and $\mathbb{Z}[C_2]$ will be denoted by I_C and I_2, respectively.
- By $[\alpha)$ we mean the right ideal generated by α; that is $[\alpha) = \{\alpha\lambda \mid \lambda \in \Lambda\}$.
- In particular, any ideal in Λ is a Λ-lattice.
- Set $\pi := (x^n - 1)(y - 1)$ and $\tilde{\rho} = (y - 1)(x - 1)$. We then define

$$P = [\pi), \;\; R = [\tilde{\rho}). \tag{2}$$

- Write $\Sigma_x = 1 + x + \cdots + x^{2n}$, which we observe is central in Λ.
- The ideal $[x - 1)$ decomposes as a direct sum $[x - 1) = P \oplus R$.

Why are P and R important?

Why are P and R important?

- We saw that I_C^* has a $\mathbb{Z}$-basis, $\{\nu^r \mid 0 \le r \le p-2\}$ where $1 + \nu + \cdots + \nu^{p-1} = 0$.
- The action of C_p on I_C^* may be extended in one of two ways to an action of the dihedral group:

Why are P and R important?

- We saw that I_C^* has a $\mathbb{Z}$-basis, $\{\nu^r \mid 0 \leq r \leq p-2\}$ where $1 + \nu + \cdots + \nu^{p-1} = 0$.
- The action of C_p on I_C^* may be extended in one of two ways to an action of the dihedral group:
  - Either: $\nu^r \cdot y = \nu^{-r} = \nu^{p-r}$ for $0 \leq r \leq p-2$;
  - or: $\nu^r \cdot y = -\nu^{-r} = -\nu^{p-r}$ for $0 \leq r \leq p-2$.
- Under the former, we denote $(I_C^*)_+$, and under the latter we denote $(I_C^*)_-$.

Why are P and R important?

- We saw that I_C^* has a $\mathbb{Z}$-basis, $\{\nu^r \mid 0 \leq r \leq p-2\}$ where $1 + \nu + \cdots + \nu^{p-1} = 0$.
- The action of C_p on I_C^* may be extended in one of two ways to an action of the dihedral group:
  - Either: $\nu^r \cdot y = \nu^{-r} = \nu^{p-r}$ for $0 \leq r \leq p-2$;
  - or: $\nu^r \cdot y = -\nu^{-r} = -\nu^{p-r}$ for $0 \leq r \leq p-2$.
- Under the former, we denote $(I_C^*)_+$, and under the latter we denote $(I_C^*)_-$.
- $P \cong (I_C^*)_-$ and $R \cong (I_C^*)_+$.

Why does this matter?

Why does this matter?

- The augmentation ideal I_G decomposes as,

$$I_G \cong P \oplus [y-1)$$

- This has the effect of separating the x, y strands, in some sense.
- As such, we end up spending more time on P and R (along with two other modules soon to be met).
- In particular, direct computation shows $P \otimes [y-1) \cong \Lambda^n$.

Why does this matter?

- The augmentation ideal I_G decomposes as,

$$I_G \cong P \oplus [y-1)$$

- This has the effect of separating the x, y strands, in some sense.
- As such, we end up spending more time on P and R (along with two other modules soon to be met).
- In particular, direct computation shows $P \otimes [y-1) \cong \Lambda^n$.
- Note, P and R are *not* isomorphic as Λ-modules, nor even stably isomorphic.
- Nevertheless, $P^* \cong R$ and $R^* \cong P$.

Why does this matter?

- The augmentation ideal I_G decomposes as,

$$I_G \cong P \oplus [y-1)$$

- This has the effect of separating the x, y strands, in some sense.
- As such, we end up spending more time on P and R (along with two other modules soon to be met).
- In particular, direct computation shows $P \otimes [y-1) \cong \Lambda^n$.
- Note, P and R are *not* isomorphic as Λ-modules, nor even stably isomorphic.
- Nevertheless, $P^* \cong R$ and $R^* \cong P$.
- Both $[y-1)$ and $[y+1)$ are self-dual, but are *not* isomorphic, as Λ-modules.

Two more modules

- Define the modules K and L to be

$$K = [\Sigma_x, y-1) \text{ and } L = [\Sigma_x, y+1). \tag{3}$$

Two more modules

- Define the modules K and L to be

$$K = [\Sigma_x, y - 1) \text{ and } L = [\Sigma_x, y + 1). \tag{3}$$

- Both K and L have $\mathbb{Z}$-rank $p + 1$.
- Further, $\Lambda/K \cong R$ and $\Lambda/L \cong P$.
- Finally, both K and L are self-dual; that is $K^* \cong K$ and $L^* \cong L$.

Spoiler for why these matter: D_{2p} has cohomological free period 4 and we have defined four modules: K, P, L, R.

How the modules interact under tensor product

- $K \otimes R(i) \cong R(i) \oplus \Lambda^n$, for $1 \leq i \leq 2$ where $R(1) \cong P$ and $R(2) \cong R$;
- $K \otimes K(i) \cong K(i) \oplus \Lambda^{n+1}$, for $1 \leq i \leq 2$ where $K(1) = L$ and $K(2) = K$;

How the modules interact under tensor product

- $K \otimes R(i) \cong R(i) \oplus \Lambda^n$, for $1 \leq i \leq 2$ where $R(1) \cong P$ and $R(2) \cong R$;
- $K \otimes K(i) \cong K(i) \oplus \Lambda^{n+1}$, for $1 \leq i \leq 2$ where $K(1) = L$ and $K(2) = K$;
- $P \otimes P \cong L \oplus \Lambda^{n-1}$;
- $P \otimes L \cong R \oplus \Lambda^n$;
- $P \otimes R \cong K \oplus \Lambda^{n-1}$.

How the modules interact under tensor product

- $K \otimes R(i) \cong R(i) \oplus \Lambda^n$, for $1 \leq i \leq 2$ where $R(1) \cong P$ and $R(2) \cong R$;
- $K \otimes K(i) \cong K(i) \oplus \Lambda^{n+1}$, for $1 \leq i \leq 2$ where $K(1) = L$ and $K(2) = K$;
- $P \otimes P \cong L \oplus \Lambda^{n-1}$;
- $P \otimes L \cong R \oplus \Lambda^n$;
- $P \otimes R \cong K \oplus \Lambda^{n-1}$.
- $[y-1) \otimes [y-1) \cong [y+1) \oplus \Lambda^n$
- $[y-1) \otimes [y+1) \cong [y-1) \oplus \Lambda^n$

Strategy for showing K acts as the identity

We show $K \otimes ? \cong ? \oplus \Lambda^r$ for some $r \geq 0$ and where $? = K,\ P,\ L,\ R$.

Strategy for showing K acts as the identity

We show $K \otimes ? \cong ? \oplus \Lambda^r$ for some $r \geq 0$ and where $? = K,\ P,\ L,\ R$.

- K has a $\mathbb{Z}$-basis given by $\{(y-1)x^i \mid 0 \leq i \leq p-1\} \cup \{\Sigma_x\}$.
- Define $K_0 = span_{\mathbb{Z}}\{(y-1), (y-1)x, \ldots, (y-1)x^{p-1}\}$.
- Then K/K_0 is represented by the class of Σ_x.

Strategy for showing K acts as the identity

We show $K \otimes ? \cong ? \oplus \Lambda^r$ for some $r \geq 0$ and where $? = K, P, L, R$.

- K has a $\mathbb{Z}$-basis given by $\{(y-1)x^i \mid 0 \leq i \leq p-1\} \cup \{\Sigma_x\}$.
- Define $K_0 = span_{\mathbb{Z}}\{(y-1), (y-1)x, \dots, (y-1)x^{p-1}\}$.
- Then K/K_0 is represented by the class of Σ_x.
- Observe $\Sigma_x \cdot x = \Sigma_x$ and $\Sigma_x \cdot y = (y-1)\Sigma_x + \Sigma_x = \Sigma_x$ in K/K_0.
- Thus, x and y act trivially on K/K_0 and it is therefore isomorphic to $\mathbb{Z}$.

Strategy for showing K acts as the identity

We show $K \otimes ? \cong ? \oplus \Lambda^r$ for some $r \geq 0$ and where $? = K,\ P,\ L,\ R$.

- K has a $\mathbb{Z}$-basis given by $\{(y-1)x^i \mid 0 \leq i \leq p-1\} \cup \{\Sigma_x\}$.
- Define $K_0 = span_{\mathbb{Z}}\{(y-1), (y-1)x, \ldots, (y-1)x^{p-1}\}$.
- Then K/K_0 is represented by the class of Σ_x.
- Observe $\Sigma_x \cdot x = \Sigma_x$ and $\Sigma_x \cdot y = (y-1)\Sigma_x + \Sigma_x = \Sigma_x$ in K/K_0.
- Thus, x and y act trivially on K/K_0 and it is therefore isomorphic to $\mathbb{Z}$.
- In particular, we have an exact sequence of the form

$$0 \to K_0 \to K \to \mathbb{Z} \to 0.$$

- As such, tensoring with any of the $P,\ R,\ K$ or L yields the exact sequence

$$0 \to K_0 \otimes ? \to K \otimes ? \to ? \to 0.$$

Strategy for showing $P \otimes P \cong L \oplus \Lambda^{n-1}$

- Recall, when looking at cyclic groups we showed

$$I_C^* \otimes I_C^* \cong_{\Lambda_0} [T) \oplus V,$$

where $V = V(1) \oplus \cdots \oplus V(2n-1)$ and

$$V(r) = span_{\mathbb{Z}}\{\nu^{r+k} \otimes \nu^k \mid 0 \leq k \leq 2n\}.$$

Strategy for showing $P \otimes P \cong L \oplus \Lambda^{n-1}$

- Recall, when looking at cyclic groups we showed

$$I_C^* \otimes I_C^* \cong_{\Lambda_0} [T) \oplus V,$$

where $V = V(1) \oplus \cdots \oplus V(2n-1)$ and

$$V(r) = span_{\mathbb{Z}}\{\nu^{r+k} \otimes \nu^k \mid 0 \leq k \leq 2n\}.$$

- We also introduced the y-action $\nu^r \cdot y = -\nu^{2n+1-r}$.
- $\{\nu^r \mid 0 \leq r \leq 2n-1\}$ is a $\mathbb{Z}$-basis for I_C^*, and under this action $(I_C^*)_- \cong P$.

Strategy for showing $P \otimes P \cong L \oplus \Lambda^{n-1}$

- Recall, when looking at cyclic groups we showed

$$I_C^* \otimes I_C^* \cong_{\Lambda_0} [T) \oplus V,$$

where $V = V(1) \oplus \cdots \oplus V(2n-1)$ and

$$V(r) = span_{\mathbb{Z}}\{\nu^{r+k} \otimes \nu^k \mid 0 \leq k \leq 2n\}.$$

- We also introduced the y-action $\nu^r \cdot y = -\nu^{2n+1-r}$.
- $\{\nu^r \mid 0 \leq r \leq 2n-1\}$ is a $\mathbb{Z}$-basis for I_C^*, and under this action $(I_C^*)_- \cong P$. So, we find the free part of V, showing that for $r \geq 2$,

$$V_r = V(r) + V(2n+1-r) \text{ is a } \Lambda\text{-module, and } V_r \cong \Lambda.$$

Strategy for showing $P \otimes P \cong L \oplus \Lambda^{n-1}$

- This leaves us with $[T) + V(1)$ and $V' = V(2) + \cdots + V(2n-1) \cong \Lambda^{n-1}$.
- We have a split short exact sequence,

$$0 \to \Lambda^{n-1} \to (I_C^*)_- \otimes (I_C^*)_- \to \natural(T + V(1)) \to 0.$$

  where $\natural : (I_C^*)_- \otimes (I_C^*)_- \to ((I_C^*)_- \otimes (I_C^*)_-)/V'$.
- $L \cong \natural(T + V(1))$.

Forming a free resolution

- Recall, the augmentation ideal of Λ splits as $I_G \cong P \oplus [y-1)$.
- We therefore have the short exact sequence,

$$0 \to P \oplus [y-1) \to \Lambda \to \mathbb{Z} \to 0.$$

Forming a free resolution

- Recall, the augmentation ideal of Λ splits as $I_G \cong P \oplus [y-1)$.
- We therefore have the short exact sequence,

$$0 \to P \oplus [y-1) \to \Lambda \to \mathbb{Z} \to 0.$$

- Tensoring with $P \oplus [y-1)$ gives us another short exact sequence,

$$0 \to (P \oplus [y-1)) \otimes (P \otimes [y-1)) \to \Lambda^{2p-1} \to P \oplus [y-1) \to 0.$$

Forming a free resolution

- Recall, the augmentation ideal of Λ splits as $I_G \cong P \oplus [y-1)$.
- We therefore have the short exact sequence,

$$0 \to P \oplus [y-1) \to \Lambda \to \mathbb{Z} \to 0.$$

- Tensoring with $P \oplus [y-1)$ gives us another short exact sequence,

$$0 \to (P \oplus [y-1)) \otimes (P \otimes [y-1)) \to \Lambda^{2p-1} \to P \oplus [y-1) \to 0.$$

- Tidying up using the earlier computations, we get the following:

$$0 \to L \oplus [y+1) \oplus \Lambda^{4n-1} \to \Lambda^{4n+1} \to P \oplus [y-1) \to 0.$$

A de-stabilization lemma

- A Λ-module M is n-coproective if $H^k(M, \Lambda) = 0$ for $1 \leq k \leq n$.
- If M is a Λ-lattice, then it is 1-coprojective.
- **Lemma:** Let $0 \to J \oplus Q_0 \xrightarrow{j} Q_1 \to M \to 0$ be an exact sequence of Λ-mdoules in which Q_0, Q_1 are projective. If M is 1-coprojective, then $Q_1/j(Q_0)$ is projective.

Applying the de-stabilization lemma

- Return to

$$0 \to L \oplus [y+1) \oplus \Lambda^{4n-1} \xrightarrow{i} \Lambda^{4n+1} \to P \oplus [y-1) \to 0.$$

and alter as follows:

$$0 \to L \oplus [y+1) \to S \to P \oplus [y-1) \to 0.$$

where $S = \Lambda^{4n+1}/i(\Lambda^{4n-1})$.

Applying the de-stabilization lemma

- Return to

$$0 \to L \oplus [y+1) \oplus \Lambda^{4n-1} \xrightarrow{i} \Lambda^{4n+1} \to P \oplus [y-1) \to 0.$$

  and alter as follows:

$$0 \to L \oplus [y+1) \to S \to P \oplus [y-1) \to 0.$$

  where $S = \Lambda^{4n+1}/i(\Lambda^{4n-1})$.
- By the de-stabilization lemma, S must be projective.
- S also occurs in the exact sequence,

$$0 \to \Lambda^{4n-1} \to \Lambda^{4n+1} \to S \to 0.$$

- As this exact sequence splits, S is stably free of rank 2. However, over D_{2p} all stably free modules are free (Eichler).

Keep tensoring with I_G

- This leaves us with the following exact sequence,

$$0 \to L \oplus [y+1) \to \Lambda^2 \to P \oplus [y-1) \to 0.$$

- Now we repeat by tensoring with $P \oplus [y-1)$ again:

$$0 \to R \oplus [y-1) \to \Lambda^2 \to L \oplus [y+1) \to 0.$$

Keep tensoring with I_G

- This leaves us with the following exact sequence,

$$0 \to L \oplus [y+1) \to \Lambda^2 \to P \oplus [y-1) \to 0.$$

- Now we repeat by tensoring with $P \oplus [y-1)$ again:

$$0 \to R \oplus [y-1) \to \Lambda^2 \to L \oplus [y+1) \to 0.$$

- And again:

$$0 \to K \oplus [y+1) \to \Lambda^2 \to R \oplus [y-1) \to 0.$$

Keep tensoring with I_G

- This leaves us with the following exact sequence,

$$0 \to L \oplus [y+1) \to \Lambda^2 \to P \oplus [y-1) \to 0.$$

- Now we repeat by tensoring with $P \oplus [y-1)$ again:

$$0 \to R \oplus [y-1) \to \Lambda^2 \to L \oplus [y+1) \to 0.$$

- And again:

$$0 \to K \oplus [y+1) \to \Lambda^2 \to R \oplus [y-1) \to 0.$$

- And again

$$0 \to P \oplus [y-1) \to \Lambda^2 \to K \oplus [y+1) \to 0.$$

A free resolution...

Splicing together the exact sequences gives us a ‘diagonalised’ free resolution:

$$\begin{tikzcd}
 & & & P \oplus [y-1) \arrow[dr] & & & & R \oplus [y-1) \arrow[dr] & & & & P \oplus [y-1) \arrow[dr] & & & \\
\cdots \arrow[r] & \Lambda^2 \arrow[r] & \Lambda^2 \arrow[ur] \arrow[rr] & & \Lambda^2 \arrow[dr] \arrow[rr] & & \Lambda^2 \arrow[ur] \arrow[rr] & & \Lambda^2 \arrow[dr] \arrow[rr] & & \Lambda^2 \arrow[ur] \arrow[rr] & & \Lambda^2 \arrow[r] & \mathbb{Z} \arrow[r] & 0 \\
 & & & & & K \oplus [y+1) \arrow[ur] & & & & L \oplus [y+1) \arrow[ur] & & & & &
\end{tikzcd}$$

...and the syzygies

Reading them off from the free resolution, we see that the syzygies are as follows:

$$\Omega_r(\mathbb{Z}) = \begin{cases} [\mathbb{Z}] = [K] \oplus [y+1], & r \equiv 0 \ (mod\ 4); \\ [P] \oplus [y-1], & r \equiv 1 \ (mod\ 4); \\ [L] \oplus [y+1], & r \equiv 2 \ (mod\ 4); \\ [R] \oplus [y-1], & r \equiv 3 \ (mod\ 4). \end{cases}$$

- Notice the paradoxical nature of stable classes. Even though $\mathbb{Z}$ is an indecomposable Λ-module, its stability class decomposes non-trivially.

Bringing this back to Wall's D(2)-problem

- From its relationship to the Realization problem, the key is to understand the third syzygy module $\Omega_3(\mathbb{Z})$.
- In particular, we want each minimal module $J \in \Omega_3(\mathbb{Z})$ to be geometrically realizable and full.
- We have shown, $(I_C)_+ \oplus [y-1) \in \Omega_3(\mathbb{Z})$ and it is in fact minimal.

Bringing this back to Wall's D(2)-problem

- From its relationship to the Realization problem, the key is to understand the third syzygy module $\Omega_3(\mathbb{Z})$.
- In particular, we want each minimal module $J \in \Omega_3(\mathbb{Z})$ to be geometrically realizable and full.
- We have shown, $(I_C)_+ \oplus [y-1) \in \Omega_3(\mathbb{Z})$ and it is in fact minimal.
- **Question:** Are there any other minimal modules?
- **Answer:** No!

Bringing this back to Wall's D(2)-problem

- From its relationship to the Realization problem, the key is to understand the third syzygy module $\Omega_3(\mathbb{Z})$.
- In particular, we want each minimal module $J \in \Omega_3(\mathbb{Z})$ to be geometrically realizable and full.
- We have shown, $(I_C)_+ \oplus [y-1) \in \Omega_3(\mathbb{Z})$ and it is in fact minimal.
- **Question:** Are there any other minimal modules?
- **Answer:** No!
- **Question:** Is this module geometrically realizable and full?
- **Answer:** Yes!
- This therefore offers us a proof that D_{2p} does in fact have the D(2)-property.