Tensor products, q-characters and R-matrices for quantum toroidal algebras

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Overview

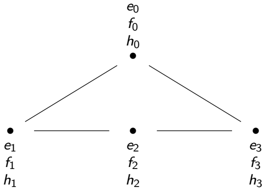
- **Quantum groups** $U_q(\mathfrak{s})$ – *definition; motivations; braid group action*
- **Quantum affine algebras** $U_q(\hat{\mathfrak{g}})$ – *two realizations*
- **Quantum toroidal algebras** $U_q(\mathfrak{g}_{\mathrm{tor}})$ – *definition; motivations & successes; difficulties*
- **Key problem** – *no tensor product* $\otimes$ *on module category* $\widehat{\mathcal{O}}$ *for* $U_q(\mathfrak{g}_{\mathrm{tor}})$*!*
- **The road to a solution...**
  1. *braid group action on* $U_q(\mathfrak{g}_{\mathrm{tor}})$
  2. *horizontal–vertical symmetries of* $U_q(\mathfrak{g}_{\mathrm{tor}})$
  3. *new topological coproduct for* $U_q(\mathfrak{g}_{\mathrm{tor}})$
- **Outcomes** – *well-defined* $\otimes$ *with nice properties; q-characters; R-matrices; transfer matrices*

Further details can be found in `arXiv:2503.08839` [L25]

(as well as `arXiv:2304.06773` [L24a] and my thesis [L24b])

Setup

Cartan matrix $A=(a_{ij})_{i,j\in I}$ = Dynkin diagram *ie. vertices I and arrows* ⇝ Kac-Moody Lie algebra $\mathfrak{s}$ *generators e_i, f_i, h_i ($i\in I$)* *relations given by arrows*



Rough idea of relations:

- Cartan subalgebra $\langle h_i \mid i \in I\rangle$
- generators at non-adjacent vertices commute
- generators at adjacent vertices do not

Examples:

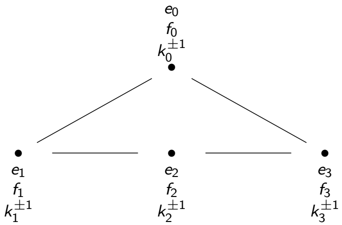
- *all finite dimensional (semi)simple Lie algebras $\mathfrak{g}$ (+ve definite A)*
- *all affine Lie algebras $\hat{\mathfrak{g}}$ (+ve semidefinite A)*

Quantum groups $U_q(\mathfrak{s})$

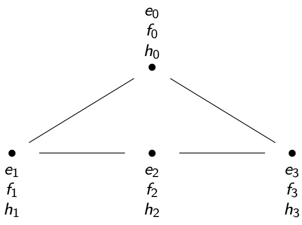
Defn: unique q-deformation of $U(\mathfrak{s})$ as a Hopf algebra

Motto: throw lots of q's into the definition of $U(\mathfrak{s})$, such that $U_q(\mathfrak{s}) \xrightarrow{q \to 1} U(\mathfrak{s})$

Quantum group $U_q(\mathfrak{s})$



Universal enveloping algebra $U(\mathfrak{s})$

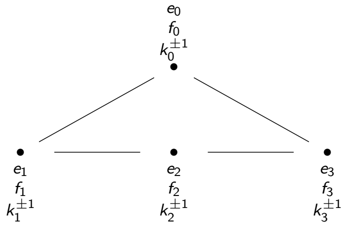


Example relation: $[e_i, f_j] = \delta_{ij} \frac{k_i - k_i^{-1}}{q - q^{-1}} \xrightarrow[(k_i \sim q^{h_i})]{q \longrightarrow 1} [e_i, f_j] = \delta_{ij} h_i$

Similarly: *Cartan subalgebra $\langle k_i^{\pm 1} \rangle$ // generators at $i \not\sim j$ commute // generators at $i \sim j$ do not*

Fundamental: *Hopf algebra w/coproduct ⇝ nice repn theory ($\otimes$ & duals), and much more ...*

Motivating quantum groups $U_q(\mathfrak{s})$

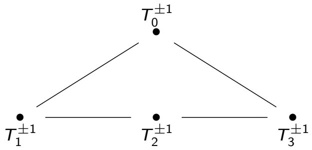


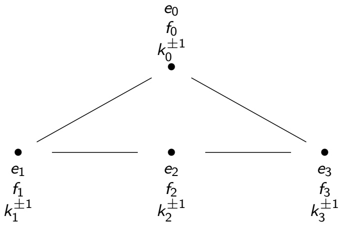
- **Algebra.** *eg. finding nice bases for (repns of) $\mathfrak{s}$ is hard, but for $U_q(\mathfrak{s})$ we have crystal & global & canonical bases*
- **Mathematical physics.** *eg.* $R : V \otimes W \xrightarrow{\sim} W \otimes V$ *as solutions to Yang-Baxter eqn*
- **Low dimensional topology.** *eg. various knot / link / 3-manifold invariants are computed using repns of quantum groups*
- **Combinatorics.** *eg. combinatorial bases for repns of quantum groups (such as Young walls) connect to partitions, Macdonald polynomials, ...*
- **Geometry.** $\widehat{U_q(\mathfrak{s})} \to K^G(\ldots)$ *connects quantum algebras with quiver varieties, Hilbert schemes, Kleinian singularities, ...*
- *... and so on ...*

Fundamental result – braid group actions

Motto: *quantization of Weyl group action $W \curvearrowright \mathfrak{s}$ on Kac-Moody Lie algebra*

$$\mathcal{B} = \left\langle T_i \mid i \in I, \underbrace{T_i T_j T_i \dots}_{\#\{i-j\}+2} = \underbrace{T_j T_i T_j \dots}_{\#\{i-j\}+2} \right\rangle \quad \curvearrowright \quad U_q(\mathfrak{s}) = \left\langle e_i,\ f_i,\ k_i^{\pm 1} \mid i \in I,\ \dots \right\rangle$$





Rough idea: *T_i intertwines generators at vertices $j \sim i$ with those at i*

Extension: *diagram automorphism group Ω acting via $\pi : e_i \mapsto e_{\pi(i)}\ ,\ f_i \mapsto f_{\pi(i)}\ ,\ k_i \mapsto k_{\pi(i)}$*

Applications:

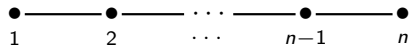
- root vectors and PBW-style bases for $U_q(\mathfrak{s})$ and $U_q(\mathfrak{s})_{\mathbb{Z}}$
- crystal and global basis theories for repns of $U_q(\mathfrak{s})$
- and more to come ...

Motto: the simplest, most well-understood part of the story

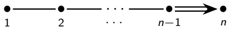
f.d. simple Lie algebras $\mathfrak{g}$ $\overset{1-1}{\longleftrightarrow}$ +ve definite Cartan matrices $\overset{1-1}{\longleftrightarrow}$ the $A_n - G_2$ Dynkin diagrams

Examples:

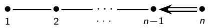
A_n



B_n



C_n



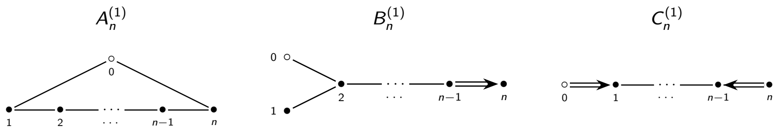
Recap:

$\mathfrak{g}$ $\xleftrightarrow{\text{equivalent to}}$ $U(\mathfrak{g})$ $\xrightarrow{q\text{-deformation}}$ $U_q(\mathfrak{g})$

$\mathfrak{g} \circlearrowleft W$ $\qquad$ $U_q(\mathfrak{g}) \circlearrowleft \mathcal{B}$

The affine case

- simplest ∞-dim Kac-Moody Lie algebras, next most complicated after f.d. semisimple case
- classified by explicit list of Dynkin diagrams
- all occur as finite A_n–G_2 Dynkin diagrams with an extra 0 vertex added in a particular way, eg.



Note: in this talk we restrict to untwisted types – the twisted case is rather different!

Two realizations for affine objects

Motto: objects on the affine level usually have two realizations ...

"object associated to affine Dynkin diagram" *vs* *"affinization of corresponding finite object"*

Example: affine Lie algebras $\hat{\mathfrak{g}}$ arise from affine Dynkin diagrams, or:

$$\text{f.d. simple Lie algebra } \mathfrak{g} \rightsquigarrow \text{loop Lie algebra } \mathfrak{g}[t, t^{-1}] \overset{\text{central extension}}{\rightsquigarrow} \mathfrak{g}[t, t^{-1}] \oplus \mathbb{C}c$$

$e_0\ f_0\ h_0$	$e_1\ f_1\ h_1\ \cdots\ e_n\ f_n\ h_n$

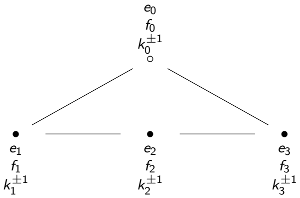
vs

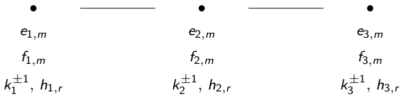
$\vdots$	$\vdots$
$e_1t^1\ f_1t^1\ h_1t^1\ \cdots\ e_nt^1\ f_nt^1\ h_nt^1$	
$e_1t^0\ f_1t^0\ h_1t^0\ \cdots\ e_nt^0\ f_nt^0\ h_nt^0$	$\mathbb{Z}$
$e_1t^{-1}\ f_1t^{-1}\ h_1t^{-1}\ \cdots\ e_nt^{-1}\ f_nt^{-1}\ h_nt^{-1}$	
$\vdots\ \ c\ \ \vdots$	

Note: similar "loop style" realizations exist for braid groups, Hecke algebras, quantum groups ...

Two presentations for affine quantum groups

	Drinfeld-Jimbo realization	**Drinfeld new realization**
Motto:	*"quantum group of the affine Lie algebra"*	*"(quantum) affinization of the finite quantum group"*
	q-deformation of $U(\hat{\mathfrak{g}})$	q-deformation of $U(\mathfrak{g}[t, t^{-1}] \oplus \mathbb{C}c)$





e_0 f_0 $k_0^{\pm 1}$	e_1 f_1 $k_1^{\pm 1}$ $\cdots$ e_n f_n $k_n^{\pm 1}$

$\vdots$ $\vdots$ $e_{1,1}$ $f_{1,1}$ $h_{1,1}$ $\cdots$ $e_{n,1}$ $f_{n,1}$ $h_{n,1}$	
$e_{1,0}$ $f_{1,0}$ $k_1^{\pm 1}$ $\cdots$ $e_{n,0}$ $f_{n,0}$ $k_n^{\pm 1}$	$\mathbb{Z}$
$e_{1,-1}$ $f_{1,-1}$ $h_{1,-1}$ $\cdots$ $e_{n,-1}$ $f_{n,-1}$ $h_{n,-1}$ $\vdots$ $C^{\pm 1}$ $\vdots$	

Uses: *coproducts, highest weight theory, nice bases, ...* | *finite dimensional repns, vertex repns, ...*

Affine braid group action

Affine braid group $\curvearrowright$ **Affine quantum group**

Affine braid group		$\curvearrowright$	Affine quantum group	
$\Omega \quad T_0$	$T_1 \quad \cdots \quad T_n$		$e_0 \; f_0 \; k_0^{\pm 1}$	$e_1 \; f_1 \; k_1^{\pm 1} \quad \cdots \quad e_n \; f_n \; k_n^{\pm 1}$

Motto: · T_i intertwines generators at vertices $j \sim i$ with those at i

· $\pi \in \Omega$ permutes generators around the affine Dynkin diagram

$\vdots \qquad \vdots$	$\vdots \qquad \vdots$
$X_{\omega_1^\vee} \quad \cdots \quad X_{\omega_n^\vee}$	$e_{1,1} \; f_{1,1} \; h_{1,1} \quad \cdots \quad e_{n,1} \; f_{n,1} \; h_{n,1}$
$T_1 \quad \cdots \quad T_n$ $\curvearrowright$	$e_{1,0} \; f_{1,0} \; k_1^{\pm 1} \quad \cdots \quad e_{n,0} \; f_{n,0} \; k_n^{\pm 1}$
$X_{-\omega_1^\vee} \quad \cdots \quad X_{-\omega_n^\vee}$	$e_{1,-1} \; f_{1,-1} \; h_{1,-1} \quad \cdots \quad e_{n,-1} \; f_{n,-1} \; h_{n,-1}$
$\vdots \qquad \vdots$	$\vdots \quad C^{\pm 1} \quad \vdots$

Motto: · X_β shifts generators up & down $\mathbb{Z}$-grading

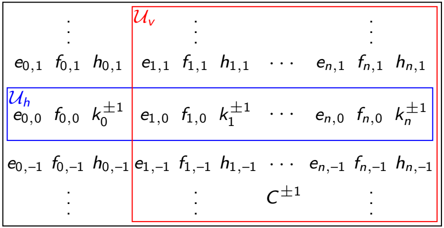
Quantum toroidal algebras $U_q(\mathfrak{g}_{\mathrm{tor}})$

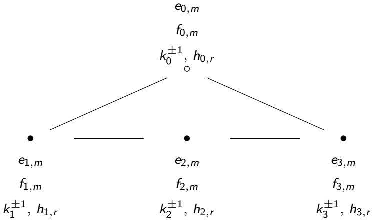
Motto:

- the double affine objects within the quantum setting
- quantize the toroidal Lie algebras $\mathfrak{g}[s^{\pm 1}, t^{\pm 1}] \oplus \mathbb{C}c_s \oplus \mathbb{C}c_t$

Construction:

- combine the two realizations for the quantum affine algebra
- specifically, index Drinfeld new presentation over *affine* Dynkin diagram





Fundamental:

- horizontal and vertical quantum affine subalgebras $\mathcal{U}_h$ and $\mathcal{U}_v$ generate $U_q(\mathfrak{g}_{\mathrm{tor}})$
- but ... not quantum groups!

Motivations and connections

	$\vdots$		$\mathcal{U}_v$	$\vdots$				$\vdots$	
$e_{0,1}$	$f_{0,1}$	$h_{0,1}$	$e_{1,1}$	$f_{1,1}$	$h_{1,1}$	$\cdots$	$e_{n,1}$	$f_{n,1}$	$h_{n,1}$
$\mathcal{U}_h$ $e_{0,0}$	$f_{0,0}$	$k_0^{\pm 1}$	$e_{1,0}$	$f_{1,0}$	$k_1^{\pm 1}$	$\cdots$	$e_{n,0}$	$f_{n,0}$	$k_n^{\pm 1}$
$e_{0,-1}$	$f_{0,-1}$	$h_{0,-1}$	$e_{1,-1}$	$f_{1,-1}$	$h_{1,-1}$	$\cdots$	$e_{n,-1}$	$f_{n,-1}$	$h_{n,-1}$
	$\vdots$			$\vdots$		$C^{\pm 1}$		$\vdots$	

- toroidal Schur-Weyl duality with double affine Hecke algebra (DAHA) in type A

- $U_q(\mathfrak{g}_{\mathrm{tor}}) \longrightarrow K^G$(quiver varieties & Hilbert schemes of Kleinian singularities)
  - *provides powerful geometric approach to repn theory*
  - *provides powerful algebraic approach to (enumerative) geometry*

- various realizations of $U_q(\mathfrak{sl}_{n+1,\mathrm{tor}})$ and an important "Fock space repn"

  algebraic // combinatorial // geometric // mathematical physics // quantum cluster // ...

  allow you to connect & pass information between the situations

- and many more ...

Difficulties

	$\vdots$		$\mathcal{U}_v$	$\vdots$				$\vdots$	
$e_{0,1}$	$f_{0,1}$	$h_{0,1}$	$e_{1,1}$	$f_{1,1}$	$h_{1,1}$	$\cdots$	$e_{n,1}$	$f_{n,1}$	$h_{n,1}$
$\mathcal{U}_h$ $e_{0,0}$	$f_{0,0}$	$k_0^{\pm 1}$	$e_{1,0}$	$f_{1,0}$	$k_1^{\pm 1}$	$\cdots$	$e_{n,0}$	$f_{n,0}$	$k_n^{\pm 1}$
$e_{0,-1}$	$f_{0,-1}$	$h_{0,-1}$	$e_{1,-1}$	$f_{1,-1}$	$h_{1,-1}$	$\cdots$	$e_{n,-1}$	$f_{n,-1}$	$h_{n,-1}$
	$\vdots$			$\vdots$		$C^{\pm 1}$		$\vdots$	

Crucially, quantum toroidal algebras are not quantum groups! So ...

- ☹ no nice basis theories for $U_q(\mathfrak{g}_{\mathrm{tor}})$ or its repns properly developed yet

- ☹ no coproducts or Hopf algebra structures known[1] for $U_q(\mathfrak{g}_{\mathrm{tor}})$

  best we have is topological coproduct $\Delta : U_q(\mathfrak{g}_{\mathrm{tor}}) \to U_q(\mathfrak{g}_{\mathrm{tor}}) \widehat{\otimes} U_q(\mathfrak{g}_{\mathrm{tor}})$

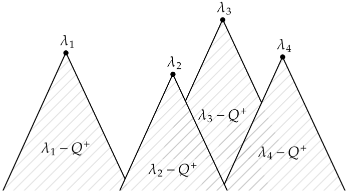
  but this has infiniteness & convergence issues with producing $\otimes$ of repns in general ...

[1] except in types $A_1^{(1)}$ and $A_2^{(1)}$ [JZ22]

Representation theory – a problem

Definition: category $\widehat{\mathcal{O}}$ of $U_q(\mathfrak{g}_{\mathrm{tor}})$-modules whose restrictions to $\mathcal{U}_h$ lie inside $\mathcal{O}$, ie.

1. all $e_{i,0}$ and $f_{i,0}$ act locally nilpotently
2. decompose as a direct sum of finite dimensional weight spaces w.r.t $\langle k_0^{\pm 1}, \dots, k_n^{\pm 1} \rangle$
3. weights lie inside a finite union of cones



Motto:

- nice & important category $\widehat{\mathcal{O}}$ of integrable representations
- toroidal analogue of finite dimensional modules for quantum affine algebras

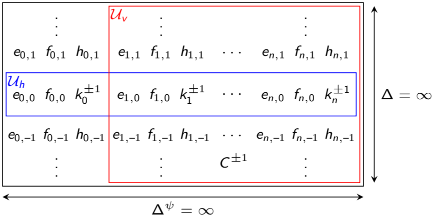
Problem: no tensor product $\otimes$ exists for category $\widehat{\mathcal{O}}$ ☹ ... *why?*

Representation theory – a problem

Problem: topological coproduct $\Delta \overset{\text{nope!}}{\rightsquigarrow} \otimes$ on category $\widehat{\mathcal{O}}$ ☹

Motto: both Δ and repns in $\widehat{\mathcal{O}}$ are *vertically infinite*, eg.

$$e_{i,0} \xrightarrow{\Delta} \sum_{k\leq 0} \alpha_k \otimes e_{i,k} \qquad \& \qquad \textit{all } e_{i,k}\cdot v \neq 0 \textit{ possible}$$



Game plan [L24a,L24b,L25][2]**:**

1. (extended) double affine braid group action $\ddot{\mathcal{B}} \curvearrowright U_q(\mathfrak{g}_{\mathrm{tor}})$
2. duality involution $\mathfrak{t}$ of $\ddot{\mathcal{B}} \overset{\text{pass across action}}{\rightsquigarrow}$ anti-involution ψ of $U_q(\mathfrak{g}_{\mathrm{tor}})$ swapping $\mathcal{U}_h \leftrightarrow \mathcal{U}_v$
3. *horizontally infinite* topological coproduct $\Delta^{\psi} = (\psi \otimes \psi) \circ \Delta \circ \psi$

[2]Note: $U_q(\mathfrak{sl}_{n+1,\mathrm{tor}})$ originally investigated by Miki

(Extended) double affine braid group $\ddot{\mathcal{B}}$:

- combine both realizations of (extended) affine braid group $\dot{\mathcal{B}}$
- braid group analogue of definition for $U_q(\mathfrak{g}_{\mathrm{tor}})$ in quantum setting

Theorem [L24a]**:** $\ddot{\mathcal{B}}$ acts on $U_q(\mathfrak{g}_{\mathrm{tor}})$ in all types

$\ddot{\mathcal{B}}$	$\mathcal{B}_v$ $\vdots$		$\vdots$
	$X_{\omega_1^\vee}$	$\cdots$	$X_{\omega_n^\vee}$
$\mathcal{B}_h$ Ω T_0	T_1	$\cdots$	T_n
	$X_{-\omega_1^\vee}$	$\cdots$	$X_{-\omega_n^\vee}$
	$\vdots$		$\vdots$

$\curvearrowright$

$U_q(\mathfrak{g}_{\mathrm{tor}})$ $\vdots$	$\mathcal{U}_v$ $\vdots$		$\vdots$
$e_{0,1}$ $f_{0,1}$ $h_{0,1}$	$e_{1,1}$ $f_{1,1}$ $h_{1,1}$	$\cdots$	$e_{n,1}$ $f_{n,1}$ $h_{n,1}$
$\mathcal{U}_h$ $e_{0,0}$ $f_{0,0}$ $k_0^{\pm 1}$	$e_{1,0}$ $f_{1,0}$ $k_1^{\pm 1}$	$\cdots$	$e_{n,0}$ $f_{n,0}$ $k_n^{\pm 1}$
$e_{0,-1}$ $f_{0,-1}$ $h_{0,-1}$	$e_{1,-1}$ $f_{1,-1}$ $h_{1,-1}$	$\cdots$	$e_{n,-1}$ $f_{n,-1}$ $h_{n,-1}$
$\vdots$	$\vdots$	$C^{\pm 1}$	$\vdots$

Rough idea: combine both realizations of affine action $\dot{\mathcal{B}} \curvearrowright U_q(\hat{\mathfrak{g}})$

Key lemma: a finite presentation for $U_q(\mathfrak{g}_{\mathrm{tor}})$

Horizontal–vertical symmetries

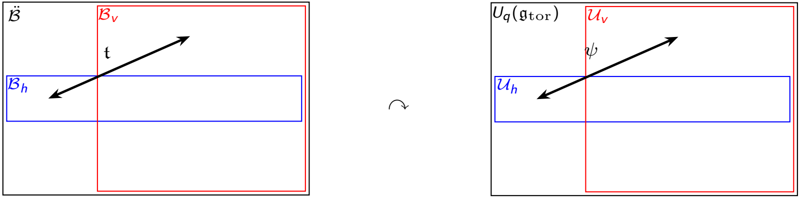
Goal: find (anti-)automorphisms of $U_q(\mathfrak{g}_{\mathrm{tor}})$ that swap $\mathcal{U}_h$ and $\mathcal{U}_v$ – surprising!

Why: only previous example (Miki automorphism in type A) <u>fundamental</u> for studying $U_q(\mathfrak{sl}_{n+1,\mathrm{tor}})$

Theorem [L24a,L24b,L25]**:** There exists an anti-involution ψ of $U_q(\mathfrak{g}_{\mathrm{tor}})$ swapping $\mathcal{U}_h \leftrightarrow \mathcal{U}_v$ in all types

Proof idea:

- consider 'duality involution' $\mathfrak{t}$ of $\ddot{\mathcal{B}}$ that swaps $\mathcal{B}_h$ and $\mathcal{B}_v$
- pass it across $\ddot{\mathcal{B}} \curvearrowright U_q(\mathfrak{g}_{\mathrm{tor}})$ to obtain $\psi = (b \cdot x \mapsto \mathfrak{t}(b) \cdot x$ for all $b \in \ddot{\mathcal{B}}$, $x \in \mathcal{U}_h \cap \mathcal{U}_v)$

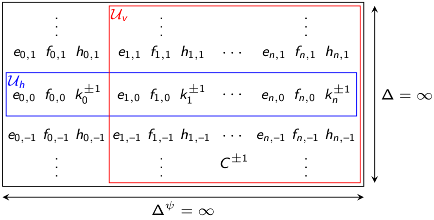


Corollary [L24b]**:** modular / congruence group action $\widetilde{SL_2(\mathbb{Z})} \curvearrowright U_q(\mathfrak{g}_{\mathrm{tor}})$

Representation theory of $U_q(\mathfrak{g}_{\mathrm{tor}})$ – a solution

Problem: topological coproduct Δ $\overset{\text{nope!}}{\rightsquigarrow}$ $\otimes$ on category $\widehat{\mathcal{O}}$ ☹

Solution [L25]: *horizontally infinite* topological coproduct $\Delta^{\psi} = (\psi \otimes \psi) \circ \Delta \circ \psi$



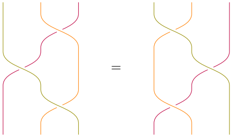
- ☺ produces well-defined $\otimes$ and monoidal structure on $\widehat{\mathcal{O}}$, hence ring structure on $K(\widehat{\mathcal{O}})$
- ☺ $\otimes$ of irreducibles is generically irreducible (w.r.t. spectral parameter)
- ☺ compatibility with Drinfeld polys that classify irreducibles: $\mathcal{P}(V \otimes W) = \mathcal{P}(V) \cdot \mathcal{P}(W)$
- ☺ $\widehat{\mathcal{O}}$ in some sense generated by $n+1$ fundamental modules

q-characters, R-matrices and transfer matrices

q-characters: $\chi_q : K(\widehat{\mathcal{O}}) \to \mathcal{Y}$ injective group homomorphism, combinatorial approach to $\widehat{\mathcal{O}}$

Theorem [L25]**:** χ_q is a ring homomorphism, ie. $\chi_q(V \otimes W) = \chi_q(V) \cdot \chi_q(W)$

R-matrices: intertwining morphisms $\mathcal{R} : V \otimes W \to W \otimes V$ satisfying Yang-Baxter equation



Theorem [L25][3]**:** There exist unique rational functions $\mathcal{R}_{V,W}(x) : V \otimes W \to W \otimes V$ such that

homomorphism except at poles // isomorphism generically // (trig. q-) YBE satisfied

Corollary: commutative family of transfer matrices $\{\mathcal{T}_{V,W}(x) \mid W\} \subset \mathrm{End}(V)(x)$

[3] for now, V, W are $\bigoplus \bigotimes$ irreducibles here

Future directions

monoidal structure
q-characters
R-matrices

→ cluster algebra structure on $\widehat{\mathcal{O}}$

→ generalised toroidal Schur-Weyl dualities

q-characters
R-matrices
transfer matrices

→ integrable systems

quantum toroidal algebras [L25] ⟷ [GTL16] affine Yangians [GNW22,AGW23]