$\mathcal{SW}$-algebras and G_2-structures with torsion

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One-slide summary (5 Ws & H)

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- **Why** do we study it?

Interplay between geometry and CFTs (e.g. mirror symmetry).

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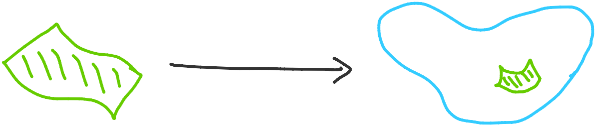
Motivation: the general idea

Setup: sigma model approach

String theory (type II or heterotic) as a sigma-model.



$$\Sigma \longrightarrow \mathcal{M} \ ,$$

- Target: d-dimensional manifold with a G-structure.
- Worldsheet: 2-dimensional superconformal field theory.

> What is the relation between them and why do we care?

Why is this interesting/useful?

- Interplay between geometry and 2d CFTs.

[Odake 89], [Shatashvili, Vafa 94], [Figueroa-O'Farrill 97]

- Worldsheet algebra provides information about the target geometry and vice versa.
  - Example: connected sum construction as algebra inclusions.

[Fiset 18], [Fiset, G. 21], [G. 23]

- Mirror symmetry.
  - Description of mirror map.
  - Also for G_2 and Spin(7) manifolds.

[Lerche, Vafa, Warner 89], [Gaberdiel, Kaste 04], [Braun, Majumder, Otto 19]

- Can be made mathematically rigorous through the chiral de Rham complex and vertex algebra language.

[Ekstrand, Heluani, Kallen, Zabzine 09, 13], [Rodríguez Díaz 16]

[Álvarez-Cónsul, De Arriba de La Hera, Garcia-Fernandez 20, 23]

The geometry side

The target manifold is equipped with a G-structure.

- Determined by a collection of characteristic tensors:
$$\{\Phi^1, \dots, \Phi^n\}\,.$$

For example, for a G_2-structure $\{\Phi^1, \Phi^2\} = \{\varphi, \psi\}$

- Exterior derivatives encode the torsion classes:
$$\mathrm{d}\varphi = \tau_0\,\psi + 3\,\tau_1 \wedge \varphi + *\tau_3\,, \qquad \mathrm{d}\psi = 4\,\tau_1 \wedge \psi + *\tau_2\,.$$
- We will require the existence of a compatible metric connection ∇^+ with totally skew torsion. This constrains the torsion classes, which can be used to write the torsion
$$\tau_2 = 0 \implies H = \frac{1}{6}\,\tau_0\,\varphi - \tau_1 \lrcorner \psi - \tau_3\,.$$

The algebra side

The worldsheet supports an $\mathcal{SW}$-algebra.

- Generated by a collection of super operators
$$\{\mathcal{J}^1, \dots, \mathcal{J}^n\}\,.$$
- When the operators come close together, their behaviour is under control via the Operator Product Expansion (OPE)
$$\mathcal{J}_{h_i}(Z_1)\mathcal{J}_{h_j}(Z_2) \ \sim C_{ij}^k \frac{1}{Z_{12}^{h_{ijk}-r/2}} D^r \mathcal{O}_k(Z_2)\,,$$
- Additional technical conditions (associativity, etc.), can be used to abstractly classify these algebras.

Mathematically: SUSY Vertex Algebra.

Meeting point: the sigma model (I)

Consider the $\mathcal{N} = (1,0)$ non-linear sigma model with target $\mathcal{M}$.
We use a superspace formalism, meaning that we repackage fields and their superpartners into superfields.

$$S[X] = \int_\Sigma \frac{\mathrm{d}^2 z\, \mathrm{d}\theta}{2\,\ell_s^2} \left[(g_{ij}(X) + B_{ij}(X))\, \bar{\partial} X^i D X^j \right] ,$$

- $(z, \bar{z})$ worldsheet coordinates, θ Grassmann variable.
- g_{ab} is a metric on $\mathcal{M}$ and B is the B-field, so $H = \mathrm{d}B$.

How do geometry and algebra connect?

Meeting point: the sigma model (II)

- Every characteristic p-form Φ of the G-structure gives rise to a new classical symmetry of the action, called $\mathcal{W}$-symmetry

$$\delta_\epsilon^\Phi X^a = \frac{\epsilon(z,\theta)}{(p-1)!}\, \Phi^a{}_{a_2\cdots a_p}\, DX^{a_2}\cdots DX^{a_p}.$$

Secretly, this is because the forms satisfy $\nabla^+\Phi = 0$.

[Howe, Papadopoulos 91, 93], [Howe, Stojevic 06], [Howe, Papadopoulos, Stojevic 10]

- Each symmetry has an associated Noether super current

$$\mathcal{J}^\Phi_{\text{cl.}} = \frac{1}{p!}\, \Phi_{a_1\cdots a_p}\, DX^{a_1}\cdots DX^{a_p}.$$

- We find that every form produces a classical current

$$\Phi^i \xrightarrow{\mathcal{W}} \mathcal{J}^i_{\text{cl.}}\,.$$

Meeting point: the sigma model (III)

String theory is a quantum theory and can be studied through a perturbative expansion around the classical theory. We will choose the string length $\ell_s = \sqrt{2\pi\alpha'}$ as our perturbative parameter.

- The $\mathcal{SW}$-algebra describes the full quantum theory.
- The sigma model describes the classical theory.

Noether currents are classical limits of the quantum operators

$$\mathcal{J}^i \xrightarrow{\text{cl.}} \mathcal{J}^i_{\text{cl.}} \, .$$

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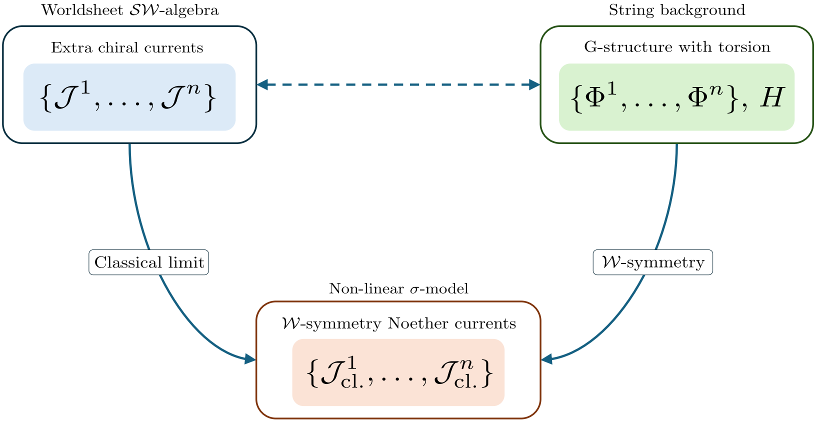
The algebra of $\mathcal{W}$-symmetries:

- is generated by the G-structure,
- and is the classical limit of the $\mathcal{SW}$-algebra.

Our strategy summarised



Warm-up: Virasoro algebra

$\mathcal{N} = 1$ Virasoro algebra (I)

We are familiar with the (super) $\mathcal{N} = 1$ Virasoro algebra:

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}(m^3 - m)\delta_{m+n,0}\,,$$

$$[L_m, G_r] = (\frac{m}{2} - r)G_{m+r}\,, \qquad \{G_r, G_s\} = 2L_{r+s} + \frac{c}{3}(r^2 - \frac{1}{4})\delta_{r+s,0}\,,$$

where c is the central charge, and L_m, G_r are the Fourier modes of the stress tensor T and the supersymmetry generator G

$$T(z) = \sum_{m\in\mathbb{Z}} L_m\, z^{m-2}\,, \qquad G(z) = \sum_{r\in\mathbb{Z}+\frac{1}{2}} G_r\, z^{r-\frac{3}{2}}\,,$$

$\mathcal{N} = 1$ Virasoro algebra (II)

The can be combined into a super stress tensor $\mathcal{T}$

$$\mathcal{T}(Z) = -\frac{1}{2}G(z) + \theta\, T(z)\,,$$

and the commutation relations can be encoded into a super OPE

$$\mathcal{T}(Z_1)\mathcal{T}(Z_2) \sim \frac{c}{6}\frac{1}{Z_{12}^3} + \frac{3}{2}\frac{\theta_{12}}{Z_{12}^2}\mathcal{T}(Z_2) + \frac{1}{2Z_{12}}D\mathcal{T}(Z_2) + \frac{\theta_{12}}{Z_{12}}\partial\mathcal{T}(Z_2) + \cdots$$

How does this manifest in the classical theory?

Superconformal symmetry

The action is invariant under superconformal transformations

$$\delta_\epsilon^{\mathcal{T}} X^i = -\epsilon\,\partial X^i - \frac{1}{2} D\epsilon\, DX^i\,,$$

the associated Noether current is the classical super stress tensor

$$\mathcal{T}_{\text{cl.}}(Z) = -\frac{1}{2} G_{\text{cl.}}(z) + \theta\, T_{\text{cl.}}(z)\,,$$

where for example $G_{\text{cl.}} = i\left(G_{ij}\partial x^i \psi^j + \frac{1}{3!}\ell_s H_{ijk}\psi^i\psi^j\psi^k\right)$.

What is the associated classical algebra of symmetries?

From classical symmetries to OPEs (I)

- The commutator of two superconformal transformations is another superconformal transformation

$$[\delta^{\mathcal{T}}_{\epsilon_1}, \delta^{\mathcal{T}}_{\epsilon_2}]X^i = \delta^{\mathcal{T}}_{\epsilon_3} X^i \,,$$

where $\epsilon_3 = \epsilon_1 \partial \epsilon_2 - \partial \epsilon_1 \, \epsilon_2 + \frac{1}{2} D\epsilon_1 D\epsilon_2$.

- Using the conformal Ward identity, we can rewrite infinitesimal transformations as contour integrations:

$$\delta^{\mathcal{T}}_{\epsilon_3} X^i(\zeta) = -\frac{1}{2\pi i} \oint_{\zeta} \mathrm{d}Z \, \epsilon_3(Z) \mathcal{T}_{\text{cl.}}(Z) X^i(\zeta) \,,$$

- The same holds for the commutator

$$[\delta^{\mathcal{T}}_{\epsilon_1}, \delta^{\mathcal{T}}_{\epsilon_2}]X^i(\zeta) = \oint_{\zeta} \frac{\mathrm{d}Z_2}{2\pi i} \oint_{\zeta_2} \frac{\mathrm{d}Z_1}{2\pi i} \, \epsilon_1(Z_1)\epsilon_2(Z_2) \, \mathcal{T}_{\text{cl.}}(Z_1)\mathcal{T}_{\text{cl.}}(Z_2) \, X^i(\zeta) \,,$$

From classical symmetries to OPEs (II)

After integrating by parts, the classical OPE can be read off.

$$\mathcal{T}_{\text{cl.}}(Z_1)\mathcal{T}_{\text{cl.}}(Z_2) \sim \frac{3}{2}\frac{\theta_{12}}{Z_{12}^2}\mathcal{T}_{\text{cl.}}(Z_2) + \frac{1}{2Z_{12}}D\mathcal{T}_{\text{cl.}}(Z_2) + \frac{\theta_{12}}{Z_{12}}\partial\mathcal{T}_{\text{cl.}}(Z_2) + \dots .$$

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We recover the $\mathcal{N} = 1$ Virasoro algebra...
except for the central charge term!

- This was to be expected: the classical OPE should only reproduce the classical version of the algebra, which in this case is the Witt algebra. The central charge is a quantum object and as such it does not appear.

$\mathcal{SW}$-algebras from G-structures

Finding the $\mathcal{SW}$-algebra candidate

- Let $\mathcal{M}$ be a Riemannian manifold equipped with a G-structure with characteristic forms $\{\Phi^1, \ldots, \Phi^n\}$.
- Each form Φ^i generates a $\mathcal{W}$-symmetry.
- We have a set of classical currents $\{\mathcal{T}, \mathcal{J}^1_{\text{cl.}}, \ldots, \mathcal{J}^n_{\text{cl.}}\}$

The underlying $\mathcal{SW}$-algebra must be generated by

$$\langle \mathbb{1}, \mathcal{T}, \mathcal{J}^1, \ldots, \mathcal{J}^n \rangle .$$

Commutator

[Howe, Stojevic 06], [Howe, Papadopoulos, Stojevic 10]

- The commutator of two $\mathcal{W}$-symmetries is
$$[\delta^{\Phi}_{\epsilon_1}, \delta^{\Psi}_{\epsilon_2}]X^i = \delta^{U}_{\epsilon_U} X^i + \delta^{N}_{\epsilon_N} X^i + \delta^{\mathcal{T}V}_{\epsilon_{\mathcal{T}V}} X^i \,.$$

It depends on contractions of the forms and the torsion:

$$U = \frac{1}{c_U}\, \Phi^i \wedge \Psi_i \ ,$$
$$V = \frac{1}{c_V}\, \Phi^{ij} \wedge \Psi_{ij} \ ,$$
$$N = \frac{\ell_s}{c_N} \left(H_{jk} \wedge \Phi^j \wedge \Psi^k - 2(-1)^p \frac{c_V}{d-(p+q-4)} H \wedge V \right) \ ,$$

- The $\delta^{\mathcal{T}V}$ symmetry is new and has a composite current
$$\mathcal{J}^{\mathcal{T}\Phi} = -\mathcal{T}\mathcal{J}^{\Phi} \,.$$

Classical super OPE

Analogous computation to the superconformal case provides a formula for the "classical" terms in the associated OPEs.

$$
\begin{aligned}
\mathcal{J}^{\Phi}_{\text{cl.}}(Z_1)\mathcal{J}^{\Psi}_{\text{cl.}}(Z_2) \sim & (-1)^{p+1}\, c_U\, \frac{\mathcal{J}^{U}_{\text{cl.}}(Z_2)}{Z_{12}} \\
& + (-1)^{p+1}\, c_U \left(\frac{p-1}{p+q-2}\right) \frac{\theta_{12}}{Z_{12}} D\mathcal{J}^{U}_{\text{cl.}}(Z_2) \\
& + (-1)^{p}\, c_N\, \frac{\theta_{12}}{Z_{12}} \mathcal{J}^{N}_{\text{cl.}}(Z_2) \\
& + c_V \left(\frac{2}{d-(p+q-4)}\right) \frac{\theta_{12}}{Z_{12}} \mathcal{T}_{\text{cl.}}(Z_2)\mathcal{J}^{V}_{\text{cl.}}(Z_2) + \dots .
\end{aligned}
$$

- The algebra is described purely by geometric data.

Our strategy

In each case we must:

- Identify characteristic forms of the G-structure.
- Propose candidate families of $\mathcal{SW}$-algebras.
- Compare classical and quantum OPEs.

We hope to identify terms and provide a geometrical meaning to the coefficients of the algebra.

Example: G_2-structures

What is known?

- In the absence of torsion, the correspondence is well-known:

$$\begin{aligned}
\text{Trivial holonomy} &\longleftrightarrow \text{Free algebra}\\
\mathrm{U}(n)\text{-holonomy} &\longleftrightarrow \mathcal{N}=2 \text{ Virasoro algebra}\\
\mathrm{SU}(n)\text{-holonomy} &\longleftrightarrow \text{Odake algebra}\\
\mathrm{G}_2\text{-holonomy} &\longleftrightarrow \mathrm{G}_2 \text{ Shatashvili–Vafa}\\
\mathrm{Spin}(7)\text{-holonomy} &\longleftrightarrow \mathrm{Spin}(7) \text{ Shatashvili–Vafa}
\end{aligned}$$

[Odake 89], [Shatashvili, Vafa 94]
[Figueroa-O'Farrill, Schrans 91, 92], [Blumenhagen 92], [Figueroa-O'Farrill 97]

- In the presence of torsion, some results for the $\mathcal{N}=2$ and G_2 cases. For example, a deformed G_2 Shatashvili-Vafa algebra can be obtained in $\mathrm{AdS}_3 \times \mathbb{S}^3 \times \mathbb{T}^4$ backgrounds.

[Álvarez-Cónsul, De Arriba de La Hera, Garcia-Fernandez 20, 23]
[Fiset, Gaberdiel 21]

Warm-up example

Let $\mathcal{M} = \mathbb{R}^n$, and take $H = 0$.

- We have n covariantly constant one-forms $\sigma^I = \mathrm{d}x^I$.
- This gives n classical currents $\mathcal{J}_I^\sigma$.
- Immediate to compute:

$$U_{IJ} = \delta_{IJ} 1\,, \qquad V = 0\,, \qquad N = 0\,.$$

- The classical OPE is:

$$\mathcal{J}_I^\sigma(Z_1)\mathcal{J}_J^\sigma(Z_2) \sim -\frac{\delta_{IJ}}{Z_{12}} + \ldots\,,$$

We do get back the OPEs of n free super fields, as we should!

G_2-structures

A G_2-structure on a seven-dimensional Riemannian manifold $\mathcal{M}$ is determined by the associative three-form φ

$$\varphi = \mathrm{d}x^{246} - \mathrm{d}x^{235} - \mathrm{d}x^{145} - \mathrm{d}x^{136} + \mathrm{d}x^{127} + \mathrm{d}x^{347} + \mathrm{d}x^{567}\,.$$

- It determines a metric and an orientation on $\mathcal{M}$.
- Gives rise to coassociative four-form $\psi = *\varphi$.

There are four torsion classes associated with a G_2-structure:

$$\mathrm{d}\varphi = \tau_0\,\psi + 3\,\tau_1\wedge\varphi + *\tau_3\,, \qquad \mathrm{d}\psi = 4\,\tau_1\wedge\psi + *\tau_2\,.$$

Demanding $\tau_2 = 0$, we find the torsion

$$H = \frac{1}{6}\,\tau_0\,\varphi - \tau_1\lrcorner\,\psi - \tau_3\,.$$

G_2 algebra candidates

- We have a 3-form φ and a 4-form ψ.
- Look at algebras generated by an operator $\mathcal{J}^{\varphi}$ of conformal weight $\frac{3}{2}$ and an operator $\mathcal{J}^{\Psi}$ of conformal weight $\frac{4}{2} = 2$.

The algebra must be a member of the $\mathcal{SW}(\frac{3}{2}, \frac{3}{2}, 2)$ family. This family depends on two free parameters:

- c, the central charge.
- λ, measuring the self-coupling $C^{\varphi}_{\varphi\varphi}$.

[Blumenhagen 91]

Are all these algebras allowed? What is the meaning of λ?

G_2 classical OPEs

The results of the classical computation are:

$\mathcal{J}^1_{\text{cl.}}$	$\mathcal{J}^2_{\text{cl.}}$	$\mathcal{J}^U_{\text{cl.}}$	$\mathcal{J}^N_{\text{cl.}}$	$\mathcal{J}^V_{\text{cl.}}$	c_U	c_N	c_V
$\mathcal{J}^\varphi_{\text{cl.}}$	$\mathcal{J}^\varphi_{\text{cl.}}$	$\mathcal{J}^\psi_{\text{cl.}}$	-	-	6	-	-
$\mathcal{J}^\varphi_{\text{cl.}}$	$\mathcal{J}^\psi_{\text{cl.}}$	-	-	$\mathcal{J}^\varphi_{\text{cl.}}$	-	-	12
$\mathcal{J}^\psi_{\text{cl.}}$	$\mathcal{J}^\psi_{\text{cl.}}$	-	$-\mathcal{J}^\varphi_{\text{cl.}}\mathcal{J}^\psi_{\text{cl.}}$	$\mathcal{J}^\psi_{\text{cl.}}$	-	$\frac{2}{3}\,\tau_0\ell_s$	12

Watch out for that scalar torsion class!!!

G_2 comparison (I)

Now for the OPE comparison:

- OPEs $\mathcal{J}^{\varphi}_{\text{cl.}}\mathcal{J}^{\varphi}_{\text{cl.}}$ and $\mathcal{J}^{\varphi}_{\text{cl.}}\mathcal{J}^{\psi}_{\text{cl.}}$ fix the normalisation.
- The only remaining OPE is

$$\mathcal{J}^{\psi}_{\text{cl.}}(Z_1)\mathcal{J}^{\psi}_{\text{cl.}}(Z_2) \sim -\frac{2}{3}\ell_s\tau_0\frac{\theta_{12}}{Z_{12}}\mathcal{J}^{\varphi}_{\text{cl.}}\mathcal{J}^{\psi}_{\text{cl.}} + 8\,\frac{\theta_{12}}{Z_{12}}\,\mathcal{T}_{\text{cl.}}\mathcal{J}^{\psi}_{\text{cl.}} + \dots .$$

However, in the quantum OPE the number 8 is instead 12.

A puzzle! Is everything lost?

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However, in the quantum OPE the number 8 is instead 12.

A puzzle! Is everything lost?

- No.

G_2 comparison (II)

- Within the two parameter family, some algebras are special: they admit a tricritical Ising model as a subalgebra.
- Through some representation theory arguments, this means there are some distinguished operators that can be quotiented out from the theory: null fields.
- In this situation, the coefficient 8 (or 12) is only well-defined up to null fields.

G_2 comparison (II)

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- Through some representation theory arguments, this means there are some distinguished operators that can be quotiented out from the theory: null fields.
- In this situation, the coefficient 8 (or 12) is only well-defined up to null fields.

Solution to the puzzle: a matching with the classical algebra is possible only in a particular locus.
In this situation, a geometric interpretation is possible.

G_2 comparison (III)

- What is this distinguished locus? Surprise surprise, it is the family found by Fiset and Gaberdiel, obtained by setting:

$$c = \frac{21}{2} - \frac{6}{k}\,, \quad \lambda^2 = \frac{32(3k-2)^2}{k^2(49k-30)}\,,$$

where k is the parameter of the family.

Physical interpretation: $\sqrt{k}$ gives you the radius R_{AdS} of the AdS_3 spacetime component of the string background in units of ℓ_s

$$k = 2\pi \left(\frac{R_{\mathrm{AdS}}}{\ell_s}\right)^2 .$$

The limit $k \to \infty$ recovers a flat spacetime.

G_2 comparison (IV)

Working with the deformed Shatashvili–Vafa algebra, we can compare the remaining OPE coefficient. This requires an expansion in powers of ℓ_s and yields

$$\tau_0 = \frac{1}{\sqrt{\pi}} \frac{6}{7} \frac{1}{R_{\mathrm{AdS}}} + O(\ell_s^2)\,.$$

This recovers the supergravity expectation!

G_2 conclusion

Note that the central charge of the algebra is now:

$$c = \frac{21}{2} - \frac{49}{12}\tau_0^2 \ell_s^2 + O(\ell_s^3)\,,$$

corrections to the central charge proportional to H^2 are expected in supergravity, and we confirm that suspicion.

- Our classical computation selects a distinguished one-parameter locus within all the space of amenable $\mathcal{SW}$-algebras.
- The parameter is directly tied to the scalar torsion class.
- Torsionless limit recovers the special holonomy case!

Vertex algebra perspective

Reformulating the statements

To make these physical ideas mathematically rigorous, we need to understand several concepts:

- SUSY Vertex Algebras: these are $\mathcal{SW}$-algebras.
- Courant algebroids: the most convenient language to describe the geometry of supergravity.
- The chiral de Rham complex: a sheaf of SUSY Vertex Algebras that can be defined from Courant algebroids.

"Having an $\mathcal{SW}$-algebra underlying a supergravity solution"
then means
"finding an embedding of a SUSY Vertex Algebra into sections of the corresponding chiral de Rham complex".

Vertex algebras (I)

A vertex algebra $(V, |0\rangle, T, Y(\cdot, z))$ is:

- A vector superspace $V = V_0 \oplus V_1$ (space of states).
- An even vector $|0\rangle \in V_0$ (vacuum vector).
- An even endomorphism $T \colon V \to V$ (infinitesimal translation).
- A parity-preserving linear map $Y \colon V \to \mathrm{End}(V)[[z^{\pm}]]$ (state-field correspondence).

In addition, three axioms must be satisfied (vacuum, translation covariance and locality).

Where are the OPEs?

In this language, a field is a formal sum

$$a(z) := Y(a, z) = \sum_{n\in\mathbb{Z}} a_{(n)} z^{-n-1},$$

with *Fourier modes* $a_{(n)} \in \mathrm{End}(V)$.

- Given two fields $a(z)$, $b(w)$, we can write $a(z)b(w)$ as a Laurent expansion in $(z - w)$.
- Ignoring the regular part, this defines the OPE of two fields:

$$a(z)b(w) \sim \sum_{n\geq 0} \frac{(a_{(n)}b)(w)}{(z-w)^{n+1}}.$$

We will call $a_{(n)}b$ the n-product of a and b.

SUSY vertex algebra

A *SUSY vertex algebra* is a tuple $(V, |0\rangle, S, Y(\cdot, z))$, where

- $(V, |0\rangle, T = S^2, Y(\cdot, z))$ is a vertex algebra,
- and $S : V \to V$ is an odd linear map which is furthermore a derivation for the n-products.

- The algebra is conformal if it includes a field $L(z)$ satisfying the Virasoro algebra.
- It is superconformal if it also has the superpartner $G(z)$.

How to operate in practice

The information of the OPEs can be recast as the λ-bracket. Furthermore, a normally ordered product can be defined

$$[a_\lambda b] = \sum_{n\in\mathbb{Z}} \frac{\lambda^n}{n!} a_{(n)} b \,, \qquad :ab: = a_{(-1)} b \,.$$

It turns out this is all one needs to define a vertex algebra:

In practice, we just need to specify the generating fields, their λ-bracket and the action of S to define the vertex algebra.

Example: the Neveu–Schwarz algebra

The Neveu–Schwarz algebra (which a physicist would call the $\mathcal{N} = 1$ super-Virasoro algebra) with central charge c is the SUSY vertex algebra freely generated by the $\mathbb{C}[T]$-module

$$\mathbb{C}G \oplus \mathbb{C}L\,,$$

the odd derivation S defined by

$$SG = 2L\,, \quad 2SL = TG\,,$$

and the λ-brackets

$$[L_\lambda L] = (T + 2\lambda)L + c\,\frac{\lambda^3}{12}\,, \qquad [L_\lambda G] = \left(T + \frac{3}{2}\lambda\right) G\,.$$

More examples

- Given a quadratic Lie algebra $(\mathfrak{g}, (\cdot|\cdot))$ and a scalar $k \in \mathbb{C}$, one can define in a similar way the universal superaffine vertex algebra with level k associated to $\mathfrak{g}$, denoted $V^k(\mathfrak{g}_{\text{super}})$.
- One can also similarly define the deformed Shatashvili–Vafa algebra SV_a with parameter $a \in \mathbb{C}$.
  - This provides a precise abstract mathematical definition of the G_2 algebra we encountered earlier.
  - The parameter a is related to the k we used before via

$$a = i\sqrt{\frac{2}{k}},$$

and $a \to 0$ recovers the Shatashvili–Vafa algebra.

Chiral de Rham complex (I)

Given M a manifold, we can construct a SUSY vertex algebra $\Omega^{\mathrm{ch}}_M(U)$ on each open subset U by taking the $\mathbb{C}[T]$-module

$$(C^\infty(U) \oplus (\mathfrak{X}(U) \oplus \Omega^1(U)) \oplus \Pi(\mathfrak{X}(U) \oplus \Omega^1(U))) \otimes \mathbb{C}[T]\,,$$

with $Tf = \mathrm{d}f$, the odd derivation S defined by

$$Sf := \Pi \mathrm{d}f\,, \quad S\Pi X := X\,, \quad S\Pi\eta := \eta\,,$$

for $X \in \mathfrak{X}(U)$, $\eta \in \Omega^1(U)$, and the λ-brackets

$$[X_\lambda f] = X(f)\,, \quad [X_\lambda \Pi Y] = \Pi[X, Y]\,, \quad [X_\lambda Y] = [X, Y]\,,$$
$$[X_\lambda \Pi\eta] = \Pi L_X\eta\,, \quad [X_\lambda \eta] = L_X\eta + \lambda\iota_X\eta\,, \quad [\Pi X_\lambda \Pi\eta] = \iota_X\eta$$

One also has to quotient by some ideals, but we skip this.

Chiral de Rham complex (II)

For a manifold M, the assignment $U \to \Omega^{\mathrm{ch}}_M(U)$ defines a sheaf of SUSY vertex algebras Ω^{ch}_M called the chiral de Rham complex.

The presence of $\mathfrak{X}(U) \oplus \Omega^1(U)$ should remind us a lot of generalised geometry... and in fact:

- There is a canonical procedure to construct a chiral de Rham complex from a Courant algebroid.

Courant algebroids

A Courant algebroid is a quadruple $(E, \langle\cdot,\cdot\rangle, [\cdot,\cdot], \pi)$ with

- E a vector bundle over M,
- $\langle\cdot,\cdot\rangle$ a non-degenerate symmetric bilinear form on E,
- $[\cdot,\cdot] : E \times E \to E$ a bilinear map on E,
- and $\pi : E \to TM$ a bundle map,

satisfying some additional conditions (Jacobi, etc). We always have the following short sequence of vector bundles:

$$0 \to T^*M \xrightarrow{\pi^*} E \xrightarrow{\pi} TM \to 0\,.$$

We call E exact if this short sequence is exact.

Any exact Courant algebroid E on M is isomorphic to the generalised tangent bundle for some closed $H \in \Omega^3 M$.

Generalised tangent bundle

The generalised tangent bundle $(\mathbb{T}M, \langle\cdot,\cdot\rangle, [\cdot,\cdot]_H, \pi)$ is given by

- $\mathbb{T}M = TM \oplus T^*M$,
- $\langle\cdot,\cdot\rangle$ the natural symmetric bilinear form
$$\langle X+\xi, Y+\eta\rangle = \frac{1}{2}\left(\eta(X)+\xi(Y)\right),$$
where $X, Y \in \Gamma(TM)$, $\xi, \eta \in \Gamma(T^*M)$.
- $[\cdot,\cdot]_H$ the H-twisted Dorfman bracket
$$[X+\xi, Y+\eta]_H := [X,Y] + \mathcal{L}_X\eta - \iota_Y \mathrm{d}\xi + H(X,Y,\cdot).$$
for some closed $H \in \Omega^3 M$,
- $\pi : \mathbb{T}M \to TM$ is the natural projection to TM.

The Lie group case

Assume now $M = K$ is a compact Lie group and focus on left-invariant exact Courant algebroids on K.

- Left-invariant sections define a quadratic Lie algebra (with the induced bracket and pairing)

$$\mathfrak{g} := \Gamma(\mathbb{T}M)^K = \mathfrak{k} \oplus \mathfrak{k}^*,$$

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Proposition

There is an embedding

$$V^2(\mathfrak{g}_{super}) \hookrightarrow \Gamma(K, \Omega^{\mathrm{ch}}_K(\mathbb{T}M))$$

of the superaffine vertex algebra $V^2(\mathfrak{g}_{super})$ of level $k = 2$ on the space of global sections $\Gamma(K, \Omega^{\mathrm{ch}}_K(\mathbb{T}K))$ of $\Omega^{\mathrm{ch}}_K(\mathbb{T}K)$.

Explicit realisations

We focus on certain supergravity backgrounds built from 7-dimensional group manifolds with different G_2-structures.

We have constructed explicit embeddings of the SV_a algebra in the corresponding superaffine vertex algebra $V^k(\mathfrak{g}_{\text{super}})$, inducing embeddings into the global sections of the chiral de Rham complex.

- $\mathbb{S}^3 \times \mathbb{T}^4$, two G_2-structures.
  - As a torus bundle over Hopf surface: $\tau_0 = 0$, algebra is SV_0.
  - As a sphere bundle over four tori: $\tau_0 \neq 0$, algebra is SV_a.
- $\mathbb{S}^3 \times \mathbb{S}^3 \times \mathbb{T}^1$, three G_2-structures.
  - Case 1: $\tau_0 = 0$, $\tau_1 \neq 0$, algebra is SV_0.
  - Case 2: $\tau_0 \neq 0$, $\tau_1 = 0$, algebra is SV_a.
  - Case 3: $\tau_0 \neq 0$, $\tau_1 \neq 0$, algebra is SV_a.

A conjecture

Let M be a 7-dimensional Riemannian manifold admitting a solution to the Killing spinor equations with parameter $\lambda \in \mathbb{R}$ and closed NS flux H (that is, a solution of NS-NS supergravity):

$$\nabla^+ \eta = 0\,, \qquad \left(\not\nabla^{1/3} - \frac{1}{2}\zeta \right) \cdot \eta = \lambda\, \eta\,, \qquad \mathrm{d}H = 0\,,$$

for a real spinor η, a three-form $H \in \Omega^3$, and a one-form $\zeta \in \Omega^1$. Here, ∇^+ and $\not\nabla^{1/3}$ are the spin connection and Dirac operator of the connections with skew torsion H and $\frac{1}{3}H$, respectively.

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Conjecture

Then, its chiral de Rham complex admits an embedding of the SUSY vertex algebra SV_a, where the value of a is determined by the eigenvalue λ of the Dirac spinor η.

Conclusion and outlook

Conclusions

- We find a procedure to compute classical OPEs in the worldsheet algebra in terms of geometric data in the target.
- Comparing classical and quantum algebras gives an interpretation of the parameters in terms of torsion classes.
- Generically, the presence of torsion modifies the OPEs and the algebra differs from the one found for special holonomy.
- We find mathematical evidence for our results using the formalism of vertex algebras.

Open questions

- Can we understand all the quantum effects on the OPEs?
- Can we find algebras for backgrounds with RR fluxes?
- Can we use the algebras with torsion to obtain new geometric information (e.g. about mirror symmetry)?
- Can we prove the embedding conjecture?
- Can we do something similar for particles or membranes?
  - Hopefully more on this soon, with Hyungrok and Leron!

Thank you!