Extremal Couplings and Gluons in AdS/CFT

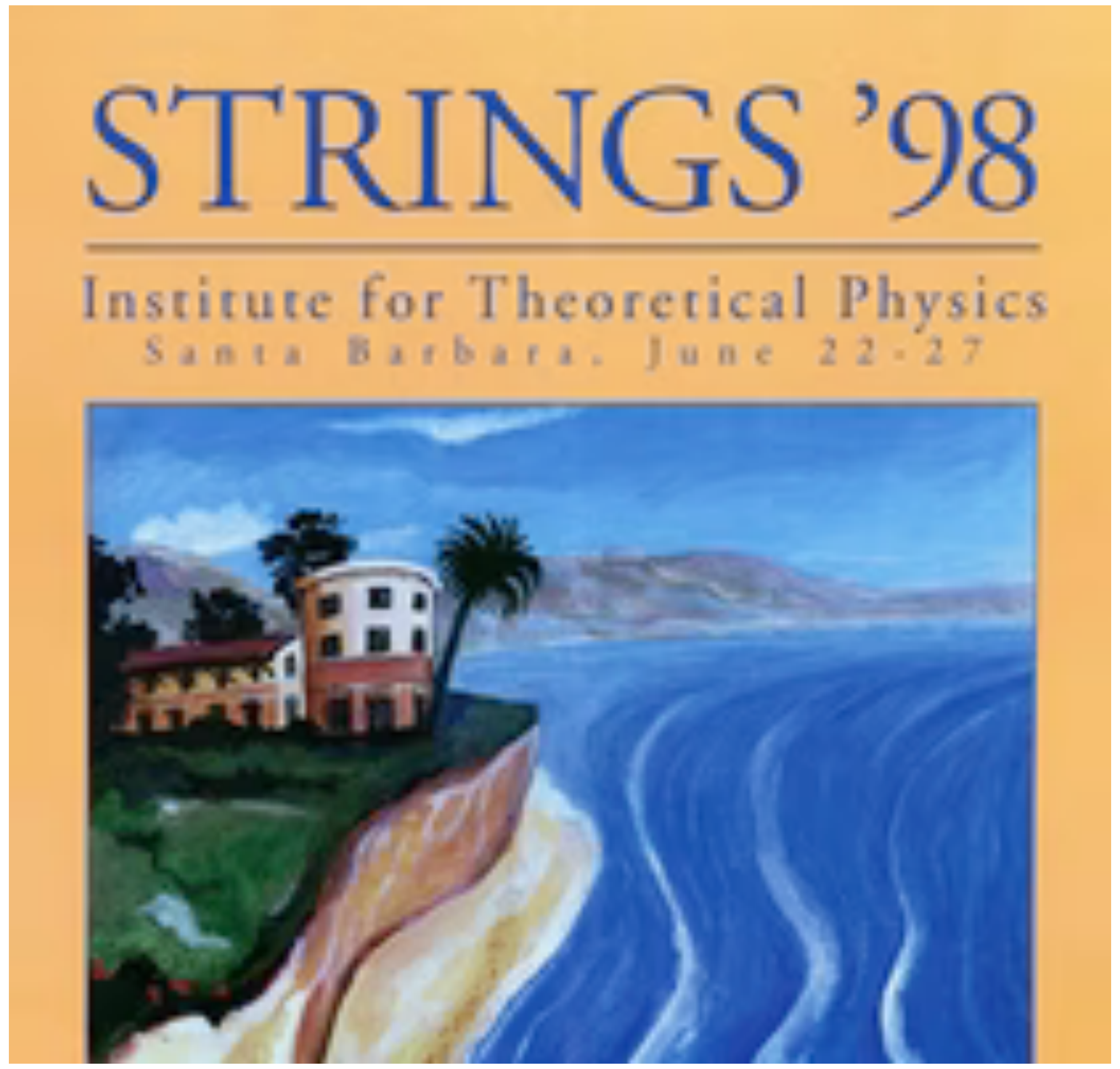
Based on:

- *Extremal Couplings and Mixing in AdS/CFT (coming soon!)*
  R. Mouland
- *Extremal couplings, graviton exchange, and gluon scattering in AdS* (2505.23948)
  S. Chester, R. Mouland, J. Van Muiden
- *An as-yet untitled paper (coming soon!)*
  S. Chester, R. Mouland, J. Van Muiden, C. Virally

Rishi Mouland, Herts, 8th October 2025

The year is 1998…

String Theory Community



Excited about:

Supergravity
Gauge/gravity duality

Me

Excited about:

Chocolate
Lego

Baby steps in AdS/CFT

Mapping the basic data

Masses m and couplings λ in supergravity Lagrangian

Scaling dimensions Δ and OPE coefficients C
(at large N and strong coupling)

How does this map work?

Compare for half-BPS operators

Scaling dimensions Δ and OPE coefficients C
(at large N and weak coupling)

Baby steps in AdS/CFT

Where the map fails

- Focus on **scalars** for this talk
- **Scaling dimensions** are easy: $m^2R^2 = \Delta(\Delta - d)$
- Bulk coupling $\lambda_{123}\phi_1\phi_2\phi_3$ leads to **OPE coefficient**

$$C_{123} = \frac{\pi^{d/2}\lambda_{123}}{2\Gamma(\Delta_1)\Gamma(\Delta_2)\Gamma(\Delta_3)}\Gamma\left(\frac{\Delta_1+\Delta_2-\Delta_3}{2}\right)\Gamma\left(\frac{\Delta_2+\Delta_3-\Delta_1}{2}\right)\Gamma\left(\frac{\Delta_3+\Delta_1-\Delta_2}{2}\right)\Gamma\left(\frac{\Delta_1+\Delta_2+\Delta_3-d}{2}\right)$$

Pole at $\Delta_3 = \Delta_1 + \Delta_2 + 2n$ for:

$n = 0 \longrightarrow$ **Extremal**

$n = 1,2,\ldots \longrightarrow$ **Super-extremal**

How do we handle (super-)extremal couplings?

- 1998 physicist says: **"Who cares!"**
  - In all maximal SUSY setups, **all (super-)extremal bulk couplings vanish**
- **New puzzle:** $\mathcal{N} = 4$ SYM has non-zero extremal OPE coefficients at weak coupling
  - **Resolution:** map between bulk fields and single-trace operators at weak coupling is subtle
  - **This talk:** always at strong coupling. No attempt to compare with weakly coupled gauge theory
- Maybe a healthy theory **cannot** have (super-)extremal bulk couplings?!

Now wait 27 years…

- **Nonsense!** We have half-maximal AdS/CFT pairs with these bulk couplings non-zero
  - So what do we do with them?!

The point of this talk

1. Non-zero (super-)extremal couplings imply **non-trivial mixing** between single- and double-trace operators in the CFT
   - Use the **conformal block expansion** of 4-point functions to diagnose this, and unmix the CFT data in a generic, bottom-up scenario

2. **(Super-)extremal couplings appear** in a class of **4d $\mathcal{N}=2$ SCFTs**, dual to IIB SUGRA in $\mathrm{AdS}_5 \times S^5$ coupled to Yang-Mills on a $\mathrm{AdS}_5 \times S^3$ brane worldvolume
   - Compute the final piece of the **gluon 4-point function** up to order $1/N^2$, by summing towers of Witten diagrams with (super-)extremal couplings
   - (Super-)extremal couplings induces mixing; **we unmix the data**
   - Determine **all stringy corrections** at order $1/N^2$ using SUSY localisation

Part I: Unmixing Data in Bottom-Up Holography

An expansion around generalised free fields

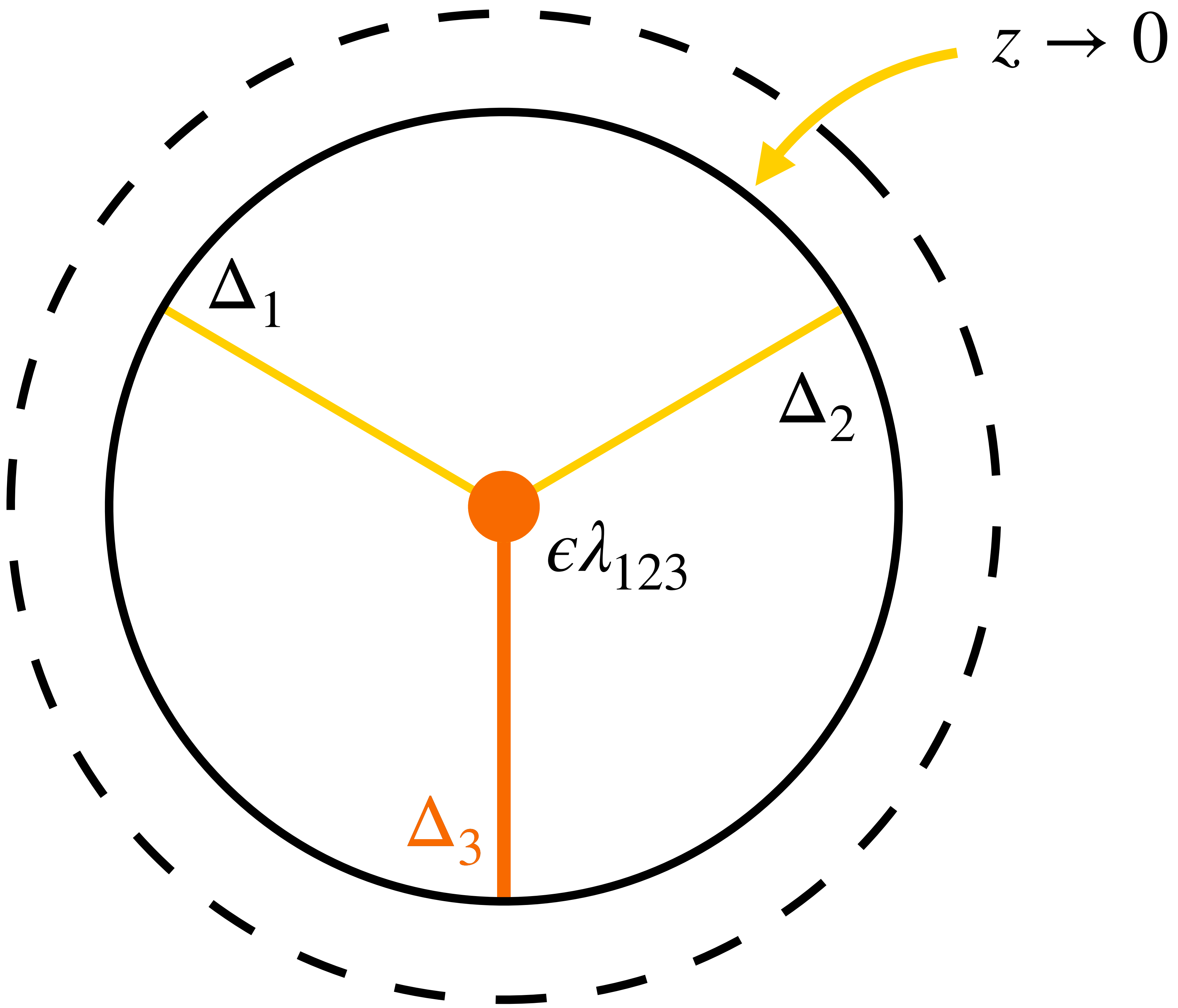
Use same notation for dual CFT operators

- Simple **bottom-up model**: a bunch of scalars ϕ_i in AdS$_{d+1}$
  - Masses m_i, cubic couplings λ_{ijk}
- Set $\lambda_{ijk} \to 0$
  - Dual CFT is a **generalised free field theory**
  - All correlation functions determined by **Wick contractions** $\overbrace{\phi_i \phi_j} \propto \delta_{ij}$
- Now turn on $\lambda_{ijk} \sim \epsilon$ with ϵ small
  - Dual CFT is now a **perturbation of GFFT**

The source of the issue

Diverging Witten diagrams

- **OPE coefficients** $C_{ijk} \sim \epsilon$ encoded in 3-point Witten diagrams



- Only converges as $z \to 0$ if

$$\Delta_3 < \Delta_1 + \Delta_2 \qquad \text{(and perms)}$$

- Γ-functions give analytic continuation to generic

$$\Delta_3 > \Delta_1 + \Delta_2 \qquad \text{(and perms)}$$

- But even this expression **diverges** when

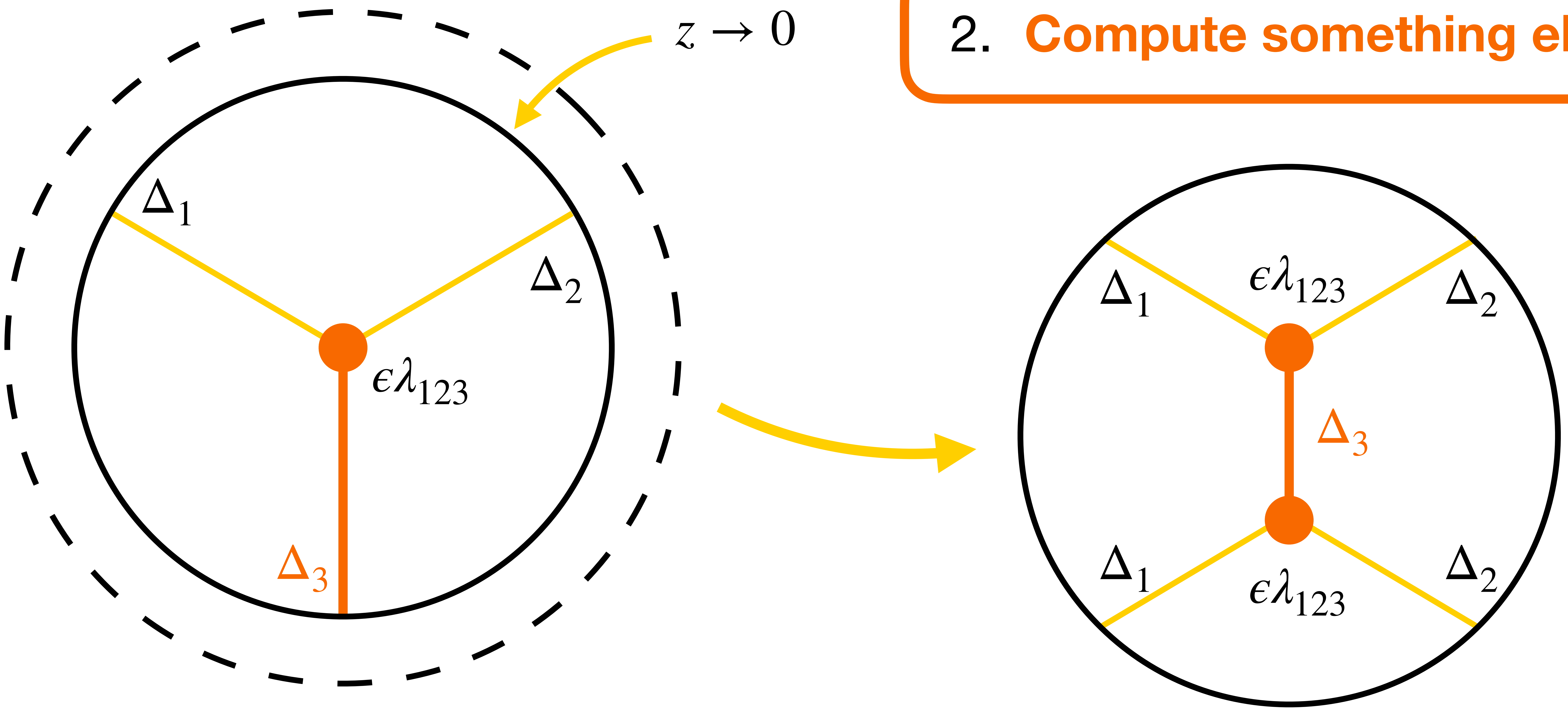
$$\Delta_3 = \Delta_1 + \Delta_2 + 2n$$

The source of the issue

Ways to proceed

1. Use **holographic renormalisation** to regularise and get something sensible

2. **Compute something else!**

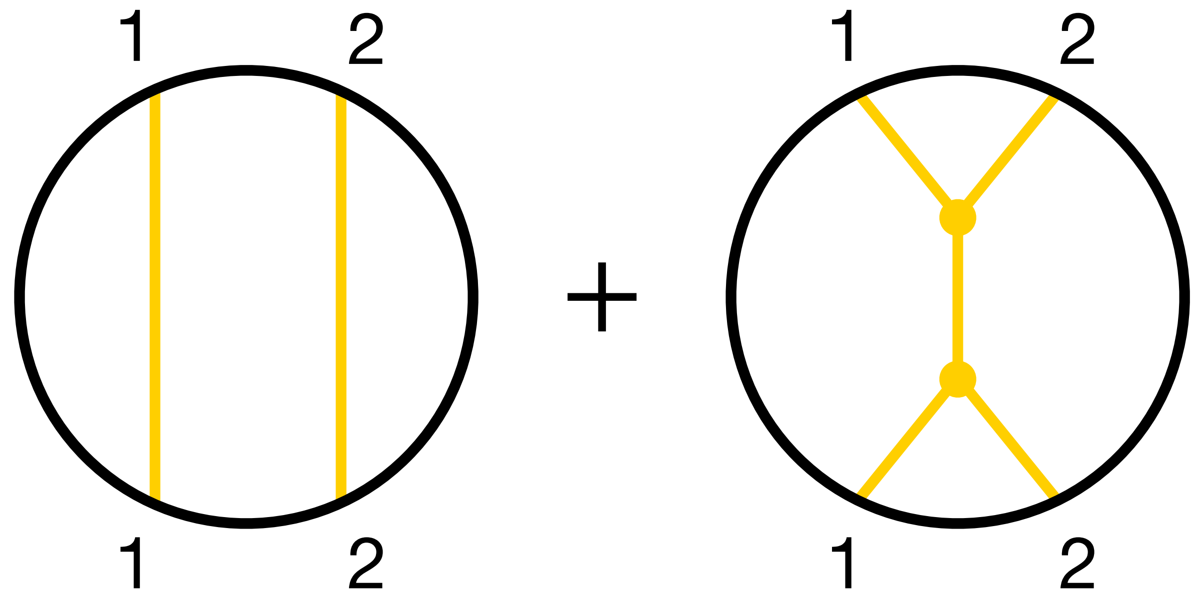


Finite even for

$$\Delta_3 = \Delta_1 + \Delta_2 + 2n$$

CFT data from 4-point functions

Conformal block expansion

- Suppose we've computed $\langle \phi_1\phi_2\phi_1\phi_2\rangle =$  + (crossed) + …

- OPE data $\phi_1\phi_2 \sim \sum_\Delta C_\Delta \mathcal{O}_\Delta$ encoded in **conformal block expansion**

$$\langle \phi_1\phi_2\phi_1\phi_2\rangle = f(x)\sum_\Delta C_\Delta^2\, g_\Delta(u,v)$$

CFT data from 4-point functions

Specialising to dominant operators

- We call an operator "**dominant**" (with respect to ϕ_1, ϕ_2) if

$$\lim_{\epsilon \to 0} C_\Delta \neq 0$$

- GFFT at $\epsilon \to 0$ means these are precisely the double traces

$$:\phi_1\phi_2: \qquad :\phi_1 \Box \phi_2: \qquad :\phi_1 \Box^2 \phi_2: \qquad \ldots$$

$$\Delta = \Delta_1 + \Delta_2 \qquad \Delta = \Delta_1 + \Delta_2 + 2 \qquad \Delta = \Delta_1 + \Delta_2 + 4$$

- How does the conformal block expansion encode their **leading anomalous dimensions?**

CFT data from 4-point functions

Anomalous dimensions of dominant operators

- Let's **expand** $(C_\Delta)^2 = a^{(0)} + \mathcal{O}(\epsilon), \qquad \Delta = \Delta^{(0)} + \epsilon\Delta^{(1)} + \mathcal{O}(\epsilon^2)$
- Fact: As $u \to 0$ we have $g_\Delta(u,v) \sim u^{\Delta/2} h_\Delta(v)$
- Expand $\langle \phi_1\phi_2\phi_1\phi_2 \rangle$ in small u and small ϵ. For each **dominant operator**, must find terms

$$(C_\Delta)^2 g_\Delta(u,v) \sim u^{\Delta^{(0)}/2}\left[\left(a^{(0)} + \dots\right) + \epsilon\left(\frac{1}{2}a^{(0)}\Delta^{(1)}\log u + \dots\right) + \epsilon^2\left(\frac{1}{8}a^{(0)}(\Delta^{(1)})^2\log^2 u + \dots\right) + \dots\right] h_{\Delta_0}(v)$$

- **Assume no (super-)extremal couplings** and compute Witten diagrams, then expand. For each dominant double trace $\Delta^{(0)} = \Delta_1 + \Delta_2 + 2n$, we indeed find terms

$$u^{\Delta^{(0)}/2}\left[\left(\# + \dots\right) + \epsilon\left(\quad 0 \quad\right) + \epsilon^2\left(\quad 0 \quad \log^2 u + \dots\right) + \dots\right] h_{\Delta_0}(v)$$

$$\longrightarrow \quad \Delta^{(1)} = 0$$

The problem with (super-)extremal couplings

$$(C_\Delta)^2 g_\Delta(u,v) \sim u^{\Delta^{(0)}/2}\left[\left(a^{(0)} + \dots\right) + \epsilon\left(\frac{1}{2}a^{(0)}\Delta^{(1)}\log u + \dots\right) + \epsilon^2\left(\frac{1}{8}a^{(0)}(\Delta^{(1)})^2\log^2 u + \dots\right) + \dots\right] h_{\Delta_0}(v)$$

- **Now suppose $\Delta_3 = \Delta_1 + \Delta_2 + 2n$**, and look at $u^{\Delta^{(0)}/2}$ terms with $\Delta^{(0)} = \Delta_1 + \Delta_2 + 2n$. We find

$$u^{\Delta^{(0)}/2}\left[\left(\# + \dots\right) + \epsilon\left(0 \right) + \epsilon^2\left(\# \log^2 u + \dots\right) + \dots\right] h_{\Delta_0}(v)$$

Non-zero!

The assumption that fails:

The only dominant operators are the double traces $:\phi_1 \Box^n \phi_2:$, $n = 0,1,\dots$

- **Resolution:** mixing between ϕ_3 and $:\phi_1 \Box^n \phi_2:$, both of which have $\Delta = \Delta_1 + \Delta_2 + 2n$

Unmixing the spectrum

Resolving the puzzle

- So now assume we have **two dominant operators $\mathcal{O}, \tilde{\mathcal{O}}$** with $\Delta^{(0)} = \Delta_1 + \Delta_2 + 2n$, which are linear combinations of ϕ_3 and $:\phi_1 \Box^n \phi_2:$

- The relevant terms in the conformal block expansion are now

$$(C_\Delta)^2 g_\Delta(u,v) + (\tilde{C}_{\tilde{\Delta}})^2 g_{\tilde{\Delta}}(u,v)$$

$$\sim u^{\Delta^{(0)}/2}\Big[\Big(\underbrace{\big(a^{(0)} + \tilde{a}^{(0)}\big)}_{\neq 0} + \dots\Big) + \epsilon\Big(\underbrace{\tfrac{1}{2}\big(a^{(0)}\Delta^{(1)} + \tilde{a}^{(0)}\tilde{\Delta}^{(1)}\big)}_{=0}\log u + \dots\Big) + \epsilon^2\Big(\underbrace{\tfrac{1}{8}\big(a^{(0)}(\Delta^{(1)})^2 + \tilde{a}^{(0)}(\tilde{\Delta}^{(1)})^2\big)}_{\neq 0}\log^2 u + \dots\Big) + \dots\Big] h_{\Delta_0}(v)$$

- Now it's fine! But **only 3 constraints for 4 pieces of data**

Unmixing the spectrum

Determining the mixing matrix

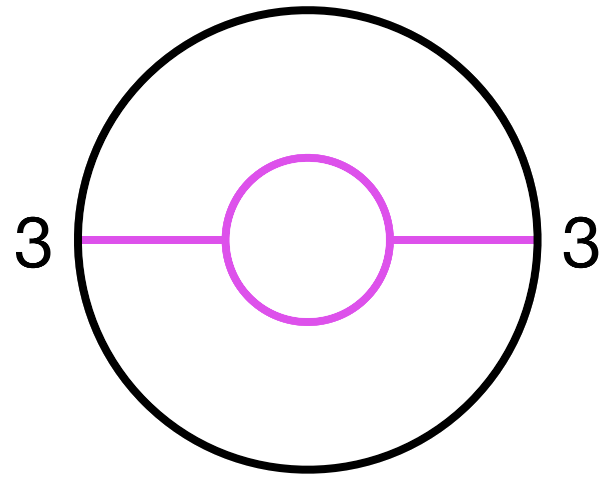
$$(C_\Delta)^2 g_\Delta(u,v) + (\tilde{C}_{\tilde{\Delta}})^2 g_{\tilde{\Delta}}(u,v)$$

$$\sim u^{\Delta^{(0)}/2}\left[\left(\left(a^{(0)}+\tilde{a}^{(0)}\right)+\dots\right)+\epsilon\left(\frac{1}{2}\left(a^{(0)}\Delta^{(1)}+\tilde{a}^{(0)}\tilde{\Delta}^{(1)}\right)\log u+\dots\right)+\epsilon^2\left(\frac{1}{8}\left(a^{(0)}(\Delta^{(1)})^2+\tilde{a}^{(0)}(\tilde{\Delta}^{(1)})^2\right)\log^2 u+\dots\right)+\dots\right]h_{\Delta_0}(v)$$

(Roughly)

$$\begin{pmatrix} \langle(\phi_1\Box^n\phi_2)\,(\phi_1\Box^n\phi_2)\rangle & \langle(\phi_1\Box^n\phi_2)\,\phi_3\rangle \\ \langle\phi_3\,(\phi_1\Box^n\phi_2)\rangle & \langle\phi_3\phi_3\rangle \end{pmatrix} = \frac{1}{|x-y|^{2\Delta^{(0)}}}\left[\mathbb{1}-\epsilon\begin{pmatrix} (*) & (*) \\ (*) & (*) \end{pmatrix}\log\left(|x-y|^2\right)+\mathcal{O}(\epsilon^2)\right]$$

Determined by Witten diagrams

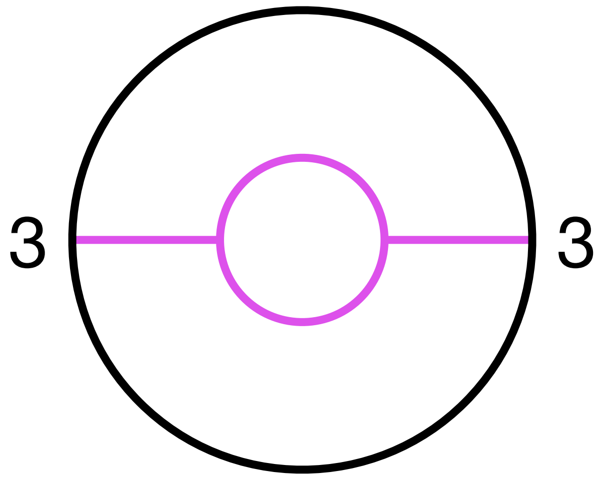


$+ \dots$

- **Eigenvalues** give you **anomalous dimensions** $\Delta_{(1)}, \tilde{\Delta}_{(1)}$
- **Eigenvectors** give you **OPE coefficients** $a_{(0)}, \tilde{a}_{(0)}$

Unmixed data for the bottom-up model

- 4-point function has no **contribution at order ϵ**
- **Leading correction to $\langle \phi_3 \phi_3 \rangle$** also comes at order ϵ^2 from



$\longrightarrow$ Mixing matrix $\begin{pmatrix} 0 & (*) \\ (*) & 0 \end{pmatrix} \longrightarrow$

Unmixed operators:

$$\frac{1}{\sqrt{2}} \left(:\phi_1 \Box^n \phi_2 : \pm \phi_3 \right)$$

$$\Delta = \Delta_1 + \Delta_2 + 2n \pm a\epsilon + \mathcal{O}(\epsilon^2)$$

$$C_\Delta = \frac{1}{\sqrt{2}} C_\Delta^{\text{GFFT}} + \mathcal{O}(\epsilon)$$

- **Key lesson:** The CFT primaries are a non-trivial mix of the generalised free field operators!

- Have also determined a independently by **holographic renormalisation** of the 3-point Witten diagram (already done in [Castro, Martinez, '24] for the extremal ($n = 0$) case)

Part II: Application to a 4d $\mathcal{N} = 2$ Setup

The theories of interest

- **String theory:** N D3-branes probing F-theory singularities $G_F = A_1, A_2, D_4, E_6, E_7, E_8$
  - All but D_4 have **constant complexified string coupling τ**. For D_4, we hold τ fixed

$\tau \sim g_s^{-1} \sim N/\lambda$

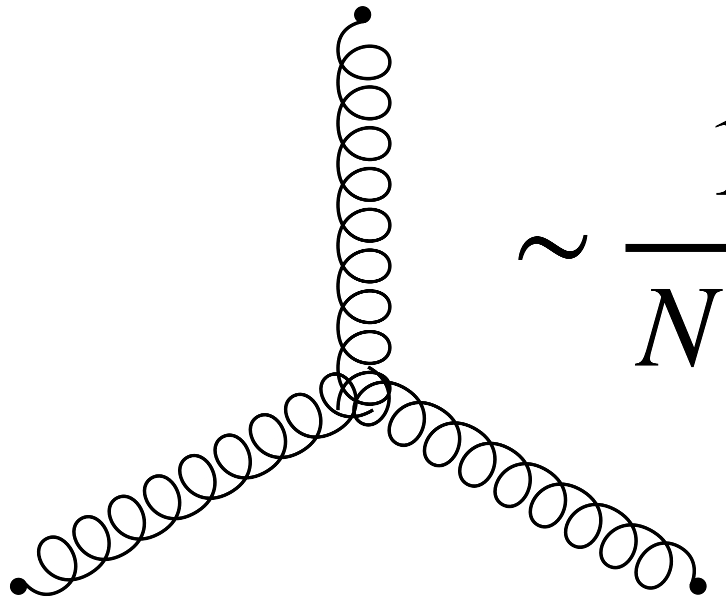
- At large N have IIB SUGRA on AdS$_5 \times (S^5/\Gamma)$
  - **8d Yang-Mills with gauge group G_F** on fixed point locus AdS$_5 \times S^3$
- $\mathcal{N} = 2$ SCFT has symmetries $G_F \times SU(2)_L \times SU(2)_R \times U(1)_R$

  Flavour: $G_F \times SU(2)_L$ — R: $SU(2)_R \times U(1)_R$

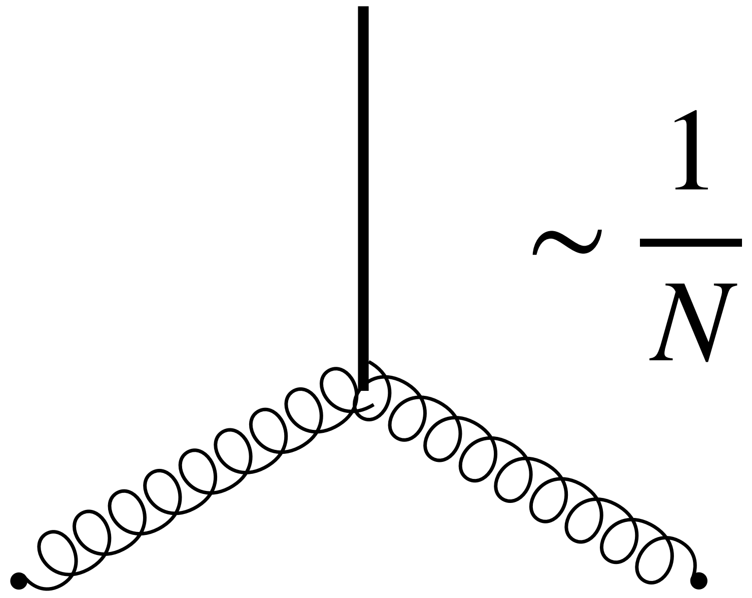
- **KK reduce to AdS$_5$** and look at scalar superprimaries:
  - 8d gauge field $\longrightarrow \phi_p$ in $(\mathbf{Adj}, \frac{p}{2}-1, \frac{p}{2})_0$ and $\Delta = p$ $\longrightarrow$ **Short multiplets**
  - 10d graviton $\longrightarrow s_{k,r}$ in $(\mathbf{1}, \frac{r}{2}-1, \frac{r}{2}-1)_0$ and $\Delta = k = r, r+2, \ldots$ $\longrightarrow$ **Long multiplets** (except $s_{2,2}$)

The correlator of interest

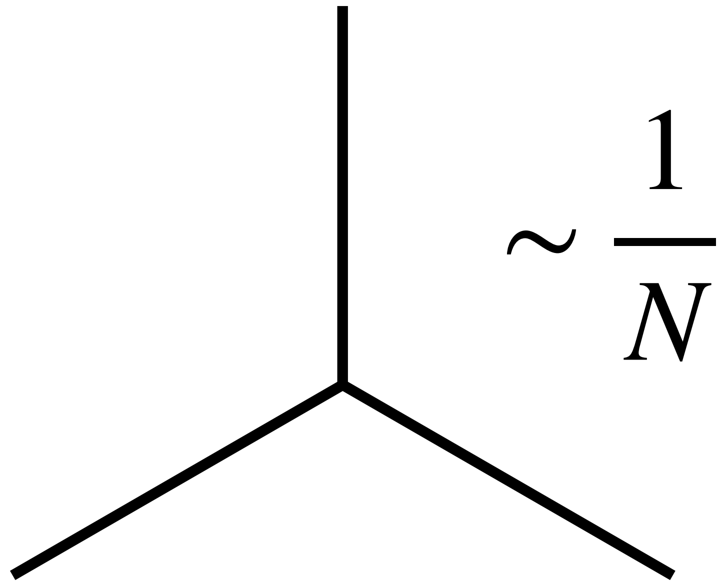
- Couplings:



$\sim \frac{1}{N^{1/2}}$

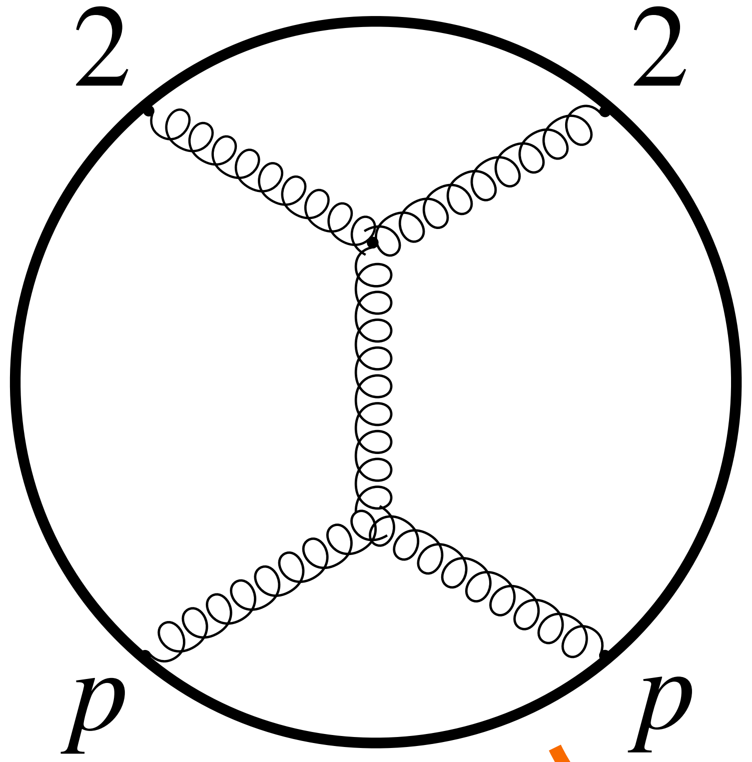
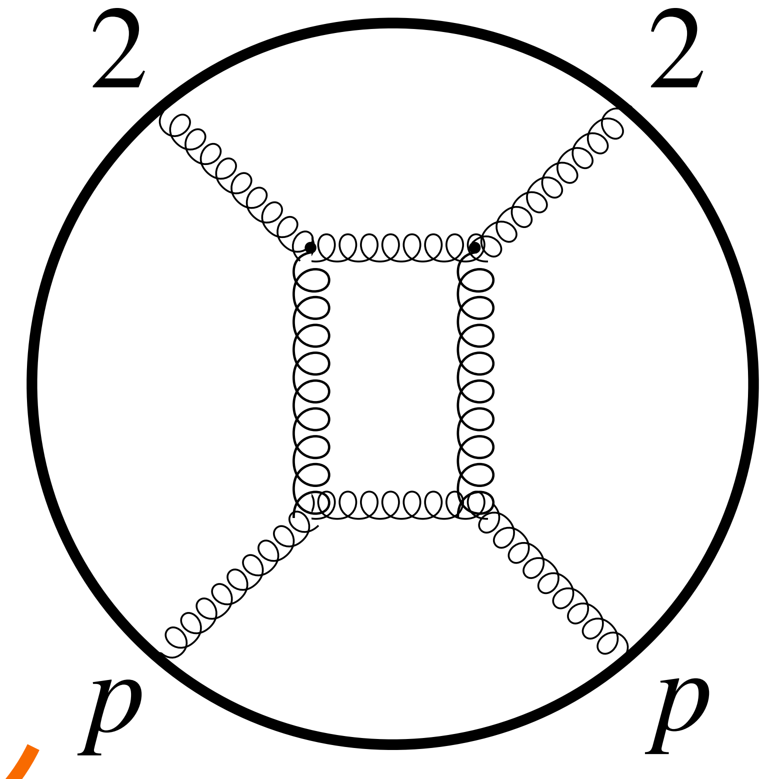
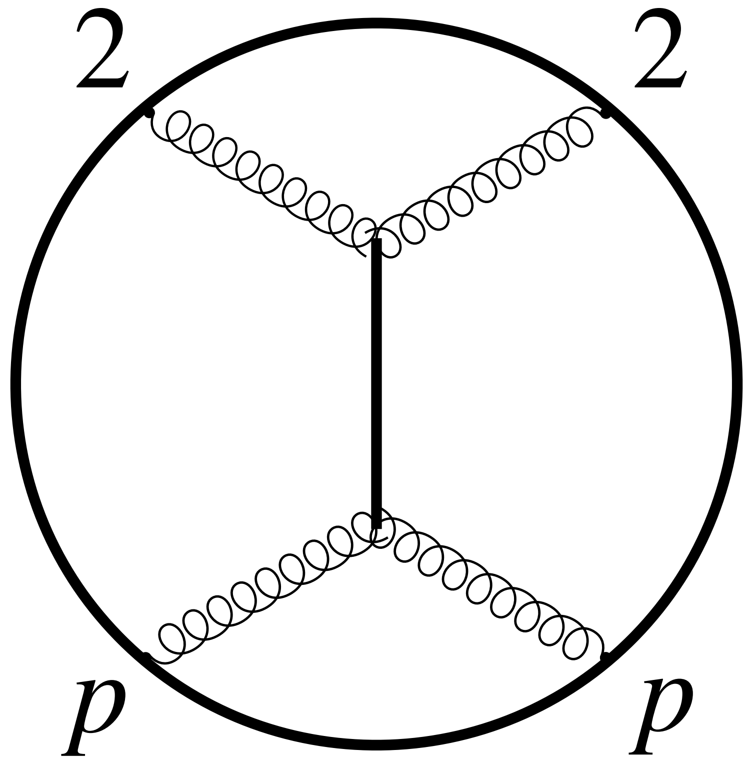


$\sim \frac{1}{N}$



$\sim \frac{1}{N}$

Plus τ-dependent contact terms

$\langle \phi_2 \phi_2 \phi_p \phi_p \rangle_{\text{conn}} = \frac{1}{N}$  $+ \frac{1}{N^2} \Big[$  $+$  $\Big]$

Involve only short multiplets
Bootstrapable! [Alday, Behan, Bissi, Ferrero, Zhou]

Long multiplet exchanges unknown

The bulk computation

- Fancy tricks (i.e. **bootstrapping using CFT consistency**) are not useful here
- Have to do things to “old” way: really just **compute the Witten diagrams!**
- Need **bulk cubic couplings:**

$$\lambda_{p,q,(k,r)} \phi_p \phi_q s_{k,r}\,, \qquad \Delta_3 - \Delta_1 - \Delta_2 = k - p - q \longrightarrow$$

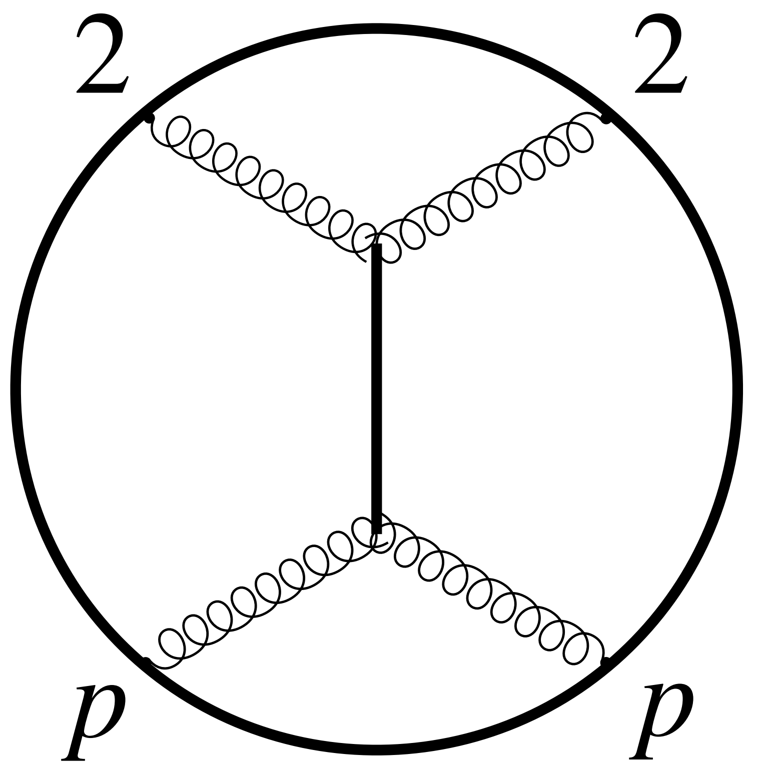
(Super-)extremal for all $k = p+q, p+q+2, \ldots$

A certain amount of pain later…

$$\lambda_{p,q,(k,r)} = \frac{\pi}{2N}\sqrt{\frac{(r-1)(p-1)(q-1)}{2\boldsymbol{\Delta} k_r(k_r-1)(k_r+1)}}\frac{\Gamma(p-1)\Gamma(q-1)\Gamma(r-1)}{\Gamma\left(\frac{p+q-r}{2}\right)\Gamma\left(\frac{r+p-q}{2}\right)\Gamma\left(\frac{r+q-p}{2}\right)\Gamma\left(\frac{p+q+r-2}{2}\right)}(k_r+p-q)(k_r+q-p)(k_r+p+q-2)(k_r+p+q-4)$$

Final answer for graviton exchange

- Work in Mellin space. Basically Fourier space for cross ratios, $(u, v) \to (s, t)$



G_F adjoint index

$$+\,(\text{crossed}) = -\frac{p}{(p-2)!\Delta}\Bigg[\left((p+1)H_{1-\frac{s}{2}} + \frac{4}{s-2}\right)\delta^{AB}\delta^{CD} + \left((p+1)H_{\frac{2+p-u}{2}} + \frac{2p}{u-p}\right)\delta^{AC}\delta^{BD} + \left((p+1)H_{\frac{2+p-t}{2}} + \frac{2p}{t-p}\right)\delta^{AD}\delta^{BC}\Bigg]$$

Harmonic number

$$H_n = \sum_k \frac{1}{n^k}$$

A more interesting mixing problem

The effect of gluon exchange

- Have mixing, for instance, between
  - Gluon double-trace $:\phi_2\phi_2:$
  - Graviton single-trace $s_{4,2}$

Both flavour and R-symmetry singlets with $\Delta = 4$

$$u^{\Delta^{(0)}/2}\left[\left(\left(a^{(0)}+\tilde{a}^{(0)}\right)+\dots\right)+\frac{1}{N}\left(\frac{1}{2}\left(a^{(0)}\Delta^{(1)}+\tilde{a}^{(0)}\tilde{\Delta}^{(1)}\right)\log u+\dots\right)+\frac{1}{N^2}\left(\frac{1}{8}\left(a^{(0)}(\Delta^{(1)})^2+\tilde{a}^{(0)}(\tilde{\Delta}^{(1)})^2\right)\log^2 u+\dots\right)+\dots\right]h_{\Delta_0}(v)$$

Now non-zero due to gluon exchange at order $1/N$

- Mixing matrix $\begin{pmatrix} (*) & (*) \\ (*) & 0 \end{pmatrix}$ $\xrightarrow[G_F = D_4]{\text{e.g.}}$

$$(C_\Delta)^2 = \frac{1}{3} \pm \sqrt{\frac{5}{381}} + \mathcal{O}\left(\frac{1}{N}\right)$$

$$\Delta = 4 - \frac{1}{N}\left(3 \pm \sqrt{\frac{381}{5}}\right) + \mathcal{O}\left(\frac{1}{N^2}\right)$$

A more interesting mixing problem

Mixing amongst double traces

- Things yet more interesting at higher scaling dimension. For $\Delta = 2n$ get mixing between

$$s_{2n,2} \qquad :\phi_n\phi_n: \qquad :\phi_{n-1}\Box\phi_{n-1}: \qquad \dots \qquad \phi_2\Box^{n-2}\phi_2$$

- Gluon exchange ⟶ **Mixing amongst double traces**
- Super-extremal graviton exchange ⟶ **Mixing between single and double traces**
- We determine mixing matrix

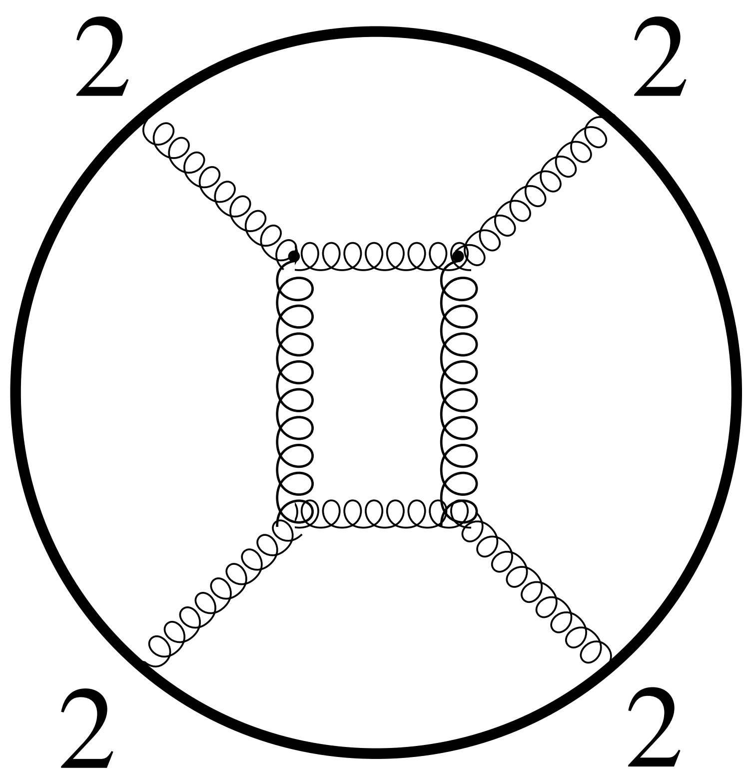
$$\overbrace{\phantom{\begin{pmatrix} & * & \end{pmatrix}}}^{\text{DT}}\;\overbrace{}^{\text{ST}}$$

$$\begin{pmatrix} \begin{pmatrix} & * & \end{pmatrix} & \begin{pmatrix} * \end{pmatrix} \\ \begin{pmatrix} & * & \end{pmatrix} & 0 \end{pmatrix} \longrightarrow \textbf{Unmixed CFT data}$$

All stringy corrections at order $1/N^2$ for $G_F = D_4$

The contact terms we want

$$\langle \phi_2\phi_2\phi_2\phi_2 \rangle_{\text{conn}} = \dots + \frac{1}{N^2}\Bigg[$$



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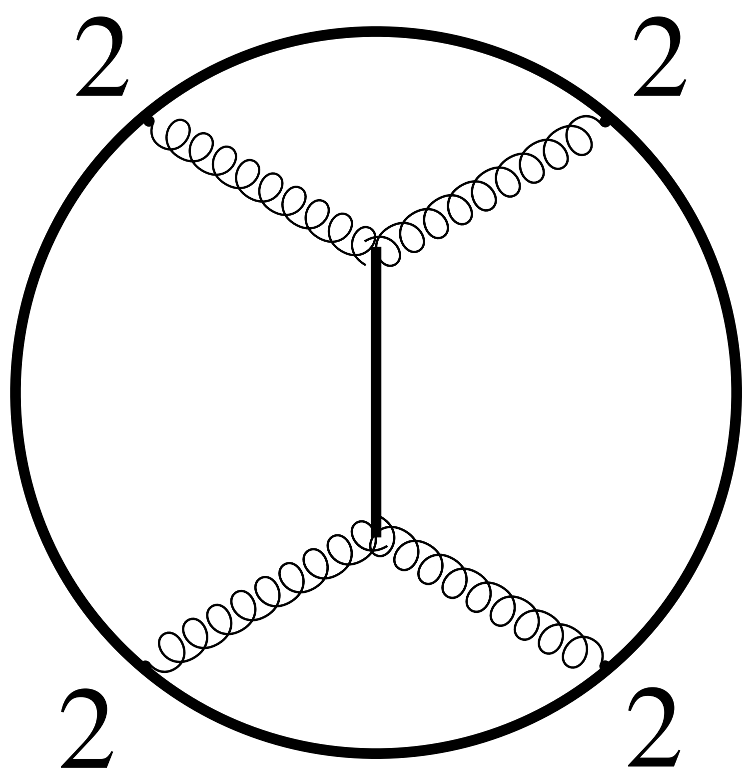

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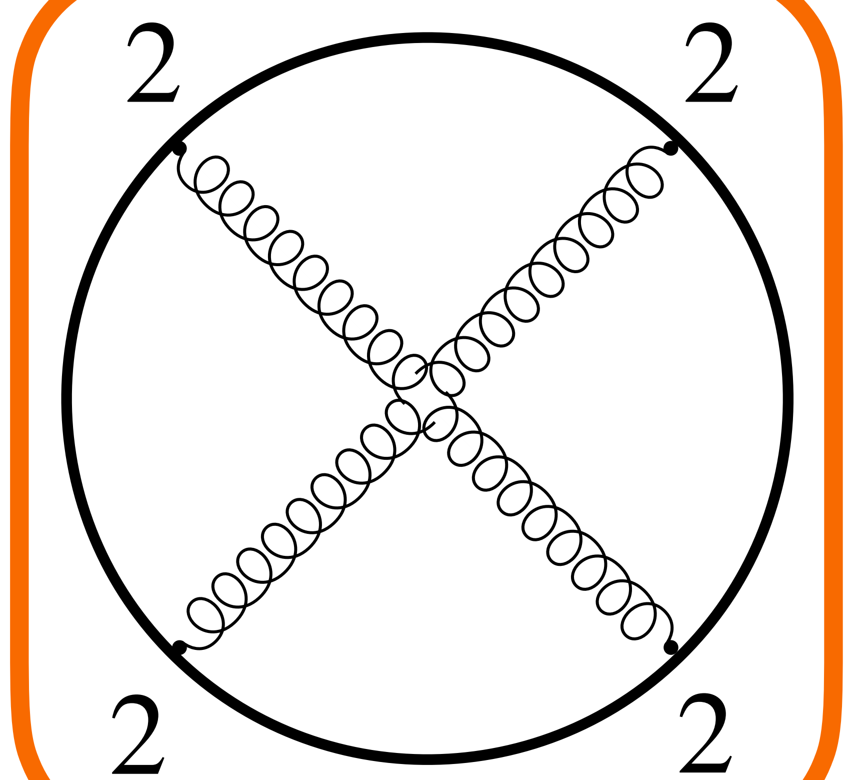

$$\Bigg]$$

- Depends on 3 functions:

$$c_i(\tau) = \tau\left(\# + \#\tau^{-1} + \#\tau^{-2} + \dots\right), \qquad i = 1, 2, 3$$

Genus expansion

Different flavour structures

- Infinite series of **stringy corrections!**

All stringy corrections at order $1/N^2$ for $G_F = D_4$

Fixing using SUSY localisation

- The $G_F = D_4$ theory has a complex exactly marginal coupling τ
  $\longrightarrow$ **Weakly coupled regime** described by $USp(2N)$ gauge theory
- Use **SUSY localisation** to compute $\mathcal{F}(m_i)$, mass-deformed free energy on S^4
  - Matrix model, evaluated to at finite τ and to all perturbative orders in $1/N$
- Get **integrated constraints** of the form

$$(\partial_{m_a})^4 \mathcal{F}(m_a) \sim \int ds\, dt\, \langle \phi_2 \phi_2 \phi_2 \phi_2 \rangle$$

- Enough to **completely fix the $c_i(\tau)$!** For example,

$$c_1(\tau) = 6(\gamma - \log(4\pi)) - 3 \log \left[\tau_2 \,|\, \theta_3(\tau) \theta_4(\tau) \,|^2 \right]$$

Some final words

Some other stuff we did

- We also compute the **flat space limit** of our graviton exchange contribution, and found an **exact match with the known result**
- **Hot off the press:** We have computed these graviton exchanges for some **other half-maximal setups:**
  - 3d $\mathcal{N} = 4$ theories dual to $AdS_4 \times (S^7/\mathbb{Z}_k)$
  - 6d $\mathcal{N} = (1,0)$ theories dual to $AdS_7 \times (S^4/\mathbb{Z}_2)$
  - Interplay between gluon and graviton exchanges **qualitatively different** due to different scalings with N (in some sense, 4d is the most interesting)

Thanks!