# Quantum and Classical Properties of the Sen Action for Self-Dual Fields

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# Introduction

Self-dual-Fields have a long history and continue to provide surprises (as I hope to show you).

Self-duality refers to a $p$-form field $P$ such that $dP = \star dP$.

$$\Longrightarrow d \star dP = 0$$

As this requires $\star^2 = 1$ it only exists for $p = 2k$ in $D = 4k + 2$ spacetime dimensions

The essential problem is that the naive action vanishes

$$dP = \star dP \Longrightarrow dP \wedge \star dP = 0$$

as $dP$ is an odd form. This complicates quantization

For example we can consider a chiral Boson in 2D ($k = 0$)

$$dP = \star dP \iff \partial_0 P - \partial_1 P = 0$$

It is well known that to have a modular invariant partition function requires $8n$ chiral Bosons.

One approach is holomorphic factorisation: embed a self-dual theory in a non-self-dual theory and assign

$$Z_{sd}(\tau) = \sqrt{Z_{nsd}(\tau, \tau^*)}$$

The problem is that

$$Z_{nsd}(\tau, \tau^*) = \sum_a Z^a_{sd}(\tau) Z^a_{asd}(\tau^*)$$

So there isn't a unique choice for $Z_{sd}(\tau)$: a relative QFT

## Over the past 40 years various actions have been proposed:

- Siegel
- Floreanini-Jackiw
- Devecchi-Henneaux
- Pasti-Sorkin-Tonin
- Belov-Moore
- Mkrtchyan

All of these actions give up something (manifest Lorentz symmetry, addition non-dynamical fields, non-quadratic actions)

Here we will look at the Sen action which came out of String Field theory. It is quadatic, Lorentz invariant, and remarkably amenable to quantisation, but has a shadow sector...

Most importantly it’s novel, enlightening and fun!

# Part I: Classical Aspects

Sen's original action in $4k+2$ dimensions is (without sources)

$$S = -\int \left(\frac{1}{2} dP \wedge \star_\eta dP + 2Q \wedge dP - Q \wedge M(Q)\right) .$$

What are the ingredients?

- $\star_\eta$ is the Hodge dual w.r.t. the Minkowski metric $\eta$
- $P$ is a $2k$-form
- $Q$ is a $(2k+1)$-form such that $Q = \star_\eta Q$
- $M(Q)$ is a map from $\star_\eta$-self-dual forms to $\star_\eta$-anti-self-dual forms constructed so that

$$Q + M(Q) = \star(Q + M(Q))$$

where $\star$ is the Hodge dual w.r.t. a metric $g$.

## So What does it do?

The equations of motion are

$$
\begin{aligned}
0 &= d\left(-\star_\eta dP + 2Q\right)\\
0 &= dP - \star_\eta dP - 2M(Q)\ .
\end{aligned}
$$

which can be re-arranged into $dF = dG = 0$ where

$$
\begin{aligned}
G &= Q - \frac{1}{2}(1+\star_\eta)dP = \star_\eta G\\
F &= Q + M(Q) = \star F
\end{aligned}
$$

So we find two self-dual fields but w.r.t. to two different metrics

Note that $P$ has the wrong sign kinetic term: In the Hamiltonian formulation $G$ has the wrong sign energy but $F$ the correct sign.

$F$ and $G$ decouple from each other (even including sources).

This action has been extended by Hull to replace $\eta$ by a generic metric $\bar{g}$ with Hodge dual $\bar{\star}$.

The previous statements all still apply with $\star_\eta \to \bar{\star}$. But:

- Now we can put the Sen action on a general manifold
- Remarkably it admits two diffeomorphism-like symmetries

$$\bar{\delta}\bar{g} = \mathcal{L}_\xi \bar{g} \qquad \bar{\delta} g = 0$$
$$\delta \bar{g} = 0 \qquad \delta g = \mathcal{L}_\zeta g$$

True diffeomorphisms, i.e. those that arise from coordinate transformations, correspond to the diagonal subgroup $\xi = \zeta$.

In the original Sen Formulation $\bar{g} = \eta$ and diffeomorphisms were obscured because you had to keep $\eta$ fixed.

Again $F$ and $G$ decouple and respond to different metrics.

Let's look at little bit more at $M$: [Andriolo, NL, Papageogakis],[Hull]

Let $\bar{\omega}_+^A$ and $\bar{\omega}_-^A$ be a basis of $(2k+1)$-forms with $\bar{\star}\bar{\omega}_\pm^A = \pm\bar{\omega}_\pm^A$

Similarly let $\omega_+^A$ and $\omega_-^A$ be a basis with $\star\omega_\pm^A = \pm\omega_\pm^A$.

It follows that we can write

$$\omega_+^A = \mathcal{N}^A{}_B\bar{\omega}_+^B + \mathcal{K}^A{}_B\bar{\omega}_-^B$$

Thus if $F = Q + M(Q)$ and $F = \star F$ then

$$\begin{aligned} F &= F_A\omega_+^A \\ &= \underbrace{F_A\mathcal{N}^A{}_B\bar{\omega}_+^B}_{Q=Q_A\bar{\omega}_-^A} + \underbrace{F_A\mathcal{K}^A\omega_-^A}_{M(Q)} \end{aligned}$$

Thus $F_A = (\mathcal{N}^{-1})^B{}_A Q_B$ and so $M(Q_A\bar{\omega}_+^A) = (\mathcal{N}^{-1}\mathcal{K})^A{}_B Q_A\bar{\omega}_-^B$

## Two metrics gives rise to two notions of EM tensors

We find [Hull,NL]

$$\Theta_{\mu\nu} = -\frac{2}{\sqrt{-g}}\frac{\delta S}{\delta g^{\mu\nu}} = \frac{1}{(2k)!}F_\mu \cdot F_\nu$$

$$\bar{\Theta}_{\mu\nu} = -\frac{2}{\sqrt{-\bar{g}}}\frac{\delta S}{\delta \bar{g}^{\mu\nu}} \cong -\frac{1}{(2k)!}G_\mu \bar{\cdot} G_\nu$$

As a consequence of the two diffeomorphism-like symmetries these are each conserved (and they are also traceless):

$$g^{\mu\nu}\Theta_{\mu\nu} = \bar{g}^{\mu\nu}\bar{\Theta}_{\mu\nu} = 0,$$

$$\nabla^\mu \Theta_{\mu\nu} = \bar{\nabla}^\mu \bar{\Theta}_{\mu\nu} = 0.$$

But true diffeomorphism invariance only requires that

$$\sqrt{-\bar{g}}\bar{\nabla}_\lambda \bar{\Theta}_\mu{}^\lambda + \sqrt{-g}\nabla_\lambda \Theta_\mu{}^\lambda = 0$$

# Part II: Quantum Aspects

## For concreteness we focus on a chiral Boson in 2D

Since the action is quadratic in fields we can evaluate the path integral using Gaussian integrals.

This was first done by [Andriolo, NL, Orchard, Papageogakis] on a torus with $\bar{g} = \eta$. The key idea is that rather than Wick rotating $t \to it$ we can analytically continue the metric $\eta$ to $\delta$.

A remarkable feature is that the wrong-sign kinetic term $dP \wedge \star_\eta dP$ remains oscillatory

$$Z \sim \int [dP][dQ] e^{iS}$$

The damping required for the path integral comes from

$$\mathrm{Im}(Q \wedge M(Q)) > 0$$

# More generally let $\bar{g}$ and $g$ be Euclidean [Hull,NL]

Consider general 2D metrics parameterised by

$$\bar{g} = e^{2\sigma}(dx + \rho dy)(dx + \tilde{\rho} dy)$$
$$g = e^{2\omega}(dx + \tau dy)(dx + \tilde{\tau} dy)$$

Lorentzian: $\tilde{\tau} \neq \tau$, $\tilde{\rho} \neq \rho$ real
Riemannian: $\tilde{\tau} = \tau^*$, $\tilde{\rho} = \rho^*$ complex

In this case, going to veilbein frame

$$\bar{e}^+ = \frac{1}{\sqrt{2}} e^{\sigma}(dx + \rho dy) \qquad \bar{e}^- = \frac{1}{\sqrt{2}} e^{\sigma}(dx + \tilde{\rho} dy)$$
$$e^+ = \frac{1}{\sqrt{2}} e^{\omega}(dx + \tau dy) \qquad e^- = \frac{1}{\sqrt{2}} e^{\omega}(dx + \tilde{\tau} dy)$$

we find $\bar{\star}\bar{e}^{\pm} = \pm\bar{e}^{\pm}$ so $Q = Q_+\bar{e}^+$ and

$$M(\bar{e}^-) = 0 \qquad M(\bar{e}^+) = M_{--}\,\bar{e}^- \qquad M_{--} = -\frac{\rho - \tau}{\tilde{\rho} - \tau}$$

$$S = -\frac{1}{2\pi}\int \det(\bar{e})\Big( -\partial_+ P\partial_- P - 2Q_+\partial_- P - Q_+Q_+M_{--}\Big) d^2x$$

$$= -\frac{1}{2\pi}\int \det(\bar{e})\Big( -\partial_+ P\partial_- P + M_{--}^{-1}\partial_- P\partial_- P$$

$$- M_{--}(Q_+ + M_{--}^{-1}\partial_- P)^2\Big) d^2x,$$

Note that here $\partial_\pm = e_\pm^{\ \mu}\partial_\mu$

$Q_+$ appears quadratically, without derivatives, and can be integrated out of the path integral

$$Z \sim \int [dP][dQ]e^{iS} \ \sim \int [dP]e^{-S_{eff}}$$

where

$$S_{eff} = -\frac{1}{2\pi}\int \det(\bar{e})\Big( -\partial_+ P\partial_- P + M_{--}^{-1}\partial_- P\partial_- P\Big)$$

We can write this as

$$S_{eff} = -\frac{1}{2\pi} \int \sqrt{h} h^{\mu\nu} \partial_\mu P \partial_\nu P$$

where there is a new metric

$$h = e^{2\sigma'} (dx + \rho dy)(dx + \tau dy)$$

i.e. it takes the holomorphic sectors of $\bar{g}$ and $g$ and puts them together to form a new complex metric $(\rho \neq \tau^*)$.

Thus the Sen action reduces to a real non-chiral Boson in a complex metric

The decoupling of the $\bar{g}$ and $g$ sectors corresponds to the familiar decoupling of left and right moving modes in a 2D CFT.

# Partition Function

I'll spare you the details but evaluating the path integral on a torus using zeta-function techniques gives

$$
\begin{aligned}
Z(\tau,\rho) &\sim \int [dP] e^{-S_{eff}} \\
&\sim \frac{1}{\eta(q)\eta(\tilde{q})} \sum_{m,k} q^{\frac{1}{4}(mR+k/R)^2} \tilde{q}^{\frac{1}{4}(mR-k/R)^2}
\end{aligned}
$$

where $q = e^{2\pi i\tau}$ and $\tilde{q} = e^{-2\pi i\rho}$ and we assumed $P$ is periodic:

$$P \sim P + 2\pi R$$

This is analogous to a normal Boson but now $\tilde{q} \neq q^*$

Crucially it is holomorphic in both $\tau$ and $\rho$ and converges so long as $\mathrm{Im}(\tau) > 0$ and $\mathrm{Im}(\rho) < 0$.

Let us compare with the computation of [Andriolo, NL, Orchard, Papageogakis] which corresponds to $\bar{g} = \eta$, i.e. $\rho = -1$

In this case $\tilde{q} = 1$ and the sums don't converge. However the divergence $\eta(1)$ is constant and discarded by zeta-function techniques.

The winding sum led to an additional delta-function imposing the constraint:

$$R^2 = r_1/r_2$$

leading to

$$Z \sim \frac{1}{\eta(q')} \sum_n q'^{r_1 r_2 n^2}$$

where $q' = e^{2\pi i \tau'}$, $\tau' = -1/(1+\tau)$

So this is in line with what we found above but it is a slightly different calculation (and can't be extended beyond genus one).

# Modular Invariance

An important feature of any 2D CFT is modular invariance.

This arises from large diffeomorphisms of the torus and is therefore expected to be symmetries in any covariant theory with an action

A classic feature of a chiral Boson is that the partition function is not modular invariant (one needs $8n$ chiral Bosons).

This has been used as an argument against a Lagrangian [Witten]

What happens here?

Recall that in the Sen action true diffeomorphisms correspond to the diagonal subgroup: $\rho$ and $\tau$ both transform:

$$\rho \to \frac{a\rho + b}{c\rho + d} \qquad \tau \to \frac{a\tau + b}{c\tau + d}$$

This is indeed a symmetry of our partition function.

For applications (e.g. the Heterotic String) we need to look for systems where $\rho$ and $\tau$ can be independently transformed.

$$Z \sim \frac{1}{\eta(q)\eta(\tilde{q})} \sum_{m,k} q^{\frac{1}{4}(mR+k/R)^2} \tilde{q}^{\frac{1}{4}(mR-k/R)^2} \to f(\tau)\bar{f}(\rho)$$

To do this we consider $n$ copies of the Sen action and include a $B$-field and constant target space metric $G_{ij}$

$$S = -\frac{1}{2\pi}\int \det(\bar{e})\Big(-\partial_+ P^i \partial_- P^j (G_{ij} + B_{ij}) - 2Q_{i+}\partial_- P^i - Q_{+i}Q_{+j}G^{ij}M_{--}\Big)d^2x$$

The Partition function can now be written in the form

$$Z(\tau,\rho) \sim \frac{1}{\eta(q)^n \eta(\tilde{q})^n} \sum_{(p_L,p_R)\in\Lambda} q^{\frac{1}{2}p_L^2}(\tilde{q})^{\frac{1}{2}p_R^2}$$

where $\Lambda$ is a $2n$-dimensional Narain lattice

At special points (rational points) in the moduli space of $B_{ij}, G_{ij}$ the partition function simplifies:

$$Z(\tau,\rho) \sim \sum_{a,\bar{b}} C_{a\bar{b}}\chi_a(\tau)\chi_{\bar{b}}(\rho)$$

One very special such case is $\Lambda = \Gamma_L \oplus \Gamma_R$ with $\Gamma_L = \Gamma_R = \Delta$ where $\Delta$ is an even self-dual lattice and hence we find.

$$Z \sim \frac{\Theta_\Delta(\tau)}{\eta^n(q)} \frac{\Theta_\Delta(\rho)}{\eta^n(\rho)}$$

where

$$\Theta_\Delta(\tau) = \sum_{p\in\Delta} q^{\frac{1}{2}p^2}$$

and famously $n$ must be a multiple of 8.

For example at $n = 8$ $\Delta$ is the root lattice of $E_8$ and

$$\Theta_\Delta(\tau) = \theta_2^8(\tau) + \theta_3^8(\tau) + \theta_4^8(\tau)$$

If $n = 16$ we find two lattices and hence we can find a partition function for the Heterotic string $E_8 \times E_8$ or $SO(32)$.

# The $\bar{g} = g$ CFT

Things simplify in the case that $\bar{g} = g$. [Hull,NL]

Here $M(Q) = 0$ and we can redefine $Q' = Q + \frac{1}{4}dP + \frac{1}{4}\bar{\star}dP$ so that

$$S = -2\int Q' \wedge dP$$

This is like a $BF$ TQFT but we have $Q' = \bar{\star}Q$ so it is a CFT.

It's analogous to a higher-dimensional holomorhpic $\beta\gamma$ ghost system (but with integer conformal weights).

A $BF$ theory describes two flat gauge fields. Here, a somewhat trivial Bosonization, leads to two self-dual fields:

$$Q' = \frac{1}{2}(dA + dC) \qquad P = \frac{1}{2}(C - A)$$

## This theory has three EM tensors:

$$\Theta_{\mu\nu} = (Q' - dP)_\mu \cdot (Q' - dP)_\nu$$
$$\bar{\Theta}_{\mu\nu} = -(Q' + dP)_\mu \cdot (Q' + dP)_\nu$$
$$T_{\mu\nu} = (Q' - dP)_\mu \cdot (Q' - dP)_\nu - (Q' + dP)_\mu \cdot (Q' + dP)_\nu$$

The gravitational anomalies decouple (in 2D):

$$\langle \Theta_{++}(z)\Theta_{++}(w) \rangle = \frac{1}{2}\frac{1}{4\pi^2}\frac{1}{(z-w)^4}$$
$$\langle \Theta_{++}(z)\bar{\Theta}_{++}(w) \rangle = 0$$
$$\langle \bar{\Theta}_{++}(z)\bar{\Theta}_{++}(w) \rangle = \frac{1}{2}\frac{1}{4\pi^2}\frac{1}{(z-w)^4}$$
$$\langle T_{++}(z)T_{++}(w) \rangle = \frac{1}{4\pi^2}\frac{1}{(z-w)^4}$$

We can now deform away from $g = \bar{g}$ by adding

$$S_{lin} = \frac{1}{2}\int d^d x\, [\bar{h}^{\mu\nu}\bar{\Theta}_{\mu\nu} + h^{\mu\nu}\Theta_{\mu\nu}]$$

This rebuilds the Sen action.

# A Democratic Action

We can extend $Q'$ and $P$ to be poly-forms:

$$Q' = Q_1 + Q_3 + Q_5 + Q_7 + Q_9 \qquad P = P_0 + P_2 + P_4 + P_6 + P_8$$

and impose $Q_{10-q} = *Q_q$.

The action becomes

$$S = \int Q' \wedge dP = \sum_{q=1}^{5} \int [Q'_q \wedge dP_{9-q} + Q'_{10-q} \wedge dP_{9-q}]$$

This gives a democratic action for two fields satisfying the field equations

$$dQ_q = 0 \qquad d * Q_q = 0 \qquad dP_{q-1} = *dP_{9-q}$$

# Conclusions

So we have had many adventures playing with the Sen action. I hope you enjoyed it. It is a novel playground in an old topic.

The action and Hamiltonian is indefinite but leads to two decoupled self-dual fields, each associated to its own metric.

The action enjoys two separate diffeomorphism-like symmetries.

The Path integral can be evaluated if the metrics, not time, are analytically continued. In this way the wrong sign kinetic term remains oscillating and the path integral converges from $Q \wedge M(Q)$

Modular invariance and a holomorphic dependence on the complex structure are manifest.

But Modular invariance is not quite what it was: both complex structures transform together and the Partition Function doesn't factorise in general. But in the special cases, such as the $E_8$ root lattice, one can find factorisation.

At $g = \bar{g}$ there is a simple and somewhat novel CFT that generalises the $\beta\gamma$ ghost system to higher dimensions.

There is a nice democratic formulation of the fields of type IIB supergravity (and other possibilities).

We only discussed the free theory here. It can be coupled to sources.

Applied to the Heterotic String we need a shadow Heterotic string to cancel all anomalies.

Should the shadow sector be taken more seriously?

You can couple it to matter fields to make a free $(2,0)$ theory in 6D or type IIB supergravity in 10D

A non-abelian $(2,0)$ Sen-like action was constructed in [NL] giving rise to the equation of motion of [NL,Papageorgakis]

Other works have focused on non-linear theories [Vanichchapongjaroen] or reduction on $\mathbb{T}^2$ in the context of S-duality [Aggarwal,Chakrabarti,Raman],[Barbagallo,Grassi].

THANK YOU