NON-INVERTIBLE SYMMETRIES IN 2d NON-LINEAR SIGMA MODELS

MTP Seminar, University of Hertfordshire
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Based on work with Chris Hull and Max Velásquez Cotini Hutt

2503.20865 , 2508.16721 , 2509.20441

Modern understanding of generalised symmetries

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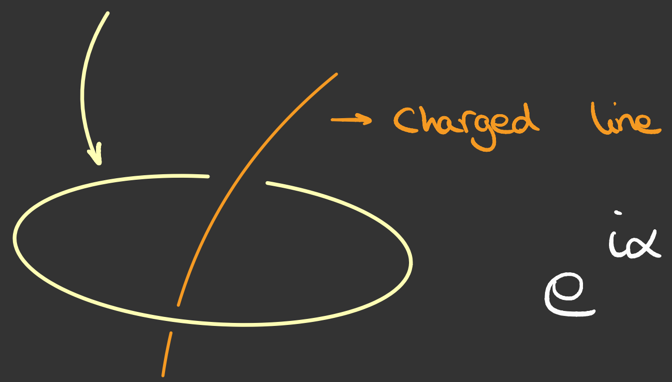
Long-standing knowledge of T-duality

GENERALISED GLOBAL SYMMETRIES

- Action on fields → topological operators (Gaiotto, Seiberg, Kapustin, Willet '14)
  - Higher form
  - Higher group
  - Non-invertible



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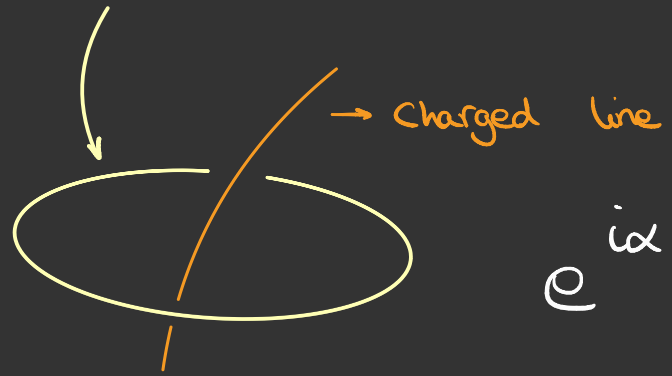

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- More systematic understanding of QFTs

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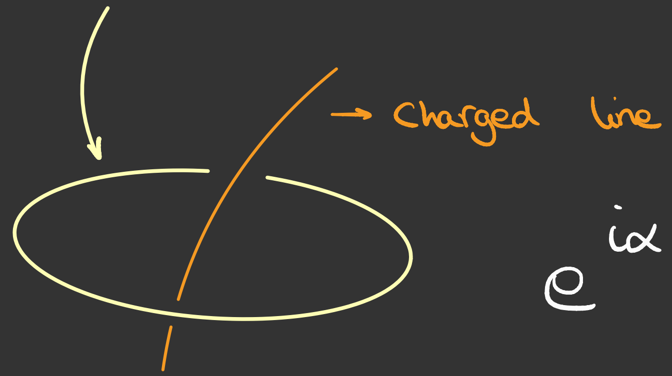

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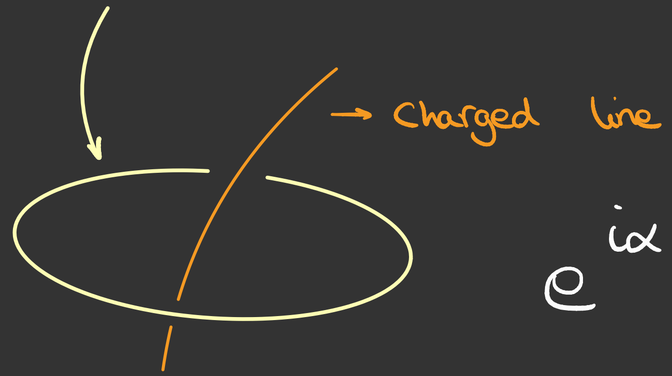

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- More systematic understanding of QFTs

- Prototypical example: free compact boson in 2d (Frolich, Fuchs, Runkel, Schweigert '06; Bachas, Brunner, Roggenkamp '12; Chang, Lin, Shao, Wang, Yin '18; Thorngren, Wang '21; Damia, Galati, Hulik, Mancani '24; Bharadwaj, Niro, Roumpedakis '24)

GOAL OF THE TALK

Non-linear sigma models in 2d have a non-invertible symmetry related to T-duality

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REASONS TO CARE

NLSMs

- Spontaneous symmetry breaking
- String Theory
- Strong coupling

NON-INVERTIBLE SYM

- Non-renormalisation of coupling constants
- New Ward identities

OUTLINE

- Introduction
- NLSMs and their grouplike symmetries
- Non-invertible defects in NLSMs
- Applications
- Conclusion and outlook

PART VI

NON LINEAR SIGMA MODELS

NON-LINEAR SIGMA MODELS WITH WZ TERM

- Theory of maps $\Phi : W \longrightarrow M$. Choose local coordinates X^i and write action

$$S = \frac{1}{2}\int_W g_{ij}\, dX^i \wedge \star dX^j + \frac{1}{3}\int_V H_{ijk}\, dX^i \wedge dX^j \wedge dX^k$$

$\partial V = W$

where:

W worldsheet

g metric on M

H closed 3-form , $\frac{1}{2\pi}[H] \in H^3(M, \mathbb{Z})$

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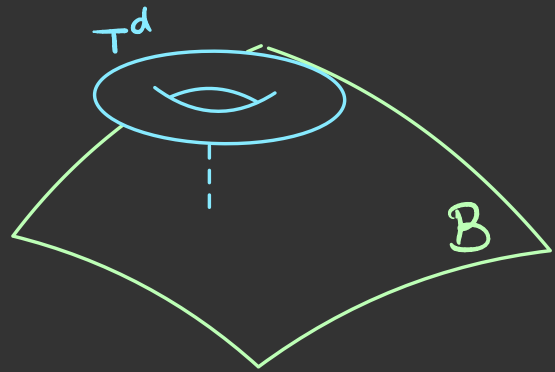
- $U(1)^d$ isometry of $M \longrightarrow$ fibration $T^d \rightarrow M \rightarrow B$



- When is $X^i \rightarrow X^i + \alpha^m k^i_m$ a global symmetry of the NLSM?

ISOMETRY SYMMETRY

$$S = \frac{1}{2}\int_W g_{ij}\, dx^i \wedge * dx^j + \frac{1}{3}\int_V H_{ijk}\, dx^i \wedge dx^j \wedge dx^k \qquad \delta x^i = \alpha k^i$$

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- Two possibilities
  - $\iota_k H = \frac{1}{2\pi} du$ exact $\longrightarrow$ $U(1)$ isometry symmetry
  - $\iota_k H$ not exact $\longrightarrow$ $\mathbb{Z}_R$ isometry symmetry

'T HOOFT ANOMALY

- Idea: gauge by promoting derivative to covariant derivative

$$DX^i = dX^i - c\,k^i$$

- Gauged action (Hull, Spence '89)

$$S^{\text{gauged}} = \frac{1}{2}\int_W g_{ij}\, DX^i \wedge * DX^j + \frac{1}{3}\int_V H_{ijk}\, DX^i \wedge DX^j \wedge DX^k + \frac{1}{2\pi}\int_V dC \wedge v_i\, DX^i$$

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$$S^{\text{gauged}} = S^{\text{gauged}}_{2d} + \frac{c_k v}{2\pi}\int_V C \wedge dC$$

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Anomaly inflow

- Coefficient $p = |\iota_k v| \in \mathbb{Z}$ and there is $\mathbb{Z}_p$ non-anomalous subgroup

EXAMPLES

1) $M = S^3$ and $H = \frac{k}{4\pi} \sin\theta \, d\theta \wedge d\psi \wedge d\phi$

$k = \partial_\phi \longrightarrow \iota_k H = \frac{k}{4\pi} \sin\theta \, d\theta \wedge d\psi \longrightarrow v = k\left(d\phi - \frac{1}{2}\cos\theta \, d\psi\right)$

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2) $M = S^2 \times S^1$ and $H = \frac{k}{4\pi} \sin\theta \, d\theta \wedge d\psi \wedge d\phi$

$k = \partial_\phi \longrightarrow \int_{S^2} \iota_k H = k \longrightarrow \delta S = \alpha k$

$\mathbb{Z}_k$ symmetry $\phi \to \phi + \frac{2\pi n}{k}$ (non-anomalous)

T-DUALITY

- Add lagrange multiplier to gauged action

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- Path integral over $\hat{X}$: $\int [d\hat{X}] = \sum_{\text{periods}} \int [d\hat{X}_0]$

  - Continuous part : $dC = 0 \longrightarrow C$ is flat

  - Periods : $\sum_{d\hat{X}} e^{iS^{\text{gauged}} + i\int_{PD(d\hat{X})} C} \longrightarrow \int_\gamma C \in 2\pi\mathbb{Z} \longrightarrow C$ is topologically trivial

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- T-dual descriptions in terms of X or $\hat{X}$ by order of path integral

- Example:

$S^1 \to S^3 \to S^2$ $H = 0$	$\longleftrightarrow$	$S^2 \times S^1$ $H \neq 0$

(Álvarez, Álvarez-Gaumé, Barbón, Lozano '93)

PART III

NON-INVERTIBLE DEFECTS IN NLSMS

TOPOLOGICAL DEFECTS

- Symmetry $\longrightarrow$ Conserved charge

$$d*j = 0 \quad \longrightarrow \quad Q = \int_{M^{d-1}} *j$$

- Symmetry topological operator

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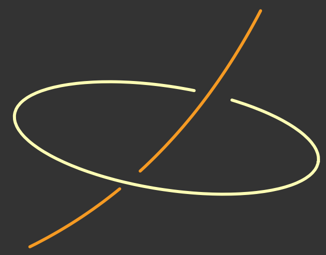
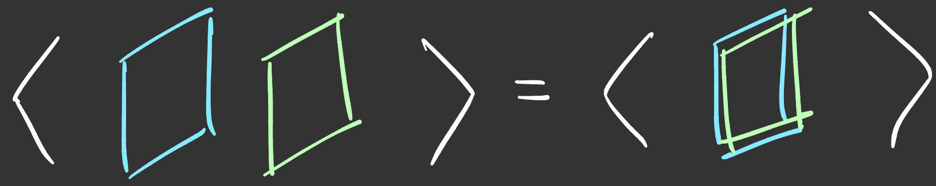
- Subsector of topological operators $\longrightarrow$ generalised symmetries

- Action by linking, shrinking, moving them around



NON-INVERTIBLE SYMMETRIES

- Fusion of topological operators: OPE in correlation function



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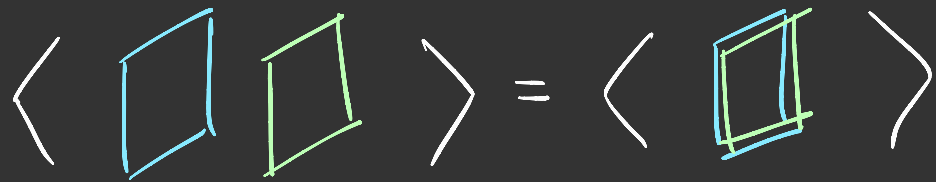
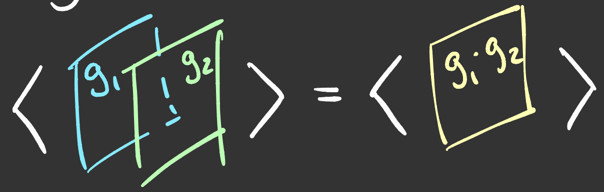


  - Grouplike symmetry:



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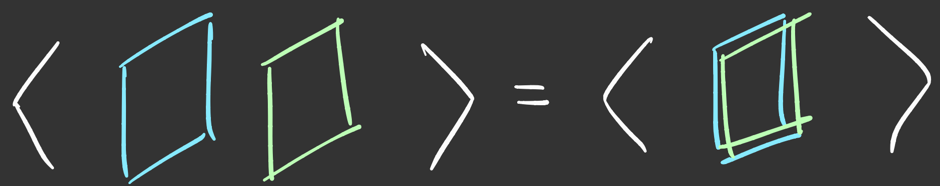

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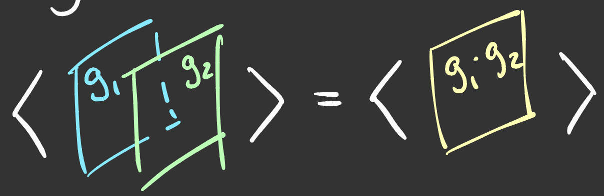

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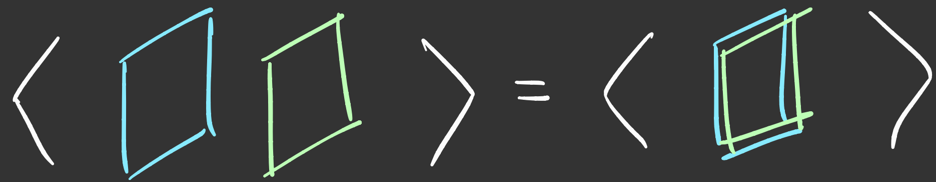

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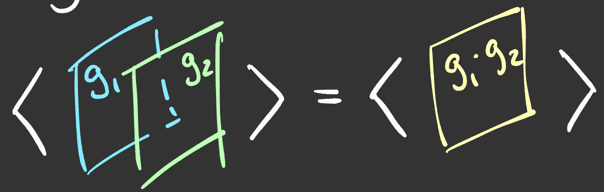

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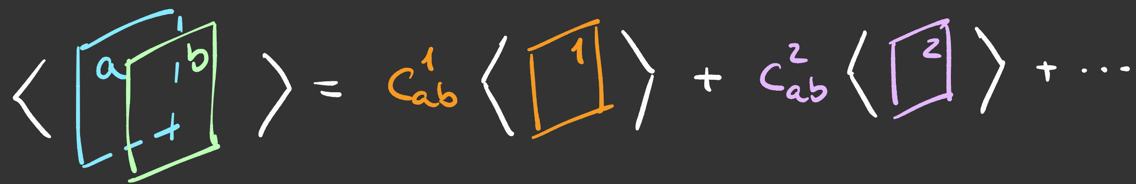

- Many ways to find non-invertible symmetries → Here, half-space gauging

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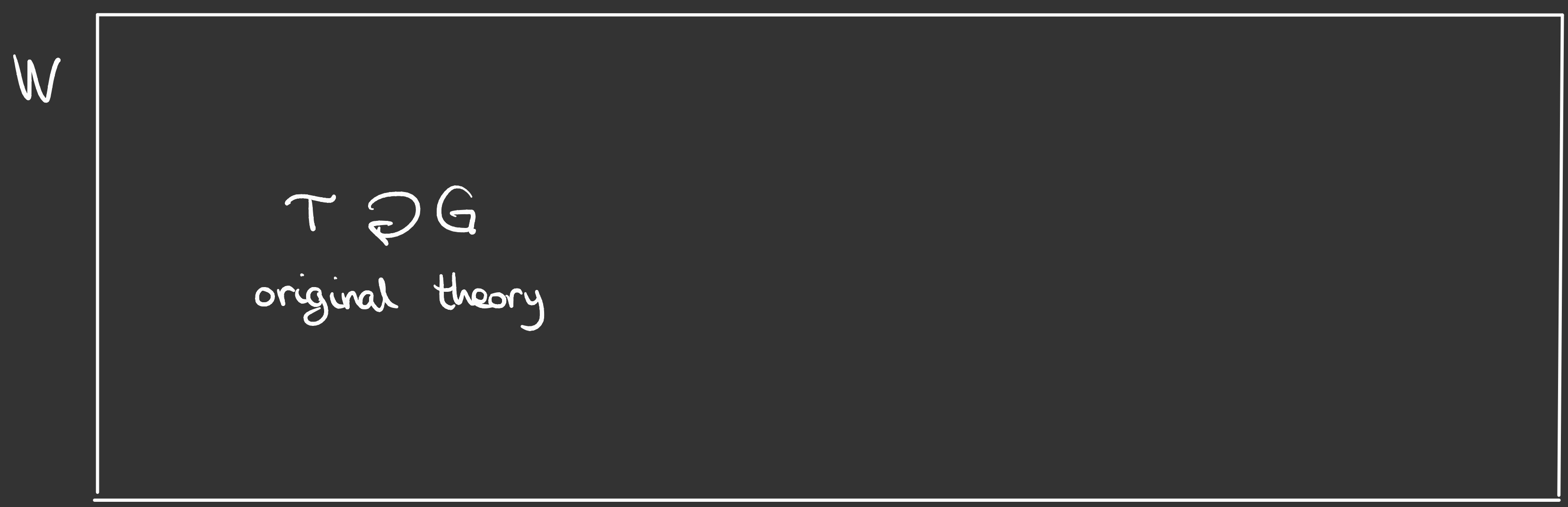

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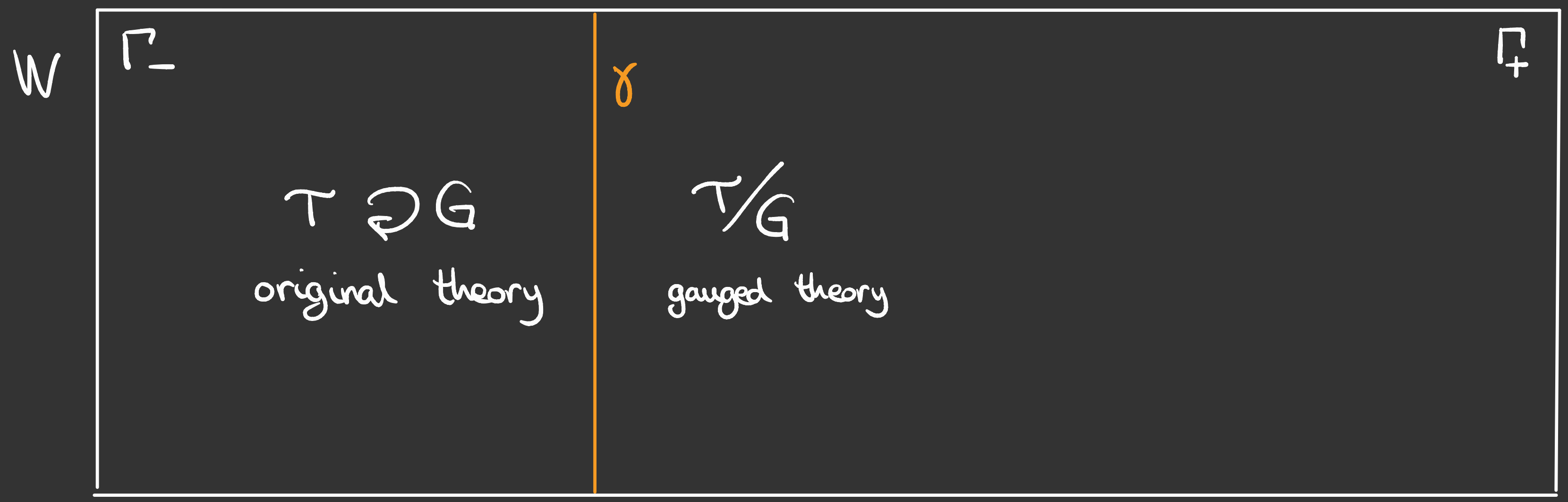

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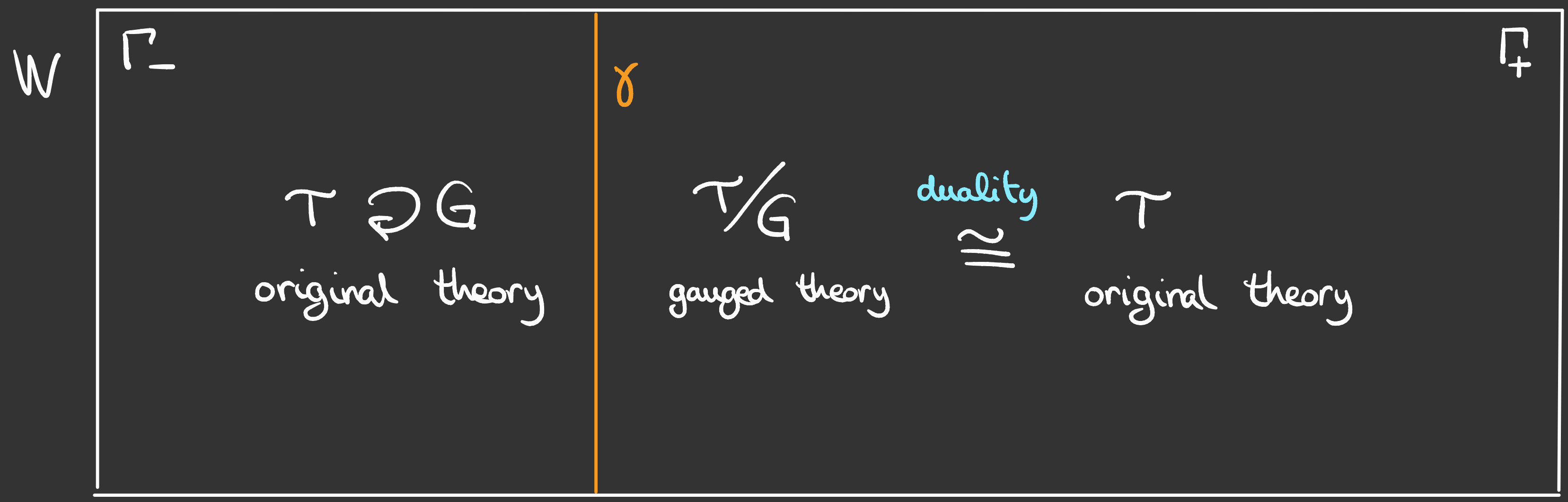

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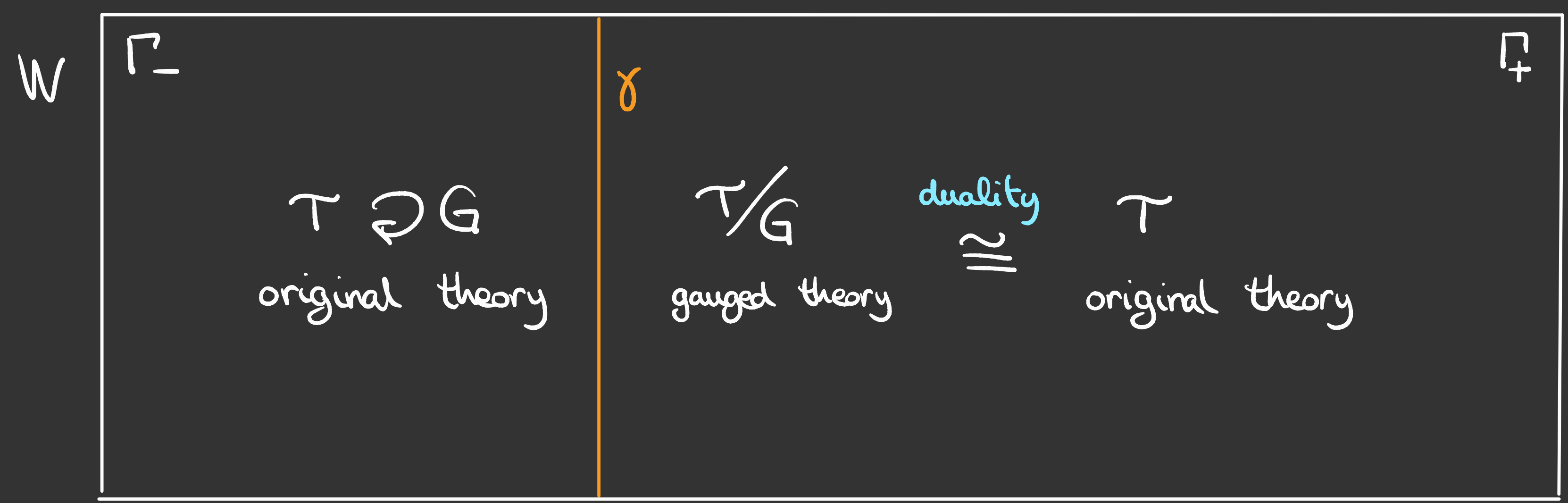
3 - Impose boundary conditions on the new gauge fields that ensure the interface is topological
4 - Sometimes, a duality implies $\mathcal{T}/G \cong \mathcal{T}$
5 - Then, we have defined a topological defect of $\mathcal{T}$, $\mathcal{N}(\gamma)$

(Choi, Córdova, Hsin, Lam, Shao '21
Kaidi, Ohmori, Zheng '21)

$$\mathcal{N} \times \mathcal{N}^\dagger = \sum_{g \in G} U_g \longrightarrow \text{Non-invertible}$$



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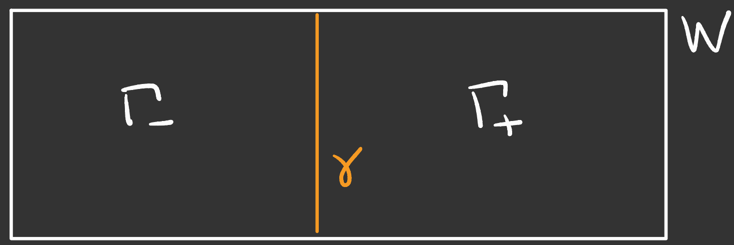

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Shift symmetry $\phi \to \phi + \text{const.}$ $\quad U(1) \supset \mathbb{Z}_p$

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Self-duality condition

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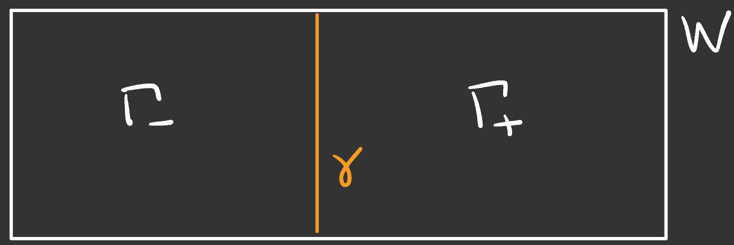

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- Model at fixed value of the radius has non-invertible symmetry

- Needed: Shift symmetry + T-duality

NEXT : THE 3-SPHERE

- In Γ_- : S^3 (as Hopf fibration) + WZ

$$S_{\Gamma_-} = \frac{R^2}{8} \int_{\Gamma_-} (2d\phi - \cos\theta \, d\psi)^2 + d\Omega^2_{S^2} - \frac{k}{4\pi} \int_{\Gamma_-} \cos\theta \; d\psi \wedge d\phi$$

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NEXT : THE 3-SPHERE

- In Γ_- : S^3 (as Hopf fibration) + WZ

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- Self-duality conditions:

$$R^2 = \frac{p}{2\pi} \qquad p = k$$

→ Non-invertible symmetry

$SU(2)_p$ WZW ?

GENERIC NON-LINEAR SIGMA MODEL

- Isometry of target space $\longrightarrow \mathbb{Z}_p$ non-anomalous subgroup of global symmetry

- Gauge with lagrange multiplier

$$\frac{1}{2}\int_W g_{ij}\, DX^i \wedge \star DX^j + \frac{1}{3}\int_V H_{ijk}\, DX^i \wedge DX^j \wedge DX^k + \frac{1}{2\pi}\int_V dC \wedge v_i DX^i + \frac{p}{2\pi}\int_W C \wedge d\hat{x}$$

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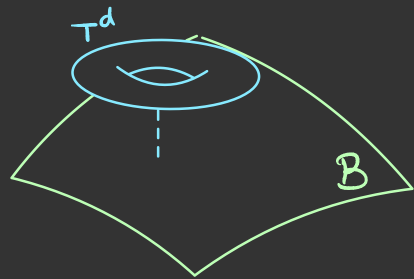


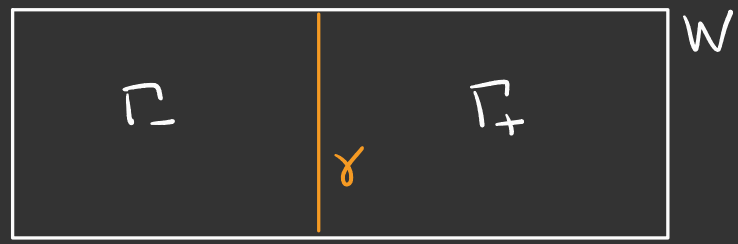
- What are the self-duality conditions?

- Two main characters : topological data of the fibration (Hull '06)

1. Connection associated with the metric $A^m \rightarrow \frac{1}{2\pi}\int dA^m$: Chern classes

2. $\iota_{k_m} H = \frac{1}{2\pi} dv_m$: also a connection 1-form $\longrightarrow \frac{1}{2\pi}\int dv_m$: H-classes

SELF-DUALITY DEFECTS IN NLSMs



- Gauge $\mathbb{Z}_p$ symmetry in Γ_+

- Self-duality conditions:

1. <u>Topological</u> $\longrightarrow$ H-class $= p \times$ Chern class

   Ensures that global symmetries are the same in Γ_- and Γ_+

2. <u>Geometrical</u>

   - Norm of Killing vectors $G_{mn} = g_{ij} K^i_m K^j_n$
   - b-field along fibre $B_{mn} = \hat{u}_m \hat{v}_n$

   $\longrightarrow$ $(G_{mn} + B_{mn})^2 = \left(\frac{p}{2\pi}\right)^2$

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     → $$(G_{mn} + B_{mn})^2 = \left(\frac{p}{2\pi}\right)^2$$

- Examples:
  - Torus, spheres
  - Lens spaces, $T^{p,q}$ manifolds
  - General Wess-Zumino-Witten models: $SU(N)_k$, $Spin(N)_k$
  - Nilmanifold, squashed S^3 (not conformal)

PART IV

APPLICATIONS

A PUZZLE

- Symmetries are preserved by RG unless a symmetry breaking deformation is introduced

- Non-conformal NLSM:

with non-invertible symmetry

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- Prediction: self-duality condition preserved by the flow

- Non-invertible symmetry protects G from quantum corrections

NON-RENORMALISATION: 1-LOOP

- 1-loop β-function for the metric of NLSMs

$$\beta^{g}_{ij} = \alpha' \left(R_{ij} - \frac{1}{4} H_{ikl} H_{j}{}^{kl} \right)$$

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- ¡ Vanishes for self-duality conditions! $G = \frac{p}{2\pi}$ $\tilde{F} = pF$

(also at 2-loops)

NON-RENORMALISATION : NON-PERTURBATIVE EXAMPLE

- Non-linear sigma model on a group manifold

$$S = -\frac{1}{2\lambda^2}\int_W \mathrm{tr}\,(g^{-1}dg)^2 + \frac{k}{12\pi}\int_V \mathrm{tr}\,(g^{-1}dg)^3$$

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- Indeed, self-duality condition = WZW condition

$$\boxed{\lambda^2 = \frac{4\pi}{k}}$$

- Conjecture : non-renormalisation at the non-perturbative level should be general (symmetry reason)

WARD IDENTITIES

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- Derived by nucleating defect and sweeping operator insertions

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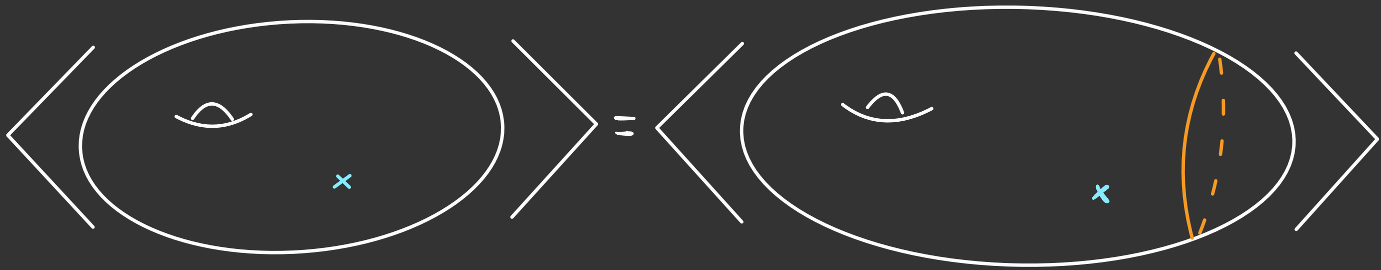

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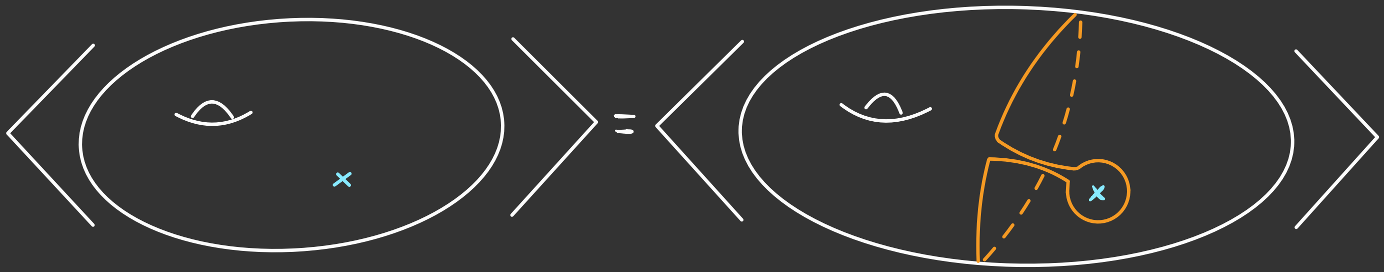

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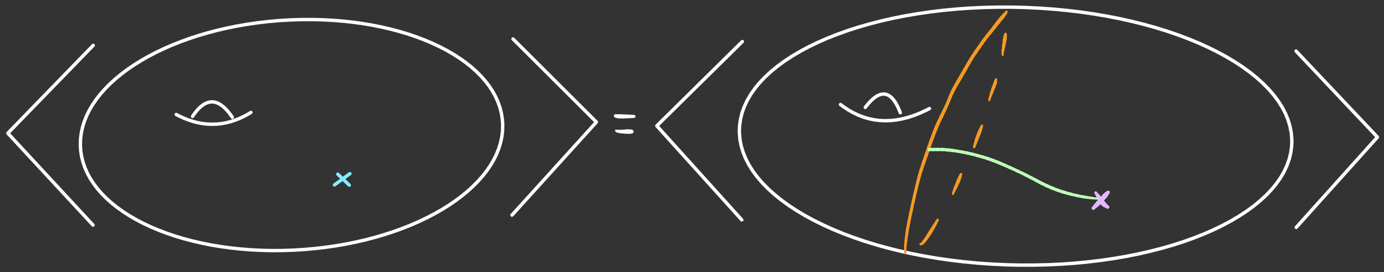

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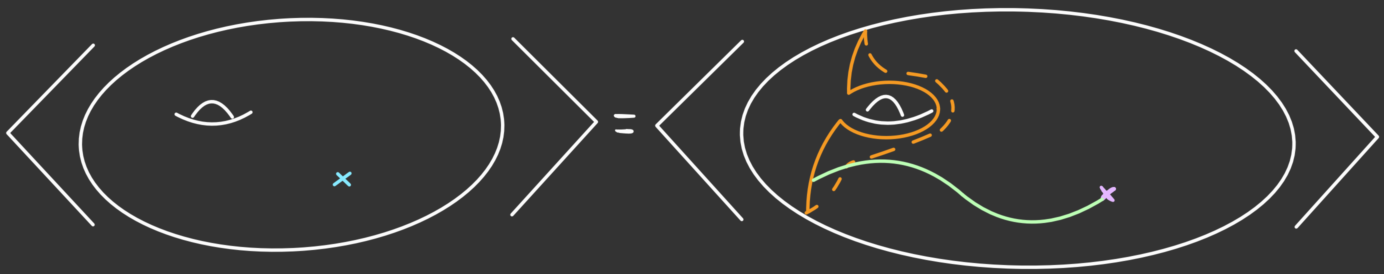

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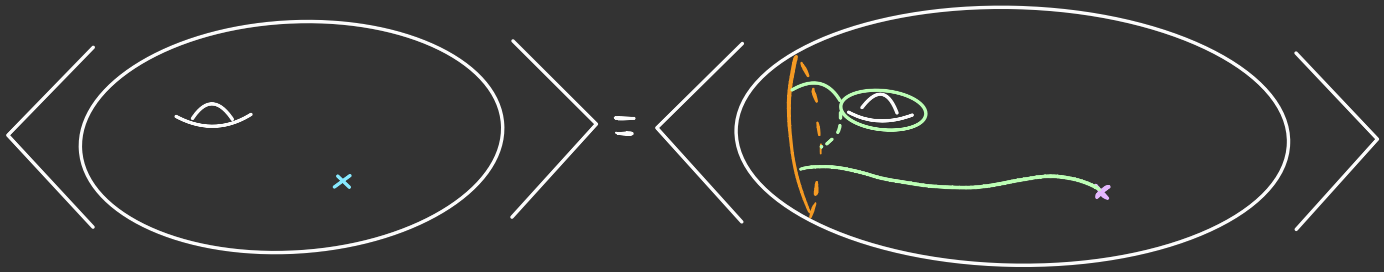

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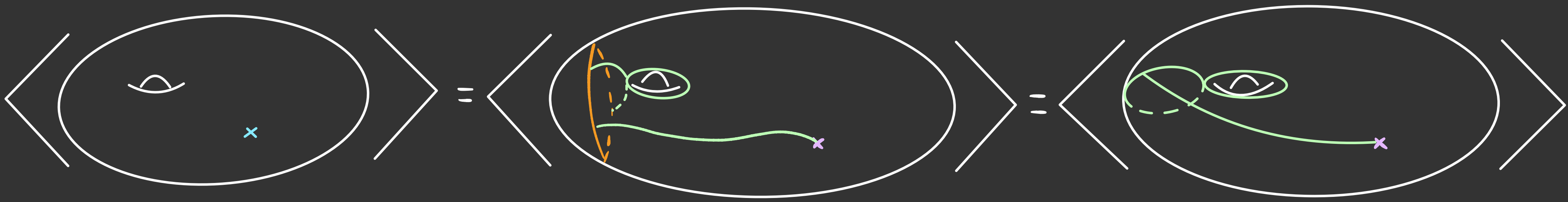

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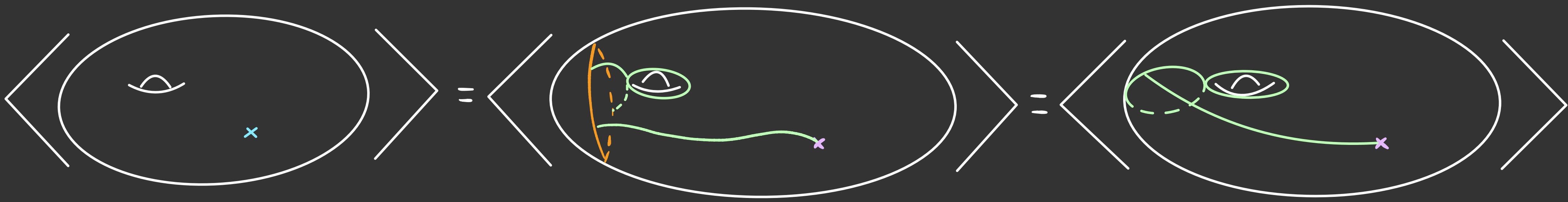

- Action on vertex operators

Compact boson: momentum ↔ winding

General NLSM: WIP

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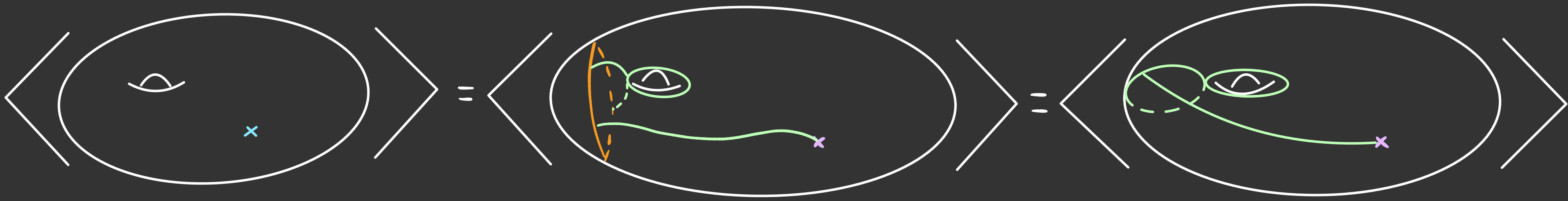

- Action on vertex operators 

  Compact boson: momentum ↔ winding

  General NLSM: WIP

- Ward identities depend on the genus of the worldsheet

- In String Theory, the non-inverible is broken by the genus expansion

(Heckman, McNamara, Montero, Sharon, Vafa, Valenzuela '24
Kaidi, Tachikawa, Zhang '24)

THE DEFECT IN TARGET SPACE

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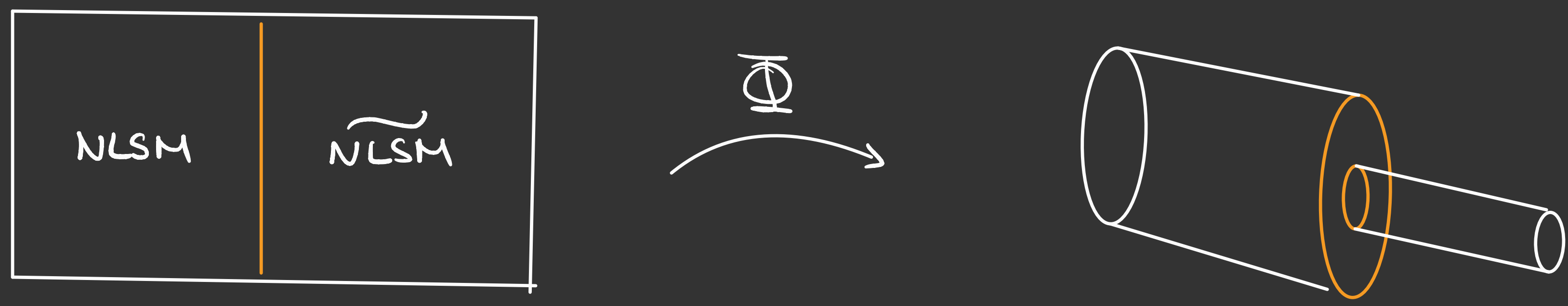
- Similar object: T-fold ~ T-duality in half space



Smooth/transparent because it's the same quantum theory

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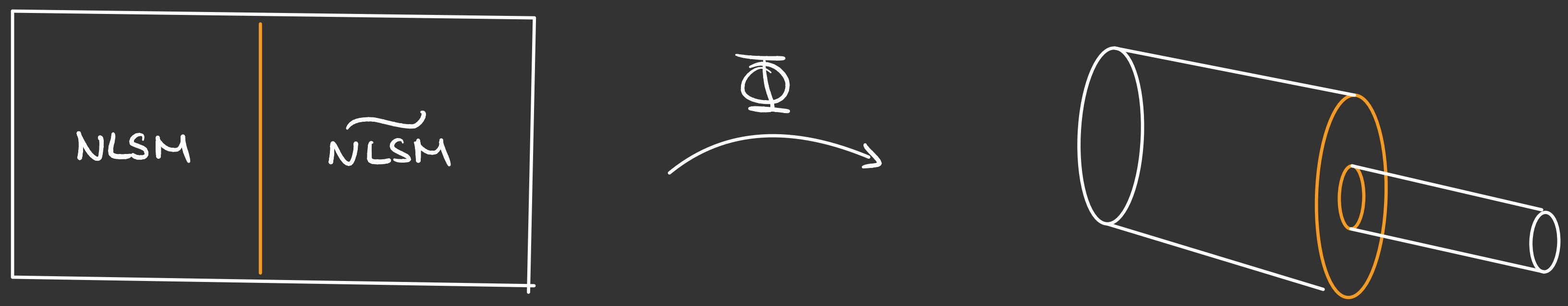
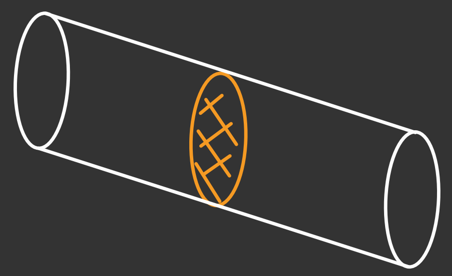


Smooth / transparent because it's the same quantum theory

- Non-invertible defect → Not a T-fold but similar idea

1) Self-duality conditions ⇝ same "radius" on both sides



2) Non-trivial action on vertex operators on the worldsheet ⇝ Not transparent

PART IV

CONCLUSION AND OUTLOOK

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- Field configuration corresponding to non-invertible defect
  - Any orientable worldsheet W
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  - Requirement: Isometry without fixed points

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- Ward identities and No Global Symmetries
- Constraints on RG flows

OUTLOOK

- Ward identities in full generality
- Target space picture
- Self-duality conditions under RG

27|27 = 1

OUTLOOK

- Ward identities in full generality
- Target space picture
- Self-duality conditions under RG
- Other string dualities? e.g. Mirror symmetry
- Constraints on IR phases after symmetry preserving deformations
- Many more...

THANK YOU FOR

YOUR ATTENTION !