Extremal Black Holes from Homotopy Algebras

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Based on arXiv:2508.08886 with Jan Gutowski and Martin Wolf

Motivation

- **BH uniqueness theorem** [Robinson; Mazur; Carter; Israel; Amsel, Horowitz, Marolf, and Roberts]: In 4D, a stationary asymptotically flat black hole is Kerr-Newman.
- The theorem breaks down in higher dimensions, e.g. in 5D, Myers–Perry solutions [Myers and Perry] and black ring solutions [Emparan and Reall] have the same set of conserved charges.
- **Deformation of near-horizon geometry** [Li and Lucietti; Katona and Lucietti; Fontanella and Gutowski]: Finiteness of linear transverse deformations. A black hole with a static and maximally symmetric near-horizon geometry is Schwarzschild–de Sitter.
- L_∞**-algebra** (review [Jurčo, Raspollini, Sämann, and Wolf]) is a powerful framework in classical and quantum field theory. The **homological perturbation lemma** allows one to solve deformation problems systematically.

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Outline

- Gaussian null coordinates and near-horizon geometries
- Deformations of near-horizon geometries
- L_∞-algebras, quasi-isomorphisms and minimal models
- Solving the deformation problem
- Summary and outlook

Gaussian Null Coordinates and Near-horizon Geometries

Gaussian null coordinates and extremal black holes

An **extremal black hole** is a black hole with a $(d-1)$-dimensional **Killing horizon** Σ (null hypersurface generated by a Killing vector) such that surface gravity vanishes, i.e.

$$\mathrm{d}(K \cdot K)|_{\Sigma} \;=\; 0\;,$$

where K is the timelike Killing vector.

Theorem ([Moncrief and Isenberg])

Near this horizon, there exist **Gaussian Null coordinates***, such that the metric takes the form*

$$g \;=\; \mathrm{d}u \odot \left[\mathrm{d}r + r\alpha_i(r,y)\mathrm{d}y^i - \tfrac{1}{2}r^2\beta(r,y)\mathrm{d}u\right] + \tfrac{1}{2}\gamma_{ij}(r,y)\mathrm{d}y^i \odot \mathrm{d}y^j\;,$$

where $K = \partial_u$. The space is foliated by $(d-2)$-dimensional submanifolds S parametrised by y^i called **spatial cross-section** *(assumed to be compact with no boundary).*

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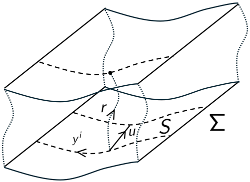

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Near-horizon geometries

Define a family of metrics via a family of diffeomorphisms $(u, r, y^i) \to (u/\epsilon, r\epsilon, y^i)$ as

$$g_\epsilon \;:=\; \mathrm{d}u \odot \left[\mathrm{d}r + r\alpha_i(\epsilon r, y)\mathrm{d}y^i - \tfrac{1}{2}r^2\beta(\epsilon r, y)\mathrm{d}u\right] + \tfrac{1}{2}\gamma_{ij}(\epsilon r, y)\mathrm{d}y^i \odot \mathrm{d}y^j$$

Definition (near-horizon geometries)

The **near-horizon limit** corresponds to $\epsilon \to 0$, and the resulting geometry is called the **near-horizon geometry**. The metric is

$$\mathring{g} \;=\; \mathrm{d}u \odot \left[\mathrm{d}r + r\mathring{\alpha}_i(y)\mathrm{d}y^i - \tfrac{1}{2}r^2\mathring{\beta}(y)\mathrm{d}u\right] + \tfrac{1}{2}\mathring{\gamma}_{ij}(y)\mathrm{d}y^i \odot \mathrm{d}y^j \;,$$

where $(\mathring{\alpha}(y^i), \mathring{\beta}(y^i), \mathring{\gamma}(y^i)) \;:=\; (\alpha(0, y^i), \beta(0, y^i), \gamma(0, y^i))$.

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Examples

The Gaussian null coordinates are analogous to the **Eddington–Finkelstein coordinates**. The Schwarzschild solution:

$$g_S = \mathrm{d}u \odot \left[\mathrm{d}r - \left(\tfrac{1}{2} - \tfrac{m}{r}\right)\mathrm{d}u\right] + r^2 \mathrm{d}\Omega^2 \ .$$

The Kerr black holes is **extremal** when the angular momentum is $J = m^2$. Its near-horizon geometry is called **extremal Kerr horizon**:

$$\begin{aligned} \mathring{g}_{\mathrm{eK}} \ &= \ \mathrm{d}u \odot \left[\mathrm{d}r + r\frac{4(1-x^2)}{(1+x^2)^2}\mathrm{d}\varphi - r\frac{2x}{1+x^2}\mathrm{d}x + r^2\frac{3-6x^2-x^4}{2m^2(1+x^2)^3}\mathrm{d}u\right] \\ &\quad + \frac{m^2(1+x^2)}{2(1-x^2)}\mathrm{d}x \odot \mathrm{d}x + 2m^2\frac{1-x^2}{1+x^2}\mathrm{d}\varphi \odot \mathrm{d}\varphi \ , \end{aligned}$$

where $x \in (-1,1)$ and $\varphi \in (0, 2\pi)$.

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Deformations of Near-horizon Geometries

Deformations of near-horizon geometries

Since **near-horizon geometries** arise as the $\epsilon \to 0$ limit of the family of diffeomorphisms, they satisfy the **Einstein field equations** and can be used as **background metrics**. We perturb these metrics to

$$g \ := \ \mathrm{d}u \odot \left[\mathrm{d}r + r\alpha_i(r,y)\mathrm{d}y^i - \tfrac{1}{2}r^2\beta(r,y)\mathrm{d}u\right] + \tfrac{1}{2}\gamma_{ij}(r,y)\mathrm{d}y^i \odot \mathrm{d}y^j \ ,$$

where

$$\begin{aligned}
\alpha_i(r,y) \ &:= \ \mathring{\alpha}_i(y) + h_i(r,y) \ , \\
\beta(r,y) \ &:= \ \mathring{\beta}(y) + h(r,y) \ , \\
\gamma_{ij}(r,y) \ &:= \ \mathring{\gamma}_{ij}(y) + h_{ij}(r,y)
\end{aligned}$$

with deformations h_i, h, and h_{ij} satisfying

$$h_i|_{r=0} \ = \ 0 \ , \quad h|_{r=0} \ = \ 0 \ , \quad \text{and} \quad h_{ij}|_{r=0} \ = \ 0 \ .$$

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Null orthonormal basis

We define an **unperturbed null orthonormal basis** as

$$\mathring{e}^{+} \;:=\; \mathrm{d}u\;, \quad \mathring{e}^{-} \;:=\; \mathrm{d}r + r\mathring{\alpha}_i \mathrm{d}y^i - \tfrac{1}{2}r^2\mathring{\beta}\mathrm{d}u\;, \quad \text{and} \quad \mathring{e}^{a} \;:=\; \mathrm{d}y^i \mathring{e}_i{}^a\;,$$

where

$$\mathring{e}_i{}^a \mathring{e}_j{}^b \delta_{ab} \;=\; \mathring{\gamma}_{ij}\;.$$

On this basis, the **near-horizon metric** is

$$\mathring{g} \;=\; \mathring{e}^{+} \odot \mathring{e}^{-} + \tfrac{1}{2}\delta_{ab}\mathring{e}^{a} \odot \mathring{e}^{b}\;,$$

and the **perturbed metric** is

$$g \;=\; \mathring{g} + \mathring{e}^{+} \odot \left[r h_a \mathring{e}^{a} - \tfrac{1}{2} r^2 h \mathring{e}^{+} \right] + \tfrac{1}{2} h_{ab} \mathring{e}^{a} \odot \mathring{e}^{b}\;,$$

where $h_a := \mathring{E}_a{}^i h_i$, etc. with $\mathring{E}_a{}^i$ the inverse of $\mathring{e}_i{}^a$.

Contracted Bianchi identities and gauge transformations

Theorem

By the **contracted Bianchi identities***, it is enough to restrict ourselves to*

$$0 \ = \ G^{++} \ , \quad 0 \ = \ G^{a+} \ , \quad \textit{and} \quad 0 \ = \ G^{ab} + \Lambda g^{ab} \ ,$$

where G is the **Einstein tensor***. The remaining Einstein field equations do not impose any further constraints.*

Theorem ([Fontanella and Gutowski])

Residual gauge transformations *are fixed by the* **gauge-fixing condition**

$$\delta^{ab}\partial_r h_{ab}|_{r=0} \ = \ \Gamma \ ,$$

where $\Gamma(y^i)$ is the unique (up to a multiplicative constant) solution of

$$\mathring{\tilde{\nabla}}_a \mathring{\tilde{\nabla}}^a \Gamma + \mathring{\alpha}^a \mathring{\tilde{\nabla}}_a \Gamma + \mathring{\tilde{\nabla}}_a \mathring{\alpha}^a \Gamma \ = \ 0 \ ,$$

and $\mathring{\tilde{\nabla}}$ is the **Levi-Civita connection** *on the* **spatial cross-section** *$\mathring{S}$ with respect to the metric $\mathring{\gamma}$.*

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L_∞-algebras and Minimal Models

Why L_∞-algebras?

$$\begin{tikzcd}
L_\infty\text{-algebra} \arrow[rr, "\text{Homological perturbation lemma}"] \arrow[d] & & \overset{\text{(minimal model)}}{L_\infty\text{-algebra}} \arrow[d] \\
\text{Maurer-Cartan equation} \leftrightarrow \text{EoM} \arrow[d] & & \text{Maurer-Cartan equation} \arrow[d] \\
\text{Solutions (modulo gauge)} \arrow[rr, leftrightarrow, "\text{Quasi-isomorphism}"] & & \text{Solutions (modulo gauge)}
\end{tikzcd}$$

L_∞-algebras

Definition (L_∞-algebra)

An L_∞-algebra (V, μ_i) is a $\mathbb{Z}$**-graded vector space** $V = \bigoplus_{k\in\mathbb{Z}} V_k$ equipped with **multi-linear maps** $\mu_i : \bigwedge^i V \to V$ satisfying the **homotopy Jacobi identities**. The first few identities are

$$
\begin{aligned}
\mu_1(\mu_1(v_1)) &= 0 \\
\mu_1(\mu_2(v_1, v_2)) &= \pm\mu_2(\mu_1(v_1), v_2) \pm \mu_2(\mu_1(v_2), v_1) \\
&\mu_2(\mu_2(v_1, v_2), v_3) \pm \text{cyclic} \\
&= \mu_1(\mu_3(v_1, v_2, v_3)) + \mu_3(\mu_1(v_1), v_2, v_3) \pm \text{cyclic} \\
&\vdots
\end{aligned}
$$

for homogeneous $v_i \in V$.

Notice that the first identity implies $\mu_1^2 = 0$. This allows us to define a **cochain complex** (V, μ_1) and its **cohomology** $H^\bullet(V)$.

Maurer–Cartan equations

Definition (curvature)

The **curvature** f is a degree 2 element, defined by

$$f := \sum_{i \geq 1} \frac{1}{i!} \mu_i(a, \ldots, a),$$

where a is a degree one element in an L_∞-algebra (V, μ_i)

Definition (Maurer–Cartan equation)

The **Maurer–Cartan equation** is defined by the vanishing of the curvature:

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Deformation of near-horizon geometry

For our purpose, our L_∞-algebra is concentrated in **degrees** 1 **and** 2, that is, $V = V_1 \oplus V_2$

$$\begin{array}{rcl} & & \Theta := (h_a, h, h_{ab}) \\ & & \quad\ \Cap \\ V_1 & := & \Omega^1_r(S) \oplus \mathscr{C}^\infty_r(S) \oplus \mathscr{S}^2_r(S)|_{\mathrm{gf}} \end{array}$$

$$\begin{array}{rcl} & & (G^{a+}, G^{++}, G^{ab} + \Lambda g^{ab}) \\ & & \qquad\quad \Cap \\ V_2 & := & \Omega^1_r(S) \oplus \mathscr{C}^\infty_r(S) \oplus \mathscr{S}^2_r(S) \end{array}$$

The **Maurer–Cartan equation** is

$$\mu_1(\Theta) + \tfrac{1}{2}\mu_2(\Theta, \Theta) + \ldots \ = \ 0 \ ,$$

where $\mu_1(\Theta)$ is the **linear part** of the Einstein field equations, $\mu_2(\Theta, \Theta)$ is the **bilinear part**, and so on. μ_i vanishes if any argument is in V_2.

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L_∞-algebra morphism

Definition (L_∞-algebra morphism)

L_∞-algebra morphism from (V, μ_i) to (V', μ'_i) is a set of maps $\phi_i : \bigwedge^i V \to V'$ satisfying the **compatibility relations**. The first few are

$$\begin{aligned} \phi_1(\mu_1(v_1)) &= \mu'_1(\phi_1(v_1)) \,, \\ \phi_1(\mu_2(v_1, v_2)) &- \mu'_2(\phi_1(v_1), \phi_1(v_2)) \\ &= \pm\phi_2(\mu_1(v_1), v_2) \pm \phi_2(\mu_1(v_2), v_1) \pm \mu'_1(\phi_2(v_1, v_2)) \\ &\vdots \end{aligned}$$

Quasi-isomorphisms

Definition (quasi-isomorphism)

A **quasi-isomorphism** is an L_∞-algebra morphism such that ϕ_1 induces an **isomorphism between cohomologies** $H^\bullet(V)$ and $H^\bullet(V')$ of the underlying cochain complexes (V, μ_1) and (V', μ'_1).

Theorem

If two L_∞-algebras are **quasi-isomorphic**, *the* **moduli spaces** *of solutions (modulo gauge) of their Maurer–Cartan equations are* **isomorphic**. *The isomorphism is given by*

$$a' = \sum_{i\geq 1} \frac{1}{i!}\phi_i(a, \ldots, a)$$

for $a \in V_1$ and $a' \in V'_1$

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Minimal model

Theorem

Given **cochain complexes** (V, μ_1) *and* $(H^\bullet(V), 0)$ *along with* **contracting homotopy** h *and cochain maps* p *and* e

$$\begin{tikzcd} (V,\mu_1) \arrow[loop left, "\mathsf{h}"] \arrow[r, shift left, two heads, "\mathsf{p}"] & (H^\bullet(V),0) \arrow[l, shift left, hook', "\mathsf{e}"] \end{tikzcd} \quad ,$$

satisfying

$$\begin{aligned} id &= \mu_1 \circ \mathsf{h} + \mathsf{h} \circ \mu_1 + \mathsf{e} \circ \mathsf{p}, \\ \mathsf{p} \circ \mathsf{e} = id, \quad \mathsf{p} \circ \mu_1 &= 0, \quad \mu_1 \circ \mathsf{e} = 0, \\ \mathsf{p} \circ \mathsf{h} = \mathsf{h} \circ \mathsf{e} &= \mathsf{h} \circ \mathsf{h} = 0\,. \end{aligned}$$

There exists an L_∞*-algebra* $(H^\bullet(V), \mu_i^\circ)$ *with* $\mu_1^\circ = 0$*, quasi-isomorphic to* (V, μ_i)*. The* **quasi-isomorphism** *is* $\mathsf{E} : (H^\bullet(V), \mu_i^\circ) \to (V, \mu_i)$ *and* $\mathsf{E}_1 = e$*.*

Minimal model contd.

Theorem (contd.)

As an example,

$$\begin{aligned}
\mathsf{E}_2(v_1^\circ, v_2^\circ) &:= -\mathsf{h}\big\{\mu_2[\mathsf{E}_1(v_1^\circ), \mathsf{E}_1(v_2^\circ)]\big\}\ , \\
\mathsf{E}_3(v_1^\circ, v_2^\circ, v_3^\circ) &:= \pm\mathsf{h}\big\{\mu_3[\mathsf{E}_1(v_1^\circ), \mathsf{E}_1(v_2^\circ), \mathsf{E}_1(v_3^\circ)] \\
&\qquad\quad \pm \mu_2[\mathsf{E}_2(v_1^\circ, v_2^\circ), \mathsf{E}_1(v_3^\circ)] \pm \mathit{cyclic}\big\}\ , \\
&\ \ \vdots \\
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&\ \ \vdots
\end{aligned}$$

for all homogeneous $v_1^\circ, \ldots, v_i^\circ \in H^\bullet(V)$.

Minimal model contd.

Maps h, p and e and their conditions are the **abstract Hodge decomposition**

$$\begin{tikzcd}
 & & & & \vdots & & & \vdots \\
V^{k-1} & = & H^{k-1} & \oplus & B^{k-1} & \oplus & I^{k-1} \arrow[dll, bend right, "\mu_1"'] & H^{k-1}(V) \\
V^{k} & = & H^{k} \arrow[rrrrr, bend right, "\mathrm{p}_k"'] & \oplus & B^{k} \arrow[urr, bend right, "\mathrm{h}_k"'] & \oplus & I^{k} & H^{k}(V) \arrow[lllll, bend right, "\mathrm{e}_k"'] \\
 & & & & \vdots & & & \vdots
\end{tikzcd}$$

where H^k, B^k and I^k are the **harmonic**, **exact** and **coexact** subspaces of V_k. p_k and h_k are the inverses of μ_1 and e_k on the restricted spaces, which are trivially extended to all of V_k.

Deformation of near-horizon geometry

In our case, $V = V_1 \oplus V_2$ we have

$$
\begin{tikzcd}
& V_1 \arrow[rrr, "\mu_1"] \arrow[dl, shift right, "\mathrm{id}|_{V_1} - \epsilon_1 \circ \pi_1"'] \arrow[dr, shift left, "\pi_1"] & & & V_2 \arrow[dl, shift right, "\mathrm{id}|_{V_2} - \epsilon_2 \circ \pi_2"'] \arrow[dr, shift left, "\pi_2"] & \\
\substack{\ker(\mu_1) \\ = H^1(V)} \arrow[ur, shift right, "\iota_1"'] & & V_1/\ker(\mu_1) \arrow[r, "\hat{\mu}_1"] \arrow[ul, shift left, "\epsilon_1"] & \mathrm{im}(\mu_1) \arrow[ur, shift right, "\iota_2"'] & & \substack{V_2/\mathrm{im}(\mu_1) \\ = H^2(V)} \arrow[ul, shift left, "\epsilon_2"]
\end{tikzcd}
$$

with $\pi_{1,2}$ the **canonical quotient projections** and $\epsilon_{1,2}$ choices of **right-inverses**, $\iota_{1,2}$ the **inclusions**, and $\hat{\mu}_1$ the **canonical isomorphism** given by the first isomorphism theorem.

We define

$$
\begin{tikzcd}
(V, \mu_1) \arrow[loop left, "\mathsf{h}"] \arrow[r, shift left, two heads, "\mathsf{p}"] & (H^\bullet(V), 0) \arrow[l, shift left, hook, "\mathsf{e}"]
\end{tikzcd} ,
$$

where

$$
\mathsf{p}|_{V_1} := \mathrm{id}|_{V_1} - \epsilon_1 \circ \pi_1 = \mathrm{id}|_{V_1} - \mathsf{h} \circ \mu_1 , \quad \mathsf{p}|_{V_2} := \pi_2 ,
$$

$$
\mathsf{e}|_{H^1(V)} := \iota_1 , \quad \mathsf{e}|_{H^2(V)} := \epsilon_2 , \quad \mathsf{h} := \epsilon_1 \circ \hat{\mu}_1^{-1} \circ (\mathrm{id}|_{V_2} - \epsilon_2 \circ \pi_2) .
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\underset{= H^1(V)}{\ker(\mu_1)} \arrow[ur, hook, shift right, "\iota_1"'] & & V_1/\ker(\mu_1) \arrow[ul, hook, shift left, "\epsilon_1"] \arrow[r, "\hat{\mu}_1"] & \mathrm{im}(\mu_1) \arrow[ur, hook, shift right, "\iota_2"'] & & \underset{= H^2(V)}{V_2/\mathrm{im}(\mu_1)} \arrow[ul, hook, shift left, "\epsilon_2"]
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Solving the deformation problem

Outline of calculation steps

- Construct an L_∞-**algebra** (V, μ_i) by reading off the μ_i from the **Einstein field equations**.
- Construct the **cochain complex** (V, μ_1) and identify its **cohomology**.
- Determine the **contracting homotopy** h in

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- Construct the **minimal model** $(H^\bullet(V), \mu_i^\circ)$ and the **quasi-isomorphism** $\mathsf{E} : (H^\bullet(V), \mu_i^\circ) \to (V, \mu_i)$
- Solve the **Maurer–Cartan equation** of the minimal model $(H^\bullet(V), \mu_i^\circ)$.
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Cochain complex and its cohomology

The **cochain complex** is

$$\underbrace{\Omega^1_r(S) \oplus \mathscr{C}^\infty_r(S) \oplus \mathscr{S}^2_r(S)|_{\mathrm{gf}}}_{=:\, V_1} \xrightarrow{\mu_1} \underbrace{\Omega^1_r(S) \oplus \mathscr{C}^\infty_r(S) \oplus \mathscr{S}^2_r(S)}_{=:\, V_2} \ .$$

We find that

$$H^1(V) \ = \ \ker(\mu_1) \ \cong \ \ker(\bar{\mathrm{d}}_{ab}{}^{cd})$$

with

$$\bar{\mathrm{d}}_{ab}{}^{cd} \ := \ r^2 \bar{\mathrm{a}}_{ab}{}^{cd}(y)\partial_r^2 + r\bar{\mathrm{b}}_{ab}{}^{cd}(y)\partial_r + \bar{\mathrm{c}}_{ab}{}^{cd}(y) \ ,$$

where $\bar{\mathrm{a}}_{ab}{}^{cd}, \bar{\mathrm{b}}_{ab}{}^{cd}$ and $\bar{\mathrm{c}}_{ab}{}^{cd}$ are **operators on the spatial cross-section** S mapping the traceless part of h_{cd} (denoted $\bar{h}_{cd}$) to traceless symmetric $(0,2)$-tensors.
Expanding $\bar{h}_{cd}(r,y) = \sum_{n>0} \frac{r^n}{n!} \bar{h}^{(n)}_{cd}(y)$, the space of solutions of

$$\left[n(n-1)\bar{\mathrm{a}}_{ab}{}^{cd}(y) + n\bar{\mathrm{b}}_{ab}{}^{cd}(y) + \bar{\mathrm{c}}_{ab}{}^{cd}(y)\right] h^{(n)}_{cd}(y) = 0$$

is **isomorphic** to $H^1(V)$.

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Cochain complex and its cohomology contd.

$$\left[n(n-1)\bar{\mathsf{a}}_{ab}{}^{cd}(y)+n\bar{\mathsf{b}}_{ab}{}^{cd}(y)\partial_r+\bar{\mathsf{c}}_{ab}{}^{cd}(y)\right]h^{(n)}_{cd}(y)=0$$

$$\begin{aligned}
\bar{\mathsf{a}}_{ab}{}^{cd} &:= \left(\delta_{(a}{}^{c}\delta_{b)}{}^{d}-\tfrac{1}{d-2}\delta_{ab}\delta^{cd}\right)\left(\tfrac{1}{4}\mathring{\alpha}^{e}\mathring{\alpha}_{e}+\tfrac{1}{4}\mathring{\bar{\nabla}}^{e}\mathring{\alpha}_{e}-\tfrac{1}{d-2}\Lambda\right)\,,\\
\bar{\mathsf{b}}_{ab}{}^{cd} &:= \delta_{(b}{}^{(c}\left[\mathring{\bar{\nabla}}^{d)}\mathring{\alpha}_{a)}-\mathring{\bar{\nabla}}_{a)}\mathring{\alpha}^{d)}+\delta_{a)}{}^{d)}\left(\tfrac{1}{2}\mathring{\alpha}_{e}\mathring{\alpha}^{e}-\mathring{\alpha}^{e}\mathring{\bar{\nabla}}_{e}-\tfrac{2}{d-2}\Lambda\right)\right]\\
&\quad -\tfrac{1}{d-2}\delta_{ab}\delta^{cd}\left(\tfrac{1}{2}\mathring{\alpha}_{e}\mathring{\alpha}^{e}-\mathring{\alpha}^{e}\mathring{\bar{\nabla}}_{e}-\tfrac{2}{d-2}\Lambda\right)\,,\\
\bar{\mathsf{c}}_{ab}{}^{cd} &:= \delta_{(b}{}^{(c}\left\{\mathring{\alpha}^{d)}\mathring{\bar{\nabla}}_{a)}-\mathring{\alpha}_{a)}\mathring{\bar{\nabla}}^{d)}-\mathring{\bar{\nabla}}^{d)}\mathring{\bar{\nabla}}_{a)}+\mathring{\bar{\nabla}}_{a)}\mathring{\bar{\nabla}}^{d)}\right.\\
&\quad \left.+\delta_{a)}{}^{d)}\left[\tfrac{1}{2}\left(\mathring{\bar{\nabla}}_{e}-\mathring{\alpha}_{e}\right)\mathring{\bar{\nabla}}^{e}+\tfrac{2}{d-2}\Lambda\right]\right\}\\
&\quad -\tfrac{1}{d-2}\delta_{ab}\delta^{cd}\left[\tfrac{1}{2}\left(\mathring{\bar{\nabla}}_{e}-\mathring{\alpha}_{e}\right)\mathring{\bar{\nabla}}^{e}+\tfrac{2}{d-2}\Lambda\right]\,.
\end{aligned}$$

For each n, the equation is **elliptic** on S (compact without boundary). Thus, at each order in r, $H^1(V)$ is **finite dimensional**, which implies that the **moduli space of all-order solutions** (smaller than $H^1(V)$) is **finite dimensional** at each order in r.

Example (extremal Kerr horizon)

$$\left[n(n-1)\bar{\mathrm{a}}_{ab}{}^{cd}(y) + n\bar{\mathrm{b}}_{ab}{}^{cd}(y)\partial_r + \bar{\mathrm{c}}_{ab}{}^{cd}(y)\right] h^{(n)}_{cd}(y) = 0$$

are difficult to solve for arbitrary near-horizon geometries. However, for **extremal Kerr near-horizon geometry**, assuming **axisymmetry**, the solutions are

$$\begin{aligned}
\bar{h}^{(1)}_{11}(x) &= \frac{A}{m}\frac{(1-x^2)(5-16x^2-5x^4)}{10(1+x^2)^2}\,, \\
\bar{h}^{(1)}_{12}(x) &= \frac{A}{m}\frac{x(1-x^2)(9+x^2)}{5(1+x^2)^2}\,, \\
\bar{h}^{(n)}_{11}(x) &= \frac{(K^{(n)}_1(1-x^2)-2K^{(n)}_3 x)\mathrm{P}^2_n(x)}{m^n(1+x^2)^{n+1}}\,, \\
\bar{h}^{(n)}_{12} &= \frac{(2K^{(n)}_1 x + K^{(n)}_3(1-x^2))\mathrm{P}^2_n(x)}{m^n(1+x^2)^{n+1}}
\end{aligned}$$

for $n>1$, where $A, K^{(n)}_1$ and $K^{(n)}_3$ are arbitrary **constants** and $\mathrm{P}^2_n(x)$ denotes the associated Legendre polynomial.

Contracting homotopy h

The **contracting homotopy** h is uniquely determined by the **Green function** of $\bar{\mathrm{d}}_{ab}{}^{cd}$, i.e.

$$\begin{aligned}\Big[n(n-1)\bar{\mathrm{a}}_{ab}{}^{cd}(y) + n\bar{\mathrm{b}}_{ab}{}^{cd}(y) + \bar{\mathrm{c}}_{ab}{}^{cd}(y)\Big]\bar{\mathrm{g}}^{(n)\,cd}_{ab}(y;y') \\ = \ \left(\delta_{(a}{}^{c}\delta_{b)}{}^{d} - \tfrac{1}{d-2}\delta_{ab}\delta^{cd}\right)\frac{\delta^{(d-2)}(y-y')}{\sqrt{\det(\mathring{\gamma}(y))}}\ .\end{aligned}$$

The relation between h and $\mathrm{g}^{(n)\,cd}_{ab}$ is lengthy and will be omitted.
For **extremal Kerr near-horizon geometry**, the Green function is

$$\begin{aligned}\bar{\mathrm{g}}^{(1)11}_{11}(x,\varphi;x',\varphi') \ &= \ -\frac{(1+x'^2)[1-x'^2+4xx'-x^2(1-x'^2)]}{24(1-x^2)(1-x'^2)(1+x^2)^2}\delta(\varphi-\varphi') \\ &\quad \times[(2-x)(1+x)^2(2+x')(1-x')^2\theta(x'-x) \\ &\quad + (1-x)^2(2+x)(2-x')(1+x')^2\theta(x-x')]\ , \\ \bar{\mathrm{g}}^{(1)12}_{22}(x,\varphi;x',\varphi') \ &= \ -\frac{(1+x'^2)(x-x')(1+xx')}{12(1-x^2)(1-x'^2)(1+x^2)^2}\delta(\varphi-\varphi') \\ &\quad \times[(2-x)(1+x)^2(2+x')(1-x')^2\theta(x'-x) \\ &\quad + (1-x)^2(2+x)(2-x')(1+x')^2\theta(x-x')]\end{aligned}$$

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for $n > 1$ and $\bar{\mathrm{g}}_{11}{}^{11} = \bar{\mathrm{g}}_{12}{}^{12}$ and $\bar{\mathrm{g}}_{11}{}^{12} = -\bar{\mathrm{g}}_{12}{}^{11}$

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The minimal model

We construct solutions only up to **next-to-lowest order**, so we consider μ_i and E_i for $i \leq 2$.

- The **minimal model** is $(H^\bullet(V) = H^1(V) \oplus H^2(V), \mu_i^\circ)$, where

$$\mu_1^\circ(\Theta_1^\circ) \;\coloneqq\; 0\;, \quad \mu_2^\circ(\Theta_1^\circ, \Theta_2^\circ) \;\coloneqq\; \mathsf{p}(\mu_2(\mathsf{E}_1(\Theta_1^\circ), \mathsf{E}_1(\Theta_2^\circ)))\;, \ldots$$

  for all homogeneous $\Theta_1^\circ, \Theta_2^\circ \in H^\bullet(V)$.

- The **quasi-isomorphism** $\mathsf{E} : (H^\bullet(V), \mu_i^\circ) \to (V, \mu_i)$ is

$$\mathsf{E}_1 \;\coloneqq\; \mathrm{id}, \quad \mathsf{E}_2(\Theta_1^\circ, \Theta_2^\circ) \;\coloneqq\; -\mathsf{h}(\mu_2(\mathsf{E}_1(\Theta_1^\circ), \mathsf{E}_1(\Theta_2^\circ)))\;, \ldots\;.$$

- The **Maurer–Cartan equation** of $(H^\bullet(V), \mu_i^\circ)$ is

$$\tfrac{1}{2}\mu_2^\circ(\Theta^\circ, \Theta^\circ) + \ldots = 0$$

  for $\Theta^\circ \in H^1(V)$.

Example (extremal Kerr horizon)

The space $H^1(V)$ is parametrised by A, $K_1^{(n)}$ and $K_3^{(n)}$. Hence, the **Maurer–Cartan equation** is algebraic,

$$\mathrm{MC}(A, K_1^{(n)}, K_3^{(n)}) = 0 \ .$$

At **next-to-lowest order**, this is **trivially satisfied**, so the space of solutions of the **minimal model** Maurer–Cartan equation is $H^1(V)$. We map these solutions to those of the **original** Maurer–Cartan equation of (V, μ_1) (the **Einstein field equations**) via

$$\Theta = E_1(\Theta^\circ) + \tfrac{1}{2} E_2(\Theta^\circ, \Theta^\circ) + \dots \ .$$

Hence, the moduli space of **next-to-lowest order deformations** of extremal Kerr near-horizon geometry is parametrised by $A, K_1^{(n)}$ and $K_3^{(n)}$. At **higher orders**, $A, K_1^{(n)}$ and $K_3^{(n)}$ may become constrained.

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Example (extremal Kerr horizon) contd.

As an example the **next-to-lowest order solutions** for h_{11} are

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\begin{aligned}
h_{11} \; &= \; \frac{Ar(5-3x^2+13x^4+5x^6)}{5m(1+x^2)^2} - \frac{3r^2(1-x^2)}{2m^2(1+x^2)^3}\big[-K_1^{(2)}(1-x^2)+2K_3^{(2)}x\big] \\
&\quad + \frac{A^2r^2}{300m^2(1+x^2)^4}(-675+92x+2752x^2-1851x^4-92x^5-268x^6+747x^8 \\
&\quad +420x^{10}+75x^{12}) + \mathcal{O}(r^3) + \dots \; .
\end{aligned}
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To locate the **extremal Kerr black hole** in the **moduli space of deformations**, we can either:

- Compare it to the **next-to-lowest order** solutions to fix A, $K_1^{(2)}$ and $K_3^{(2)}$.
- Projects it onto $H^1(V)$ using p_1, then compare it to the **lowest-order** solution to fix all $A, K_1^{(n)}$ and $K_3^{(n)}$.

We find

$$
A = \tfrac{15}{7}\,, \quad K_1^{(2)} = \tfrac{277}{49}\,, \quad K_3^{(2)} = \tfrac{23}{49}\,, \quad K_1^{(3)} = \tfrac{2684}{1029}\,, \quad K_3^{(3)} = -\tfrac{254}{343}\,.
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Summary and outlook

- The deformation problem of near-horizon geometry is encoded as an L_∞-algebra. The Einstein field equations are the Maurer–Cartan equation.
- The problem reduces to solving the Maurer–Cartan equation of the minimal model $(H^\bullet(V), \mu_i^\circ)$.
- In extremal Kerr near-horizon geometry, the equation is algebraic and trivially satisfied (up to the next-to-lowest order) for all $\Theta^\circ \in H^1(V)$.
- The formalism gives an upper bound for the degrees of freedom of the moduli space of deformations, which can be lowered by including higher-order terms in the Maurer–Cartan equation of the minimal model.
- Explore warp-product of AdS space can be expressed in Gaussian-null coordinates (connection to holography)
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