Semigroups Of Generalised Symmetries

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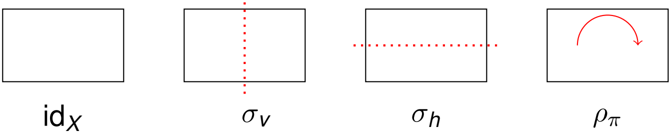
Classical Symmetry

The concept of “symmetry” originally belongs in Geometry.

Example: A rectangle has four symmetries:



The symmetries of a geometric shape X are the distance preserving bijections $f : X \to X$.

Beyond Geometry

The symmetries of any object X are the bijections $f : X \to X$ which preserve the structure of X.

object X	**Symmetries of X**	**preserved structure**
metric space	isometries	distance
topological space	homeomorphisms	open sets
relational structure	automorphisms	tuples
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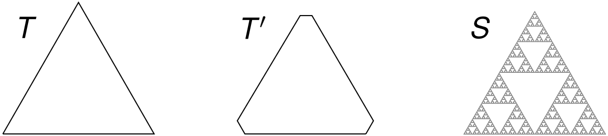
The symmetries of any X form a subgroup of the symmetric group $\mathrm{Sym}(X)$.

Three shapes with the same symmetry groups:



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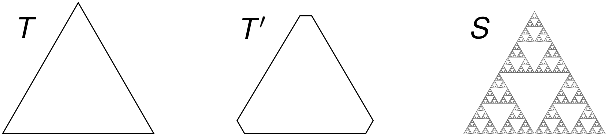

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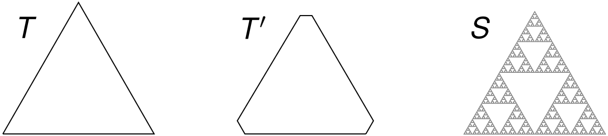

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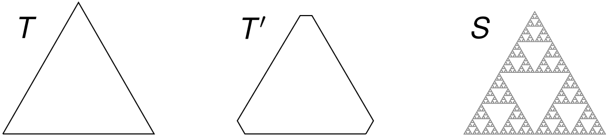

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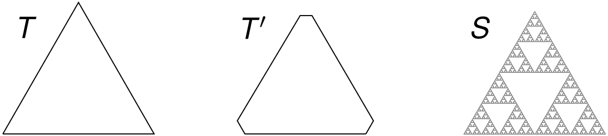

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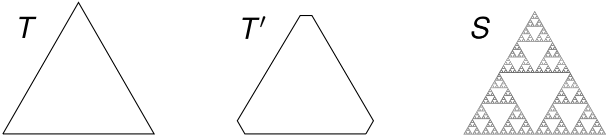

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+ Maybe, intuitively, T' is not a hexagon but a “triangle with cut-off corners”.
+ And the Sierpiǹski **triangle** S is very “triangly”.
- But, also intuitively, S seems to have a lot more symmetry (self-similarity). This is not captured by groups.

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- A *transformation* of X is a function $f : X \to X$.
- A *partial bijection* on X is a bijection $f : Y \to Z$ where Y and Z are subsets of X.
- The *Full Transformation Semigroup* X^X consists of all transformations of X.
- The *Symmetric Inverse Semigroup* I_X consists of all partial bijections on X.

Some abstract definitions. . .

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- ***monoid***: a semigroup with an identity element.

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Recall (or see for the first time):

- ***semigroup***: a set with an associative binary operation.
- ***monoid***: a semigroup with an identity element.
- ***inverse semigroup***: a semigroup S such that $(\forall x \in S)(\exists! x^{-1} \in S) : xx^{-1}x = x$ and $x^{-1}xx^{-1} = x^{-1}$.

$$\begin{tikzcd} & & \text{monoid} \arrow[dr, Rightarrow] & \\ \text{group} \arrow[r, Rightarrow] & \text{inverse monoid} \arrow[ur, Rightarrow] \arrow[dr, Rightarrow] & & \text{semigroup} \\ & & \text{inverse semigroup} \arrow[ur, Rightarrow] & \end{tikzcd}$$

. . . and why you may immediately forget them.

Theorem (A. Cayley, 1854)

Every group is isomorphic to a subgroup of some $Sym(X)$.

Theorem (Folklore, 20th century)

Every semigroup is isomorphic to a subsemigroup of some X^X.

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The rest of this talk is mostly about $\mathrm{Sym}(X)$, X^X, and I_X when X is an infinite set.

Joint work with. . .

- **Serhii Bardyla**, University of Vienna, Austria
- **Igor Dolinka**, University of Novi Sad, Serbia
- **James East**, Western Sydney University, Australia
- **Luna Elliott**, University of Manchester, UK
- **Martin Hampenberg**, University of Hertfordshire, UK
- **James Hyde**, Cornell University, USA
- **Julius Jonušas**, enspired GmbH, Austria
- **Zak Mesyan**, University of Colorado, USA
- **James D. Mitchell**, University of St Andrews, UK
- **Michał Morayne**, Technical University of Wroclaw, Poland
- **Michael Pinsker**, Technical University of Vienna, Austria
- **Martin Quick**, University of St Andrews, UK

Cardinality

If X is a finite set, then

$$\mathrm{Sym}(X) = |X|! \qquad \left|X^X\right| = |X|^{|X|} \qquad |I_X| = \sum_{k=0}^{|X|} \binom{|X|}{k}^2 k!$$

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In the smallest infinite case $X = \mathbb{N} = \{0, 1, 2, 3, \ldots\}$,

$$|\mathrm{Sym}(\mathbb{N})| = |\mathbb{N}^{\mathbb{N}}| = |I_{\mathbb{N}}| = 2^{\aleph_0} = \mathfrak{c}$$

where $\mathfrak{c} = |\mathbb{R}|$ is the *continuum*.

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Is there a better notion of rank for uncountable semigroups?

Sierpiǹski rank

The *Sierpiǹski rank* of a semigroup S is the least $n \in \mathbb{N} \cup \{\infty\}$ such that every countable subset of S is contained in an n-generated subsemigroup of S.

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If X is any infinite set, then:

Theorem (W. Sierpiǹski, 1935)

The Sierpiǹski rank of X^X is 2.

Theorem (F. Galvin, 1995)

The Sierpiǹski rank of $Sym(X)$ is 2.

Theorem (J. Hyde, YP, 2012)

The Sierpiǹski rank of I_X is 2.

Other semigroups with finite Sierpiǹski rank

rank 2 The homeomorphism groups of certain totally disconnected spaces including the rational numbers, the Baire space, and the Cantor space. (F. Galvin's proof)

rank 2 the continuous functions on any of the rational numbers $\mathbb{Q}$, the m-dimensional unit square $[0, 1]^m$, or the Hilbert cube $[0, 1]^{\mathbb{N}}$ (Cook, Ingram, 1969);

rank 2 the linear maps on an infinite dimensional vector space (Magill, 1988);

rank 3 the order endomorphisms of the unit interval $[0, 1]$ (Mitchell, YP, Quick, 2007).

rank 2 the endomorphisms of certain Fraïssé limits such as the Random Graph (Dolinka, YP, 2014);

rank 2 The order automorphisms of the rational numbers $\mathbb{Q}$ (J. Hyde, J. Jonusas, J.D. Mitchell, YP, 2016)

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Consequences of finite Sierpiǹski rank 1: embeddings

From the facts that $\mathrm{Sym}(X)$, X^X, and I_X...

1. ...have Sierpiǹski rank 2;
2. ...contain all groups, semigroups, and inverse semigroups, respectively;

we get the following corollary.

Corollary

Every countable group (semigroup, inverse semigroup) is a subgroup (subsemigroup, inverse subsemigroup) of a 2-generator group (semigroup, inverse semigroup).

Definition (relative rank)

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Relative rank is an algebraic measure of the size of a subsemigroup T. Big subsemigroups have small relative rank.

Consequences of finite Sierpiǹski rank 2: relative rank

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Proposition

Let S be a semigroup with finite Sierpiǹski rank n and T a subsemigroup of S. Then the relative rank of T in S is either at most n or uncountable.

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Theorem (M. Hampenberg, YP, 2025)

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So $\mathrm{Sym}(X)$, I_X, and X^X have as many maximal substructures as they have subsets!
Let us borrow some extra tools from Topology.

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- An *inverse semigroup topology* for an inverse semigroup $(I, \cdot)$ is any topology on I under which the maps
$$(a, b) \mapsto a \cdot b \quad \text{and} \quad a \mapsto a^{-1}$$
are continuous.
- An inverse semigroup topology on a group is called a *group topology*.

Why Topological Algebra?

Some good ways of using Topology in Semigroup Theory:

1. Fix a semigroup S. What kind of topologies does S admit?
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    - What are the subsemigroups of S which are closed (or open, compact, ...) under τ?
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For point 3 to be interesting and meaningful, we need to agree on a τ which is (i) 'nice' in a topological sense and (ii) 'natural' for S.

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Properties that make topologists happy

Part 1. Having **enough** open sets

There are nine, increasingly stronger, “separation axioms”

$$T_0 \Leftarrow T_1 \Leftarrow T_2 \Leftarrow T_{2\frac{1}{2}} \Leftarrow T_3 \Leftarrow T_{3\frac{1}{2}} \Leftarrow T_4 \Leftarrow T_5 \Leftarrow T_6.$$

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They describe ways in which points in the space may be separated by open sets. For example, a topological space S is...

- ... T_1 (Fréchet) if for all distinct $x, y \in S$, there exists an open neighbourhood of x which does not contain y;
- ... T_2 (Hausdorff) if all distinct $x, y \in S$ have disjoint open neighbourhoods.

$U \subseteq S$ is a *neighbourhood* of $x \in S$ if $x \in V \subseteq U$ for some open $V \subseteq S$.

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Part 2. Having **not too many** open sets

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- ...*separable* if S has a countable dense subset;
- ...*compact* if every open cover of S may be reduced to a finite subcover;
- ...*connected* if no open set (other than $\emptyset$ and S) is also closed;

Properties that make topologists happy

Part 3. Being like $\mathbb{R}$

A topological space (S, τ) is…

- …*second-countable* if τ has a countable basis;
- …*metrizable* if τ is induced by a metric on S;
- …*completely metrizable* if τ is induced by a complete metric on S;
- …*Polish* if S is completely metrizable and separable.
- …*locally compact* if every point in S has a compact neighbourhood.

Natural topologies for a semigroup

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Two approaches:

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2. **Purely algebraic:** Ignore the context (if any) of the set S as an object and consider topologies that may be defined on any abstract semigroup $(S, \cdot)$. Examples:
   - Minimal topologies which are T_1, Hausdorff, ...
   - Maximal topologies which are compact, second-countable,...
   - Topologies defined via algebraic equations (Zariski topologies).

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- $\mathrm{Sym}(\mathbb{N})$ is a topological group;
- $\mathbb{N}^{\mathbb{N}}$ and $\mathrm{Sym}(\mathbb{N})$ are Polish (completely metrizable & separable);
- $M \leq \mathbb{N}^{\mathbb{N}}$ is closed $\iff$ $M = \mathrm{End}(R)$ for some relational structure R on $\mathbb{N}$;
- $G \leq \mathrm{Sym}(\mathbb{N})$ is closed $\iff$ $G = \mathrm{Aut}(R)$ for some relational structure R on $\mathbb{N}$.

Unique topologies: Sym($\mathbb{N}$)

Theorem (Gaughan 1967)

The pointwise topology is the least Hausdorff group topology on Sym($\mathbb{N}$).

Theorem (Kechris, Rosendal 2004)

- *The pointwise topology is the unique non-trivial separable group topology on Sym($\mathbb{N}$).*
- *Every homomorphism from Sym($\mathbb{N}$) to a separable group is continuous (automatic continuity).*

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Unique topologies: $\mathbb{N}^\mathbb{N}$ and friends

Theorem (L. Elliott, J. Jonušas, J.D. Mitchell, Z. Mesyan, M. Morayne, YP 2023)

The following transformation monoids have a unique Polish semigroup topology:

- $\mathbb{N}^\mathbb{N}$*;*
- *The monoid* $P(\mathbb{N})$ *of partial transformations of* $\mathbb{N}$*;*
- *The monoid* $C([0,1]^\mathbb{N})$ *of continuous transformations of the Hilbert cube* $[0,1]^\mathbb{N}$*;*
- *The monoid* $C(2^\mathbb{N})$ *of continuous transformations of the Cantor set* $2^\mathbb{N}$*.*

Moreover, $\mathbb{N}^\mathbb{N}$, $P(\mathbb{N})$*, and* $C(2^\mathbb{N})$ *have automatic continuity: every homomorphism to a second-countable semigroup is continuous.*

Unique topologies: endomorphism semigroups

Theorem (L. Elliott, J. Jonušas, J.D. Mitchell, YP, M. Pinsker 2023)

The pointwise topology is the unique Polish semigroup topology on the endomorphism monoids of the following relational structures:

1. *the random graph;*
2. *the random directed graph;*
3. *infinitely many copies of a finite complete graph K_n;*
4. *a partition of $\mathbb{N}$ into finitely many infinite classes;*
5. *the random strict partial order.*

The endomorphism monoids of 1-4 have automatic continuity.

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- *a unique Polish inverse semigroup topology (the maximal Polish semigroup topology).*

Every finite partial order embeds into the partial order of Polish semigroup topologies on $I_{\mathbb{N}}$.

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Do closed subsemigroups of $I_{\mathbb{N}}$ correspond to relational structures like they do in $\mathrm{Sym}(\mathbb{N})$ and $\mathbb{N}^{\mathbb{N}}$?

Model Theory on $I_{\mathbb{N}}$

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Every partial automorphism monoid must be *full*; that is closed under taking restrictions of functions.

Theorem (M. Hampenberg, YP 2024)

Let M be a full inverse submonoid of $I_{\mathbb{N}}$. Then the following are equivalent:

- *M is closed in a Polish semigroup topology on $I_{\mathbb{N}}$;*
- *M is closed in every Polish semigroup topology on $I_{\mathbb{N}}$;*
- *M is the partial automorphism monoid of a relational structure defined on $\mathbb{N}$.*

The Bergman-Shelah preorder

Recall: The relative rank of a subsemigroup T of S is the least number of extra elements needed to turn T into a generating set for S.

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Inspired by this dichotomy, Bergman and Shelah introduced the following pre-order on subgroups of $\mathrm{Sym}(X)$:

$$G \preccurlyeq H \iff G \subseteq \langle A, H \rangle \text{ for some finite } A \subseteq \mathrm{Sym}(X).$$

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Notation:

$$G \approx H \iff G \preccurlyeq H \text{ and } H \preccurlyeq G$$
$$G \prec H \iff G \preccurlyeq H \text{ and } H \not\preccurlyeq G.$$

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Four classes of closed subgroups of Sym($\mathbb{N}$)

Theorem (G. Bergman, S. Shelah, 2006)

Every topologically closed subgroup of Sym($\mathbb{N}$) *is* $\approx$*-equivalent to one of the following:*

1. *the entire symmetric group Sym*($\mathbb{N}$)
2. *the infinite direct product* $G_\omega := S_2 \times S_3 \times S_4 \times \dots$ *in its natural action on the partition* $\{0,1\},\{2,3,4\},\{4,5,6,7\},\dots$
3. *the infinite direct product* $G_2 := S_2 \times S_2 \times S_2 \times \dots$ *in its natural action on the partition* $\{0,1\},\{2,3\},\{4,5\},\dots$
4. *the trivial subgroup* $\{1_{\mathbb{N}}\}$.

Moreover, $\{1_{\mathbb{N}}\} \prec G_2 \prec G_\omega \prec Sym(\mathbb{N})$

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Relative ranks of closed subgroups of Sym($\mathbb{N}$)

Using the result of Bergman and Shelah, we can classify the relative ranks of closed subgroups of Sym($\mathbb{N}$).

Theorem (J.D. Mitchell, M. Morayne, YP, 2010)

Let G be a proper subgroup of Sym($\mathbb{N}$) which is closed in the pointwise topology. Then the relative rank of G in Sym($\mathbb{N}$) is one of the following: 1*, the dominating number* $\mathfrak{d}$*, or the continuum* $2^{\aleph_0}$.

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The Bergman-Shelah preorder on $I_{\mathbb{N}}$

Theorem (M. Hampenberg, YP, 2024)

Let M be the monoid of automorphisms between substructures of a relational structure R defined on $\mathbb{N}$. If R has finitely many relations, then $M \approx I_{\mathbb{N}}$.

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Further results:

- The Bergman-Shelah preorder on $I_{\mathbb{N}}$ has at least 4 classes containing closed full inverse monoids.
- We conjecture that these are the only ones.

Theorem (Z. Mesyan, J.D. Mitchell, M.Morayne, YP, 2012)

The Bergman-Shelah preorder on $\mathbb{N}^{\mathbb{N}}$ *has*

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- *at least $\aleph_0$ classes containing closed submonoids of $\mathbb{N}^{\mathbb{N}}$;*
- *arbitrarily large finite anti-chains of $\approx$-classes containing closed submonoids of $\mathbb{N}^{\mathbb{N}}$.*

Moreover, the following are equivalent.

1. *There exists a subsemigroup S of $\mathbb{N}^{\mathbb{N}}$ such that $S \approx \mathbb{N}^{\mathbb{N}}$ and every subsemigroup T of S satisfies $T \approx S$ or $T \approx \{1_{\mathbb{N}}\}$.*
2. *The continuum hypothesis.*

Thank you for listening!