Correlators of Multiple Light-Like Wilson Loops in Planar N=4 Super Yang-Mills Theory,

Rowan Wright, 21st January 2026, University of Hertfordshire

Based on

2512.12655 [Drummond, Gürdoğan, Rochford, RW]

+ 2601.23210 [Drummond, Rochford, RW]

+ forthcoming paper on one-loop leading singularities [Drummond, Rochford, RW]

+ forthcoming paper on BCFW-like recursion [Drummond, Rochford, RW]

Outline

- Motivations
- Review of duality between amplitudes and Wilson loops in N=4 super Yang-Mills, and simple Wilson loop calculations
- Review of the holomorphic Wilson loop formalism of Mason and Skinner
- Computations of multiple Wilson loop correlators at tree level and one-loop
- Summary of some interesting properties (Q-bar equation, BCFW, solution to the general one-loop problem)
- Wild speculations

0. Motivations

Motivation: Scattering Amplitudes

We are in the **precision-era:** the LHC provides an abundance of high-precision data (most cross-sections ~1% accuracy!) but theory lags behind – hard to get precise predictions out of the model!

e.g. ATLAS W+ jets

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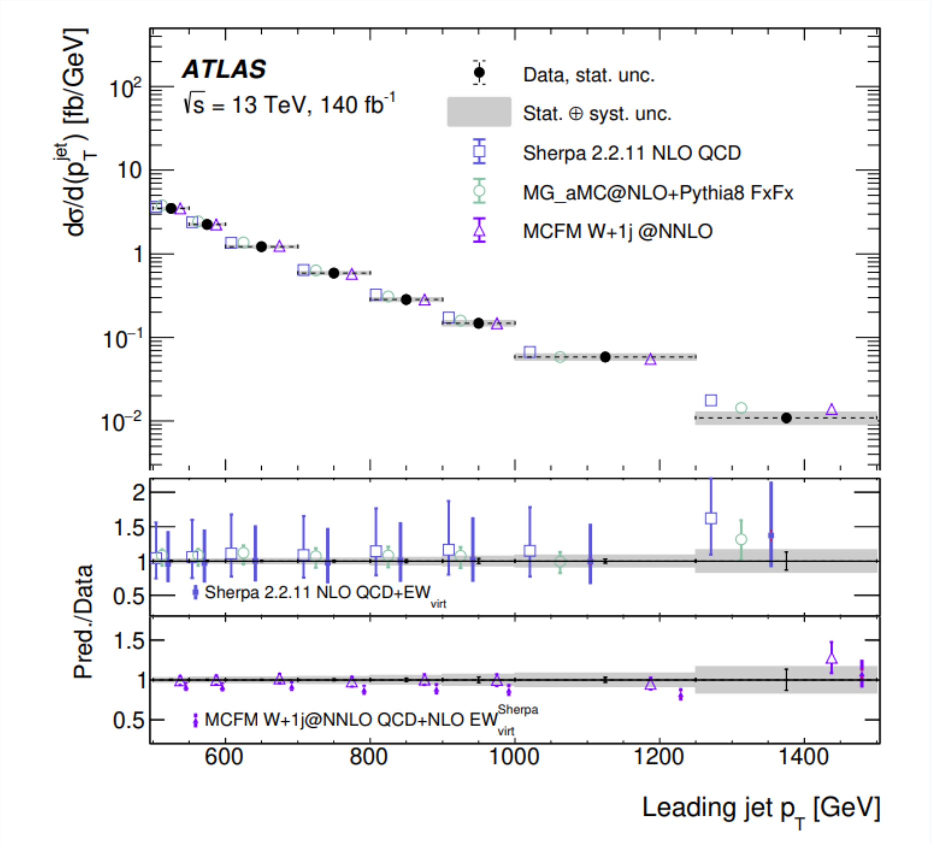
Experimental uncertainty hardly visible! Theory error bars far wider…

Need ultra-precise predictions for QCD background to disentangle signal from noise major and pressing challenge.



Motivation: Scattering Amplitudes

Amplitudes 2024 (Institute for Advanced Study), Maria O'Leary

A whole subdiscipline has emerged around the development of new techniques for computing these objects.

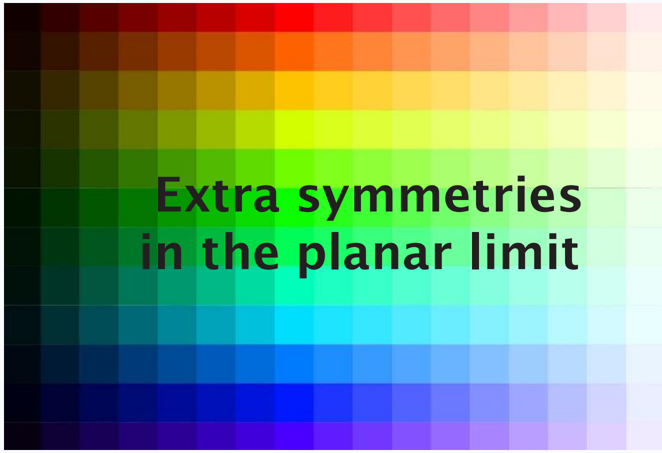
Motivation: Scattering Amplitudes

Computing loop corrections to QCD amplitudes is hard! Simpler, toy models can provide a useful crucible for the development of new techniques.

A natural choice: maximally supersymmetric Yang-Mills theory.
- 4D; conformal theory; no masses.
- Contains gauge boson, 4 (complex) fermions, 6 real scalars (all in adjoint representation)

Tree-level gluon amplitudes match QCD; other amplitudes can also be extracted (takes work!).

Also contains most complicated piece of QCD gluon amplitude at all loop-order!

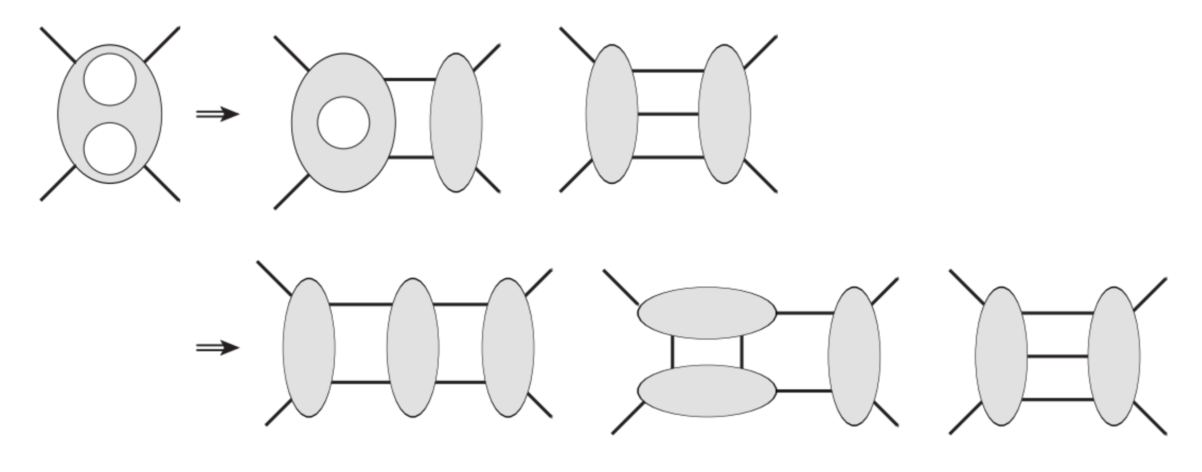


Motivation: Scattering Amplitudes

Many important and widely applicable techniques were originally developed in the context of N=4 SYM.

- Unitarity method [Bern, Dixon, Kosower, 1994] – take cuts to reconstruct the answer in terms of tree-level data.

[Review, Bern and Huang, 2011]



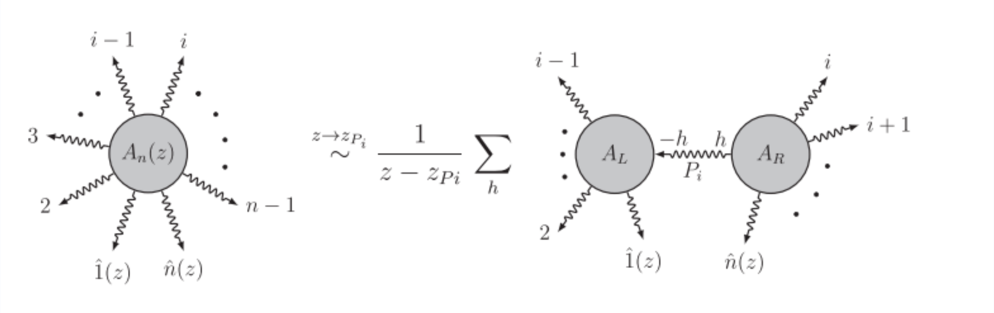
Used e.g. for the first NLO QCD calculations with W + 4 jets [Berger et. al, 2011]

Motivation: Scattering Amplitudes

Many important and widely applicable techniques were originally developed in the context of N=4 SYM.

- BCFW recursion relations [Britto, Cachazo, Feng, 2004] [+Witten, 2005]; originally for N=4 and used to express all tree-level N=4 amplitudes in closed form [Drummond, Henn, 2009].

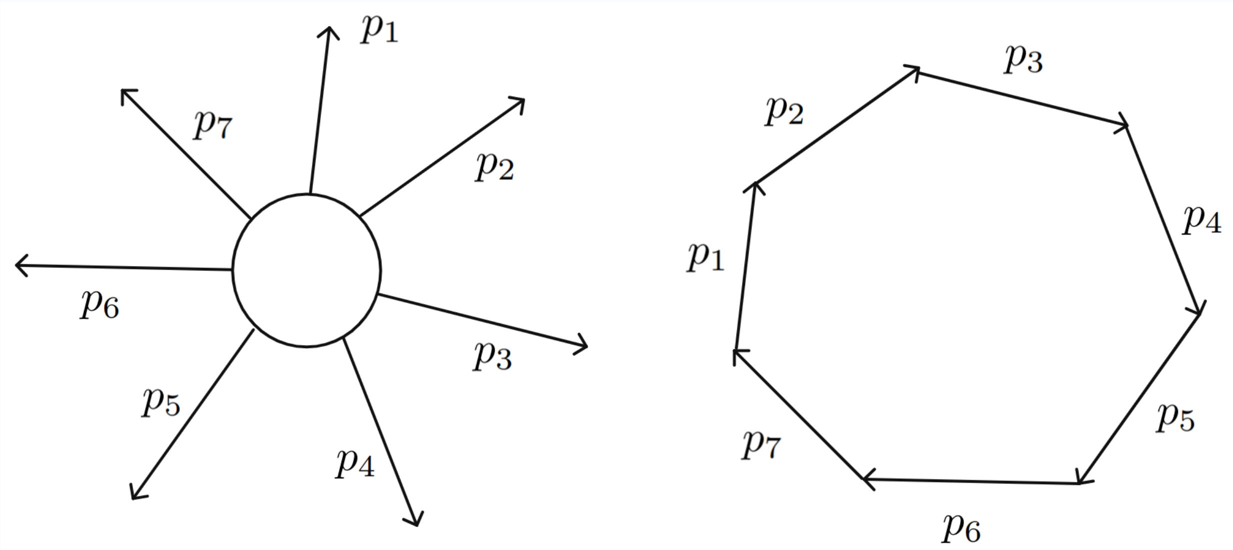
Exploits factorisation of amplitude into simpler pieces on poles, leading to very compact expressions for tree-level amplitudes.



Can also be used for loop integrands in N=4 [Arkani-Hamed, Bourjaily, Cachazo, Caron-Huot, Trnka, 2010].

Motivation: Scattering Amplitudes

Key observation (reviewed later): planar N=4 amplitudes are dual to the expectation values of null polygonal Wilson loops computed in the same theory!



Any property of the Wilson loops must also hold for the amplitudes (and vice versa); more tools overall!

Motivation: Scattering Amplitudes

Many properties more natural on Wilson loop side of the duality:

- Dual conformal symmetry [Drummond, Korchemsky, Sokatchev, Henn, 2007]; part of the infinite-dimensional Yangian symmetry underpinning integrability [Drummond, Henn, Plefka, 2009] [Drummond, Ferro, Ragoucy, 2010].
- Q-bar equation [Caron-Huot, 2010, He] (relates amplitudes of different loop-orders and helicities; used to compute MHV octagon amplitudes at three loops [Li, Zhang, 2021])
- Flux-tube OPE expansion; used to obtain collinear expansions at very high loop-order [Basso, Viera, 2014]
- Amplituhedron (interprets the amplitude/Wilson loop as the volume of a polytope) [Arkani-Hamed, Trnka, 2012]

Even properties more native to amplitudes have interpretations in terms of Wilson loops.e.g. BCFW from Makeenko-Migdal loop equations for Wilson loops [Bullimore, Skinner, 2011]!

Motivation: Scattering Amplitudes

To what extent do these mathematical properties generalise to more general objects related to Wilson loops? Can we learn more about amplitudes by studying such generalisations?

Obvious candidates:

1. "Wilson loop with Lagrangian operator insertion"; currently of much-interest in the literature (e.g. recently [Carrolo, Chicherin, Henn, Yang, Zhang, 2025] – hexagon at two loops)

2. Correlation functions of multiple Wilson loops

(in principle should contain the former object as a limit, and separately may also be related to spectrum of local operators)

1. Review of amplitude-Wilson loop duality

Perturbative expansion of Wilson loops

Planar N=4 Super Yang-Mills; gauge group G (usually SU(N)); action

$$S = \frac{1}{g^2_{\mathrm{YM}}} \int d^4x \left[-\frac{1}{2} \operatorname{tr} F^{\mu\nu} F_{\mu\nu} + \ldots \right],$$

Interested in Wilson loop operators (more specifically their correlators!) defined by

$$\mathcal{L}(C) = \frac{1}{N} \operatorname{tr} \mathcal{P} \exp i \oint_C dx^\mu A_\mu$$

Key gauge-invariant observables of much interest beyond application to amplitudes!

Standard, textbook computation. Curve $x^\mu : [0,1] \to \mathbb{R}^{1,3}$

$$\mathcal{L}(C) = \frac{1}{N} \operatorname{tr} \left[\mathbb{1} + i \int_0^1 dt_1 \dot{x}^\mu(t_1) A_\mu\big(x(t_1)\big) - \int_0^1 dt_1 \dot{x}^\mu(t_1) A_\mu\big(x(t_1)\big) \int_0^{t_1} dt_2 \dot{x}^\nu(t_2) A_\nu\big(x(t_2)\big) + \ldots \right]$$

Feynman rules for Wilson loops

Now consider expectation value of this Wilson loop. In the SU(N) theory $A_\mu = A_\mu^a t^a$

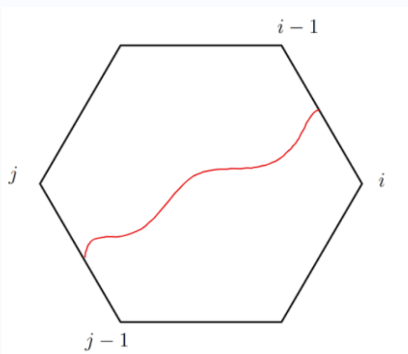
$$\begin{aligned}\langle \mathcal{L}(C) \rangle = 1 &- \frac{1}{N} \int_{t_1>t_2} \dot{x}_1^\mu \dot{x}_2^\nu \left[G^{1,2}_{\mu\nu,ab} \,\mathrm{tr}(t_a t_b) + \ldots \right] \\ &- \frac{i}{N} \int_{t_1>t_2>t_3} \dot{x}_1^\mu \dot{x}_2^\nu \dot{x}_3^\rho \left[V^{1,2,3}_{\mu\nu\rho,abc} \,\mathrm{tr}(t_a t_b t_c) + \ldots \right] \\ &+ \frac{1}{N} \int_{t_1>t_2>t_3>t_4} \dot{x}_1^\mu \dot{x}_2^\nu \dot{x}_3^\rho \dot{x}_4^\sigma \left[\left[G^{1,2}_{\mu\nu,ab} G^{3,4}_{\rho\sigma,cd} + G^{1,3}_{\mu\rho,ac} G^{2,4}_{\nu\sigma,bd} + G^{1,4}_{\mu\sigma,ad} G^{2,3}_{\nu\rho,bc} \right] \mathrm{tr}(t_a t_b t_c t_d) + \ldots \right] \\ &+ \ldots .\end{aligned}$$

Propagators/vertices for perturbative calculations:

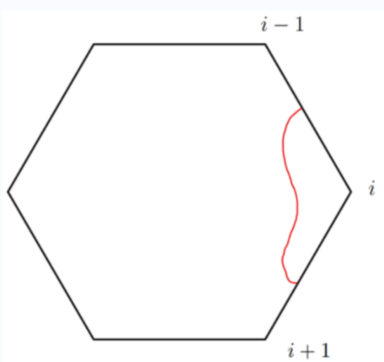
$$\begin{aligned} G^{i,j}_{\mu\nu,ab} &= -\frac{g^2_{\mathrm{YM}}}{4\pi^2} \eta_{\mu\nu} \delta_{ab} \frac{(\pi \tilde{\mu}^2)^\epsilon \Gamma(1-\epsilon)}{[-(x_i - x_j)^2 + i\varepsilon]^{(1-\epsilon)}} , \\ V^{i,j,k}_{\mu\nu\rho,abc} &= \frac{1}{g^2_{\mathrm{YM}}} f_{a'b'c'} \tilde{\mu}^{-2\epsilon} \int d^{4-2\epsilon} x_0 \mathcal{D}^{ijk}_{\mu'\nu'\rho'} [G^{i,0}_{\mu\mu',aa'} G^{j,0}_{\nu\nu',bb'} G^{k,0}_{\rho\rho',cc'}] , \\ \mathcal{D}^{ijk}_{\mu\nu\rho} &= \eta_{\mu\nu}(\partial_{i\rho} - \partial_{j\rho}) + \eta_{\nu\rho}(\partial_{j\nu} - \partial_{k\nu}) + \eta_{\rho\mu}(\partial_{k\mu} - \partial_{i\mu}) . \end{aligned}$$

Example diagrams

Can now perform diagrammatic calculations of Wilson loop correlators (nodal, light-like curves) at weak coupling. For instance, for one loop...



$$
\begin{aligned}
I_{ij} =\;& 2\int_0^1 dt_1 \int_0^1 dt_2 \frac{x_{i+1,i}\cdot x_{j+1,j}}{[-(x_i(t_1)-x_j(t_2))^2+i\varepsilon]} \\
=\;& \mathrm{Li}_2\left[\frac{(x_{ij}^2-x_{i,j+1}^2)(x_{i,j+1}^2-x_{i+1,j+1}^2)}{x_{i,j+1}^2x_{i+1,j}^2-x_{ij}^2x_{i+1,j+1}^2}\right]+\mathrm{Li}_2\left[\frac{(x_{ij}^2-x_{i+1,j}^2)(x_{i+1,j}^2-x_{i+1,j+1}^2)}{x_{i,j+1}^2x_{i+1,j}^2-x_{ij}^2x_{i+1,j+1}^2}\right] \\
&-\mathrm{Li}_2\left[\frac{(x_{ij}^2-x_{i,j+1}^2)(x_{ij}^2-x_{i+1,j}^2)}{x_{i,j+1}^2x_{i+1,j}^2-x_{ij}^2x_{i+1,j+1}^2}\right]-\mathrm{Li}_2\left[\frac{(x_{i,j+1}^2-x_{i+1,j+1}^2)(x_{i+1,j}^2-x_{i+1,j+1}^2)}{x_{i,j+1}^2x_{i+1,j}^2-x_{ij}^2x_{i+1,j+1}^2}\right]
\end{aligned}
$$



$$
I_{j-1,j} = 2(\pi\tilde{\mu}^2)^\epsilon\Gamma(1-\epsilon)\int_0^1 dt_1\int_0^1 dt_2 \frac{x_{j,j-1}\cdot x_{j+1,j}}{[-(x_{j-1}(t_1)-x_j(t_2))^2+i\varepsilon]^{(1-\epsilon)}}
$$

$$
= -\left(\mu^2(-x_{j-1,j+1}^2)\right)^\epsilon\left[\frac{1}{\epsilon^2}+\frac{1}{2}\zeta_2+O(\epsilon)\right]
$$

Amplitude-Wilson Loop Duality

Key observation [Drummond, Korchemsky, Sokatchev, Henn 2007, 2008] [Brandhuber, Heslop, Travaglini, 2007]: these objects factorise as

$$W_n = \langle \mathcal{L}(C) \rangle = \left[\prod_{i=1}^{n} D_i\right] F_n R_n$$

- D_i Is a UV divergent factor associated to each cusp

$$D_i = \exp\left\{-\frac{1}{4}\sum_{l=1}^{\infty} g^{2l}(-\mu^2 x_{i-1,i+1}^2)^{l\epsilon}\left[\frac{\Gamma_{\text{cusp}}^{(l)}}{(l\epsilon)^2} + \frac{\Gamma_{\text{sub}}^{(l)}}{l\epsilon}\right]\right\} \qquad \Gamma_{\text{cusp}}^{(1)} = 4 \cdot \frac{2C_F}{N}\,, \qquad \Gamma_{\text{sub}}^{(1)} = 0$$

- F_n a particular choice of finite part chosen such that it fixes the contribution at one-loop (obeys conformal Ward identity)

$$F_n = 1 + g^2 \cdot \frac{2C_F}{N} f_n + O(g^4)\,, \qquad f_n = \sum_{\{i,j\}} I_{ij} - \frac{n}{2}\zeta_2 \qquad K^\mu f_n = 2\sum_{i=1}^{n}(x_{i-1}^\mu - 2x_i^\mu + x_{i+1}^\mu)\log x_{i-1,i+1}^2$$

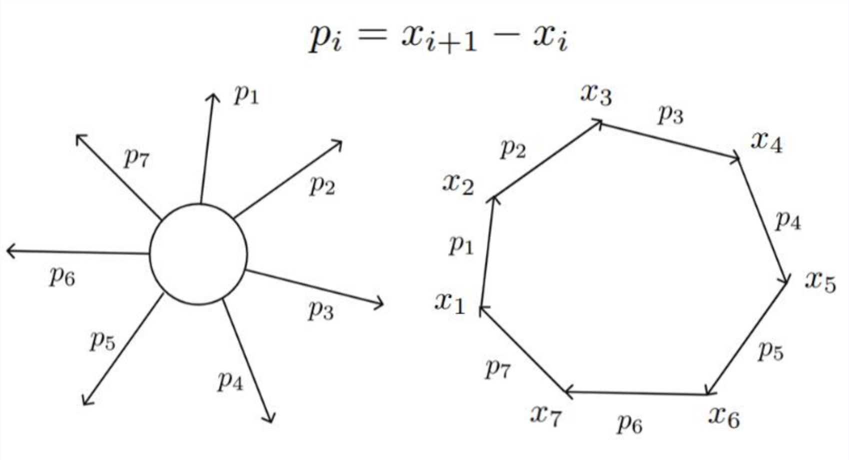
- R_n's log is the **finite and conformally invariant 'remainder function'**

$$F_n = \exp\left\{\frac{1}{4}\Gamma_{\text{cusp}}(g,N) f_n\right\} \qquad R_n = 1 + O(g^4)$$

Matches MHV gluon amplitude! Similar observations at strong coupling came first [Alday, Maldacena, 2007].

Dual conformal symmetry revealed

- Remarkable property of planar N=4 amplitudes: **dual conformal symmetry** [Drummond, Korchemsky, Sokatchev, Henn, 2007] – conformal symmetry in dual coordinates.



- Appears mysterious on the amplitude side of the duality - simply the ordinary conformal symmetry of the Wilson loop!
- Part of the infinite-dimensional **Yangian symmetry** [Drummond, Henn, Plefka, 2009], [Drummond, Ferro, Ragoucy, 2010], which underpins integrability.

2. Amplitudes beyond MHV

The on-shell superspace formalism

Recall that in N=4 Super Yang-Mills we can use the **on-shell superspace formalism** to write an **on-shell superstate**

$$\begin{aligned}\Phi(p,\eta) &= G^+(p) + \eta^A\Gamma_A(p) + \frac{1}{2}\eta^A\eta^B S_{AB}(p) + \frac{1}{3!}\eta^A\eta^B\eta^C\epsilon_{ABCD}\bar{\Gamma}^D(p) \\ &\quad + \frac{1}{4!}\eta^A\eta^B\eta^C\eta^D\epsilon_{ABCD}G^-(p)\,,\end{aligned}$$

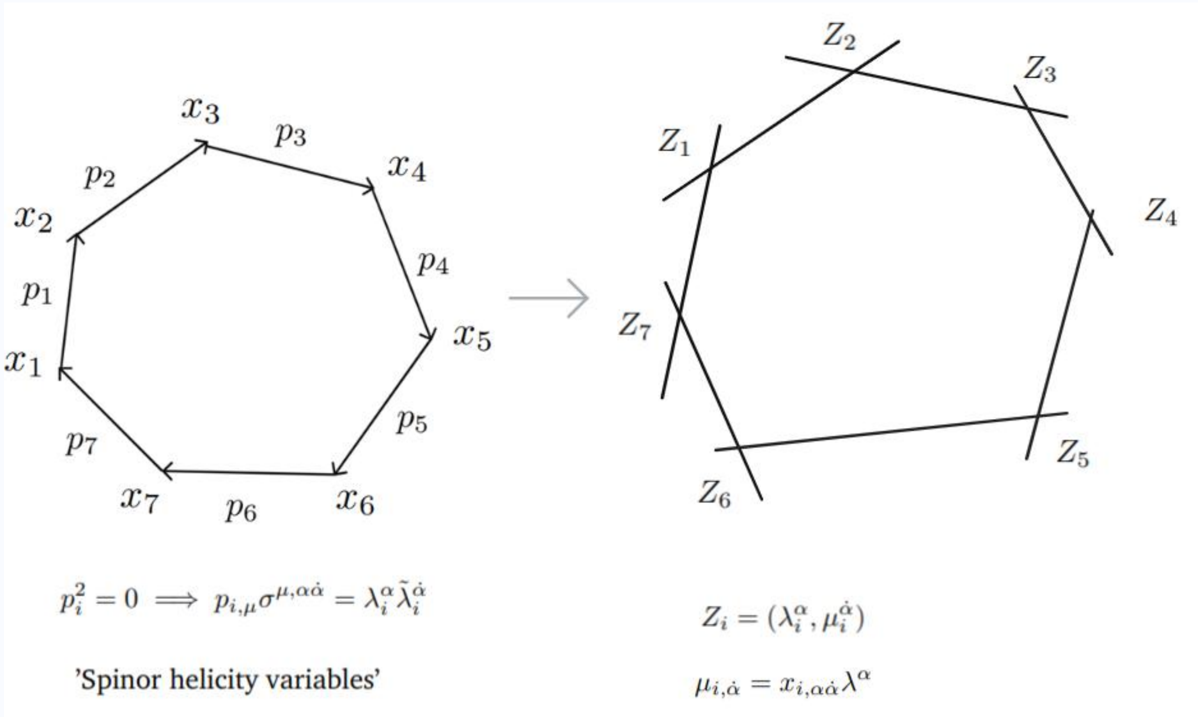
We can then consider a **super-amplitude**

$$\mathcal{A}\left(\Phi_1 \ldots \Phi_n\right)$$

Polynomial in (Grassmann) η variables; the coefficient of each η monomial is the amplitude associated to a specific particle/helicity configuration.

Switching to twistor variables

Convenient to reparametrise kinematics using **momentum supertwistors**, replacing momenta p_i and Grassmann η_i with momentum twistors Z_i and Grassmann variables χ_i.



Supertwistors

Analogously: replace Grassmann variables η_i with dual variables θ_i and then twistor variables χ_i. Now the kinematics are encoded in **momentum super-twistors**.

$$\mathcal{Z}_i^A = (Z_i^A, \chi_i^A)$$

A convenient way to parametrize the kinematic data!

The super-amplitude (now polynomial in χ) separates into 'MHV sectors' each with four more powers of the Grassmann variables χ than the last. Convenient to factor off the MHV tree.

$$A_n = A_n^{tree,MHV}(1 + A_n^{NMHV} + A_n^{N^2MHV} + \cdots)$$

Example: five particle NMHV amplitude

Example: the five-particle NMHV superamplitude admits an expression as a single 'R-invariant'

$$[1,2,3,4,5] = \frac{(\chi_1 < 2345 > \ + cyclic)^4}{< 1234 >< 2345 >< 3451 >< 4512 >< 5123 >}$$

$$< abcd > = Det\left[Z_i Z_j Z_k Z_l\right]$$

Extending the duality

Question: can amplitude-Wilson loop duality be extended beyond the MHV sector?

Yes! Two main (and equivalent!) approaches:

1) **Mason and Skinner's 'holomorphic Wilson loop' [2010]**
2) Caron-Huot's spacetime super Wilson loop [2010]

3. Review of the twistor Wilson loop

N=4 in twistor space

In short: supersymmetrise the gauge field to obtain the gauge superfield

$$\mathcal{A} = a + \chi^a \tilde{\psi}_a + \frac{1}{2}\chi^a \chi^b \phi_{ab} + \epsilon_{abcd}\chi^a \chi^b \chi^c \left(\frac{1}{3!}\psi^d + \frac{1}{4!}\chi^d g\right)$$

Holomorphic Chern-Simons theory corresponds to the self-dual theory which we expand around

$$S(\mathcal{A}) = S_1(\mathcal{A}) + S_2(\mathcal{A})$$

$$S_1(\mathcal{A}) = \frac{iN}{8\pi^3}\int D^{3|4}\mathcal{Z} \wedge \mathrm{tr}\left(\mathcal{A} \wedge \bar{\partial}\mathcal{A} + \frac{2}{3}\mathcal{A} \wedge \mathcal{A} \wedge \mathcal{A}\right)$$

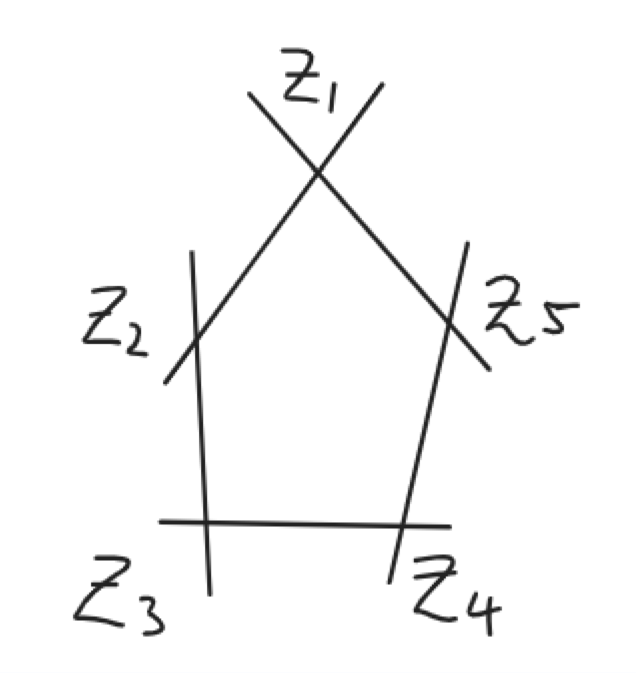
$$S_2(\mathcal{A}) = \frac{g^2 N}{\pi^2}\int d^{4|8}X \log\det\left(\bar{\partial} + \mathcal{A}\right)_X$$

The twistor Wilson loop

Now consider a closed polygon in twistor space! The contour can be described by a sequence of intersecting lines X_i via

$$Z_i(s) = sZ_{i-1} + Z_i$$

$$Z_i(0) = Z_i \text{ and } Z_i(\infty) = Z_{i-1}$$



Supersymmetric loop operators

Define the supersymmetric loop operator as in spacetime, but replacing gauge field with super-gauge field

$$\mathcal{L}(C) = \frac{1}{N} \operatorname{tr} \mathcal{P} \prod_{i=1}^{n} \sum_{l_i=0}^{\infty} \left(-\bar{\partial}_i^{-1} \mathcal{A}(Z_i(0))\right)^{l_i}$$

$$(\bar{\partial}_i^{-1}\omega)(s) = \int_{X_i} G(s,s') \wedge \omega(s') \qquad G(s,s') = \frac{1}{2\pi i} \frac{ds'}{(s-s')}$$

$$D^2 c = c_1 dc_2 \wedge dc_3 + \text{ cyc.}$$

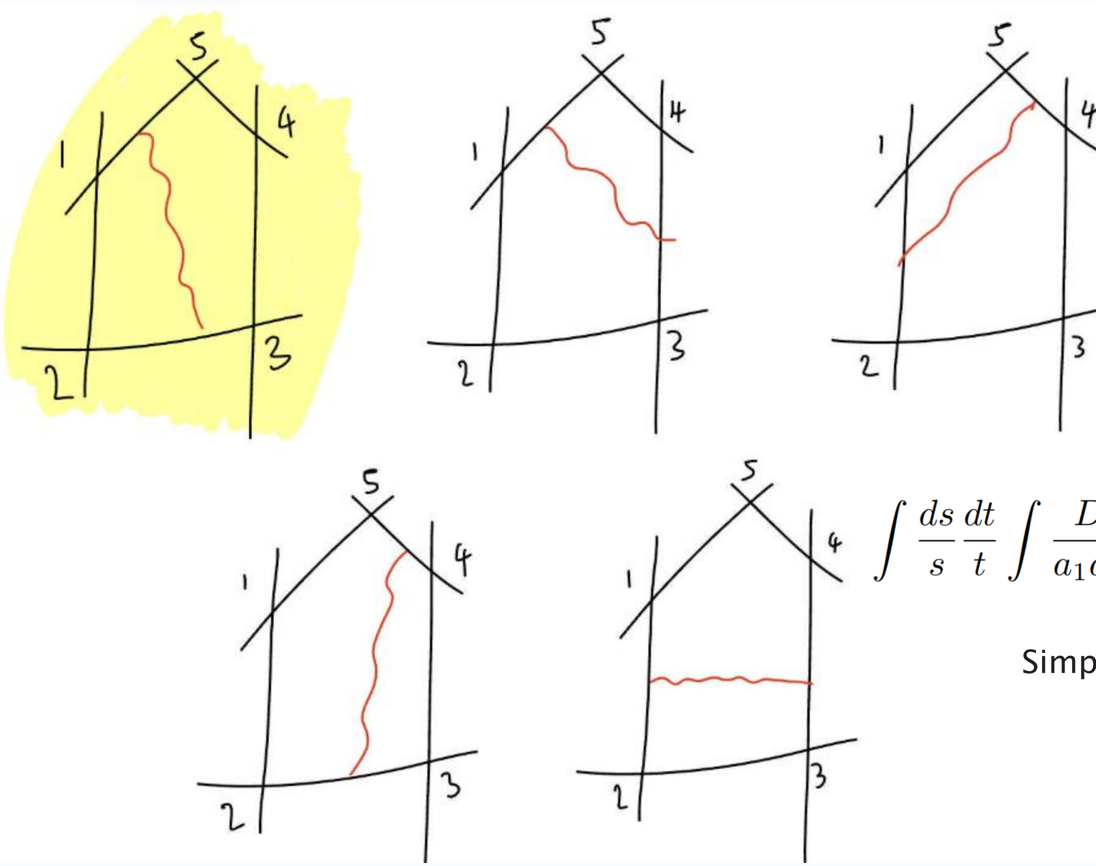
Expand and take the expectation value!

$$\langle \mathcal{A}_a(Z_i(s)) \mathcal{A}_b(Z_j(t)) \rangle^{\text{CS}} = -\frac{8\pi^2}{N} \delta_{ab} \Delta_*(Z_i(s), Z_j(t))$$

$$= -\frac{8\pi^2}{N} \delta_{ab} \int \frac{D^2 c}{c_1 c_2 c_3} \bar{\delta}^{4|4}(c_1 Z_* + c_2 Z_i(s) + c_3 Z_j(t))$$

Example: pentagonal NHMV contribution (tree-level, planar)

Number of propagators matches MHV degree!



Reference twistor; consequence of choosing a gauge

The line (15)

The line (23)

$$\int \frac{ds}{s}\frac{dt}{t}\int \frac{D^2 a}{a_1 a_2 a_3}\bar{\delta}^{4|4}(a_1\mathcal{Z}_* + sa_2\mathcal{Z}_1 + a_2\mathcal{Z}_5 + ta_3\mathcal{Z}_2 + a_3\mathcal{Z}_3$$

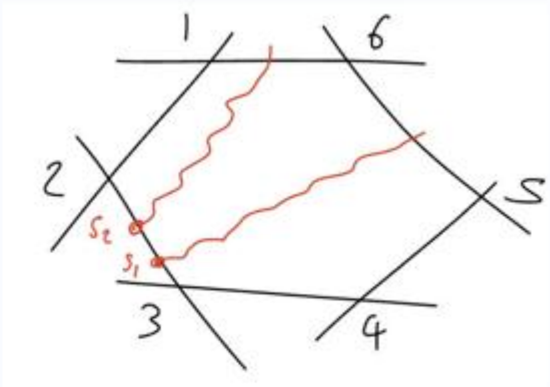
Simply integrate out the delta functions!

$$[*, 5, 1, 2, 3]$$

Sums to give the five-particle NMHV amplitude!

Example: hexagonal NNHMV contribution (tree-level, planar)

Measure slightly modified when two propagators end on the same twistor line



$$\int \frac{ds_1}{s_1}\frac{ds_2}{s_1 - s_2}\frac{du}{u}\frac{dv}{v}\int\frac{D^2a}{a_1a_2a_3}\int\frac{D^2b}{b_1b_2b_3}\bar{\delta}^{4|4}(a_1\mathcal{Z}_* + ua_2\mathcal{Z}_6 + a_2\mathcal{Z}_1 + s_2a_3\mathcal{Z}_2 + a_3\mathcal{Z}_3)$$
$$\times\, \bar{\delta}^{4|4}(b_1\mathcal{Z}_* + vb_2\mathcal{Z}_5 + b_2\mathcal{Z}_6 + s_1b_3\mathcal{Z}_2 + b_3\mathcal{Z}_3)$$

Due to double insertion on the line (23)

More complicated due to shifted denominator!

$$[*,5,6,2,3][*,1,6,2,(23)\cap(*56)]$$

$$(23)\cap(*56) = \langle 356*\rangle Z_2 - \langle 256*\rangle Z_3$$

Sum again reproduces the tree-level amplitude.

Beyond tree level

So far we have neglected the interaction term to reproduce **tree-level amplitudes**. We can calculate loop-level contributions using the method of **Lagrangian insertions.**

For instance, at one loop…

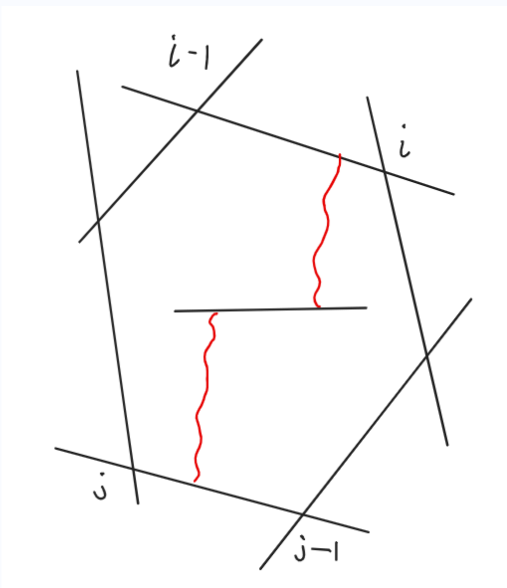
$$\int d^{4|8}X \langle \mathcal{W}(\mathcal{Z}_1, \mathcal{Z}_2, \ldots, \mathcal{Z}_n)\mathcal{L}(X)\rangle$$

The 'log-det' interaction term expands to give an infinite tower of interaction vertices

$$g^2 \sum_{n=2}^{\infty} d^{4|8}X \frac{1}{n}(-1)^n \int_{X^n} \frac{du_n du_{n-1} du_{n-2} \ldots du_2 du_1}{(u_1-u_2)(u_2-u_3)(u_3-u_4)\ldots(u_{n-1}-u_n)(u_n-u_1)} \mathrm{Tr}\mathcal{A}(u_n)\mathcal{A}(u_{n-1})\ldots\mathcal{A}(u_1).$$

Example: a one-loop MHV diagram

The integral associated to the Lagrangian removes 8 degrees in the Grassmann variables, so we need two more propagators than at tree level. At MHV, one-loop: ‘Kermit diagrams’



$$\int d^{4|8}x_{AB} \int \frac{du_1}{u_1 - u_2} \frac{du_2}{u_2 - u_1} \frac{ds}{s} \frac{dt}{t}$$
$$\times \bar{\delta}^{4|4}(a_1 \mathcal{Z}_* + a_2 u_1 \mathcal{Z}_A + a_2 \mathcal{Z}_B + a_3 s \mathcal{Z}_{i-1} + a_3 \mathcal{Z}_i)$$
$$\times \bar{\delta}^{4|4}(b_1 \mathcal{Z}_* + b_2 u_2 \mathcal{Z}_A + b_2 \mathcal{Z}_B + b_3 t \mathcal{Z}_{j-1} + b_3 \mathcal{Z}_j)$$

$$\int d^{4|8}x_{AB}[*, i-1, i, A, B'][*, j-1, j, A, B'']$$

Loop integral: easiest to rewrite in basis of known, local, ‘chiral pentagons’ [Arkani-Hamed, Bourjaily, Cachazo, Trnka, 2010]

Matches bosonic calculation →

The bottom line?

Summing over all diagrams is easy and reproduces the result on the amplitude side; tree level amplitude, or loop integrand!

We can consider higher loop order by simply including more Lagrangian insertions.

In this way the twistor Wilson loop extends amplitude-Wilson loop duality to **all MHV degrees and all loop orders**!

4. Correlators of multiple light-like Wilson loops

University of Southampton

Correlators of multiple Wilson loops

[Drummond, Gürdogan, Rochford, RW, 2025]

$$W_{n_1,\ldots,n_m} = \langle \mathcal{L}(C_1)\ldots\mathcal{L}(C_m)\rangle = \prod_{r=1}^{m}\left[\left[\prod_{i=1_r}^{n_r} D_{i_r}\right] F_{n_r}\right] R_{n_1,\ldots,n_m}$$

(ordinary or super!)

Leading N contribution is in fact the disconnected part – we focus on the connected part e.g. for three Wilson loops

$$R_{n_1,n_2,n_3} = R_{n_1}R_{n_2}R_{n_3} + R_{n_1}R^{\text{conn}}_{n_2,n_3} + R_{n_2}R^{\text{conn}}_{n_1,n_3} + R_{n_3}R^{\text{conn}}_{n_1,n_2} + R^{\text{conn}}_{n_1,n_2,n_3}$$

Example: one-loop (MHV) contribution

Consider the one-loop contribution for two MHV Wilson loops.

No contribution for SU(N); focus on the Abelian theory. Only diagrams are those where one propagator crosses from one Wilson to the other:

$$I_{ij} = \quad 2\int_0^1 dt_1 \int_0^1 dt_2 \frac{x_{i+1,i}\cdot x_{j+1,j}}{[-(x_i(t_1)-x_j(t_2))^2+i\varepsilon]}$$

$$\begin{aligned}
&\mathrm{Li}_2\left[\frac{(x_{ij}^2-x_{i,j+1}^2)(x_{i,j+1}^2-x_{i+1,j+1}^2)}{x_{i,j+1}^2x_{i+1,j}^2-x_{ij}^2x_{i+1,j+1}^2}\right]+\mathrm{Li}_2\left[\frac{(x_{ij}^2-x_{i+1,j}^2)(x_{i+1,j}^2-x_{i+1,j+1}^2)}{x_{i,j+1}^2x_{i+1,j}^2-x_{ij}^2x_{i+1,j+1}^2}\right]\\
-&\mathrm{Li}_2\left[\frac{(x_{ij}^2-x_{i,j+1}^2)(x_{ij}^2-x_{i+1,j}^2)}{x_{i,j+1}^2x_{i+1,j}^2-x_{ij}^2x_{i+1,j+1}^2}\right]-\mathrm{Li}_2\left[\frac{(x_{i,j+1}^2-x_{i+1,j+1}^2)(x_{i+1,j}^2-x_{i+1,j+1}^2)}{x_{i,j+1}^2x_{i+1,j}^2-x_{ij}^2x_{i+1,j+1}^2}\right]
\end{aligned}$$

Example: one-loop (MHV) contribution

Take the sum, manipulate into a manifestly conformally invariant form:

$$f_{n_1,n_2} = \sum_{i,j}^{n_1,n_2} \mathrm{Li}_2(1 - v_{ij}) + \sum_{\substack{k \leq i \\ j \leq l}} \log v_{ij} \log v_{kl} \, .$$

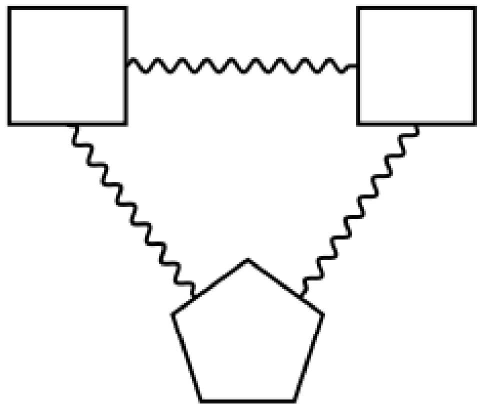
$$u_{i,j,k,l} = \frac{x_{ij}^2 x_{kl}^2}{x_{il}^2 x_{kj}^2} \, , \qquad v_{ij} = u_{i,j,i+1,j+1}$$

N.B. Using the twistor formulation, sum over Kermit diagrams crossing the Wilson loops gives the same answer!

Leading order contribution in the SU(N) theory

$$f_{n_1,n_2} = \sum_{i,j}^{n_1,n_2} \mathrm{Li}_2(1 - v_{ij}) + \sum_{\substack{k\leq i\\ j\leq l}} \log v_{ij} \log v_{kl} \,.$$

This also lets us write down the leading contribution in the SU(N) theory, for which we have 'chain'-type contributions

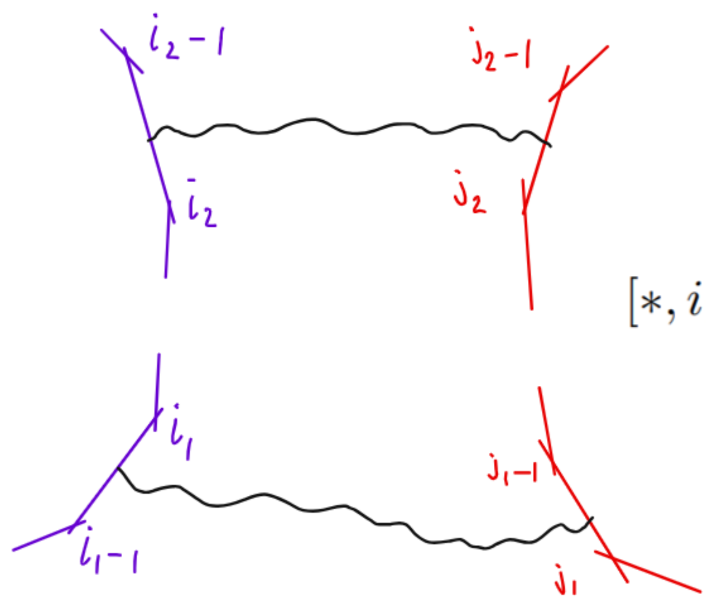


$$W^{\text{conn}}_{n_1,\ldots,n_m} = g^{2m} \frac{2NC_F}{N^{2m}} \prod_{r=1}^{m} f_{n_r,n_{r+1}} + (\text{non-dihedral permutations}) + O(g^{2m+2})$$

Example: tree-level (NNMHV) contribution

Now consider tree level, NNMHV contribution using the twistor Wilson loop (NMHV vanishes for trace reasons!).

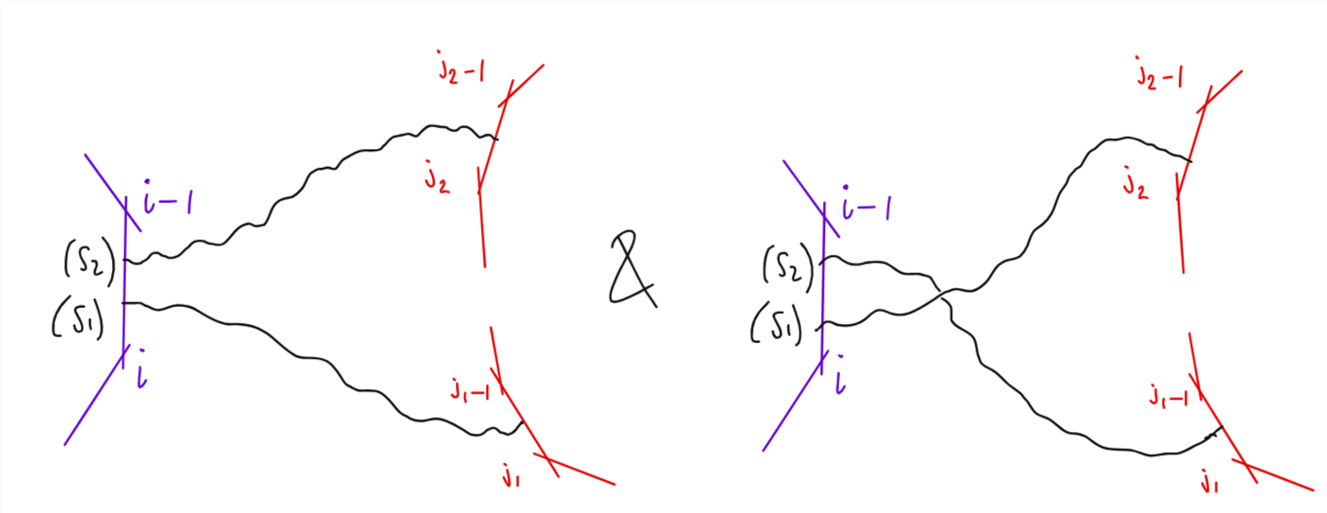
Two types of diagram; those without a double insertion on a twistor line…



$$[*, i_1-1, i_1, j_1-1, j_1][*, i_2-1, i_2, j_2-1, j_2]$$

Example: tree-level (NNMHV) contribution

… and those with a double insertion, which come in two variants (both planar!)



$$\frac{ds_1}{s_1}\frac{ds_2}{s_1 - s_2}$$

$$[*, i-1, i, j_1-1, j_1][*, j_2-1, j_2, i-1, (i-1,i)\cap(j_1-1, j_1 \;+\; [*, i-1, i, j_2-1, j_2][*, j_1-1, j_1, i-1, (i-1,i)\cap(j_2-1, j_2, *)]$$

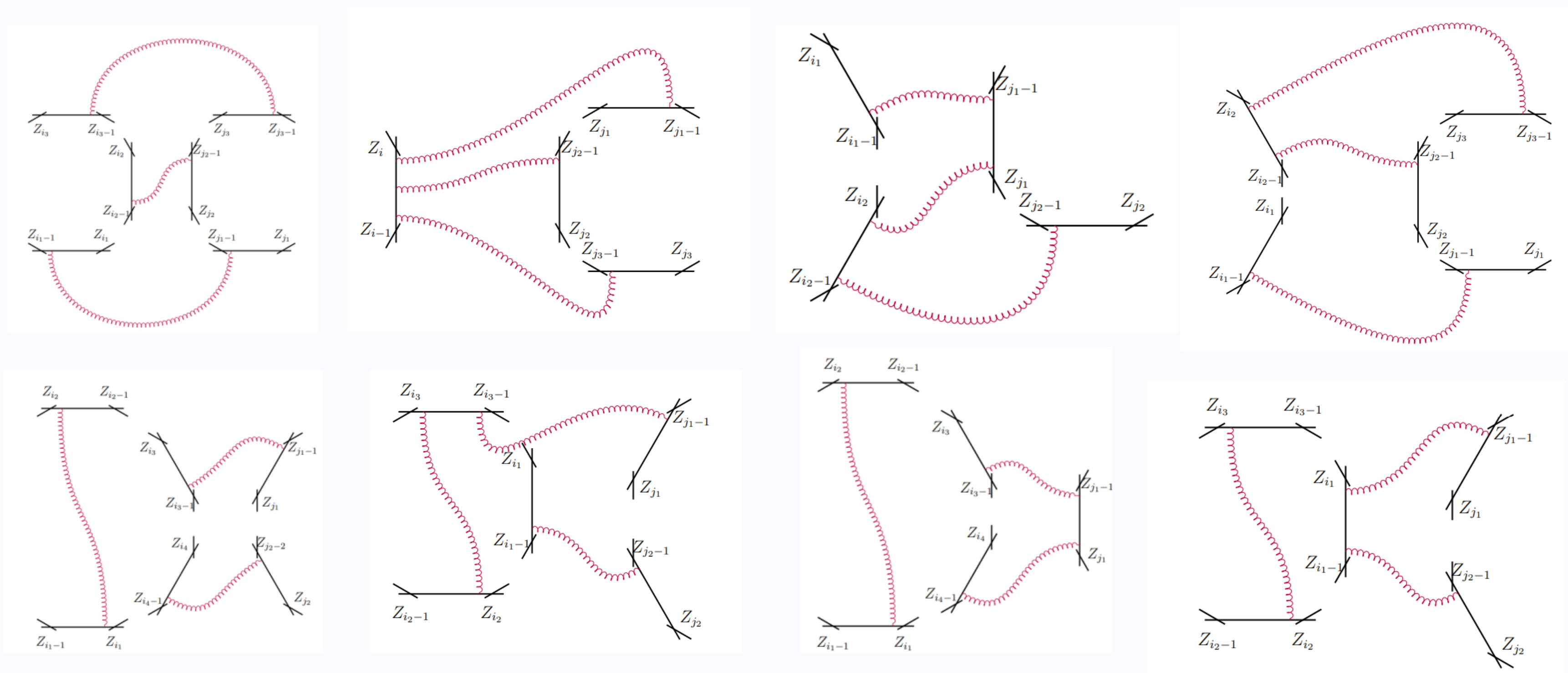
$$= [*, i-1, i, j_1-1, j_1][*, i-1, i, j_2-1, j_2]$$

Cancellation of shifts means the NNMHV answer is (half) the square of the (Abelian) NMHV answer. **Simpler than for a single Wilson loop!**

University of Southampton

Example: tree-level (NNNMHV) contribution

Higher MHV degree at tree-level simply means adding more propagators; easy enough to automate!



Likewise no obstruction in generating loop-level integrands diagrammatically.

4. One-loop integrated correlators

Loop integration for Wilson loops

Easy to generate loop **integrands** for general multiplicity and MHV degree, integrating diagrammatic expression is challenging.

Can use the **chiral box expansion** [Bourjaily, Caron-Huot, Trnka, 2013] as a basis of one-loop integrals, all of which are known! The coefficients of each integral will be the residue of the integrand on a particular maximal cut (on one of two singular configurations: ‘Schubert solutions’).

Thus **knowing all leading singularities is enough to write down the integrated answer at one-loop!**

Computing leading singularities

Computing the residues manually from diagrams is possible, but slow and ugly!

Many ways to calculate the leading singularities directly, e.g.

- On-shell diagram formalism
- Generalised unitarity

… very much native to the **amplitude** side of the duality.

End result: leading singularities factorise into a product of tree-level amplitudes. Can we see this from the perspective of Wilson loops?

Computing leading singularities

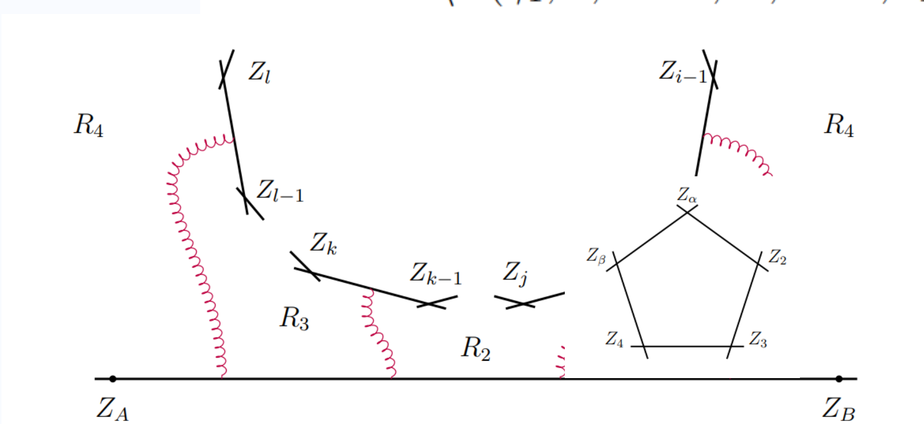
Short answer: yes, but no time for the details!

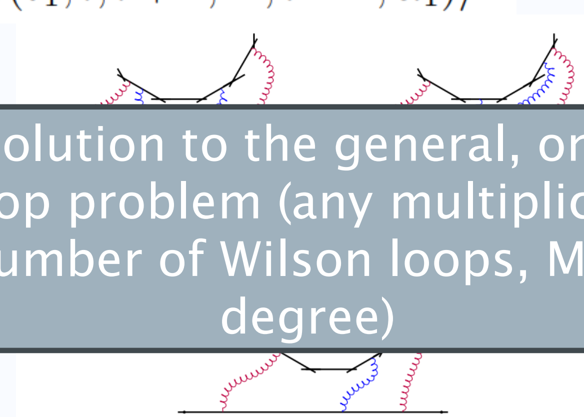
[Drummond, Rochford, RW, to appear]

In forthcoming work, we present formulae for the residues on all maximal cuts in terms of simple, tree-level objects.

$$\langle A\,B\,i-1\,i\rangle = \langle A\,B\,j-1\,j\rangle = \langle A\,B\,k-1\,k\rangle = \langle A\,B\,l-1\,l\rangle = 0$$

$$\begin{aligned} C_1(i,j,k,l)\langle\mathcal{L}(\alpha_1,i,i+1,...,j-1,\beta_1)\rangle \times \langle\mathcal{L}(\beta_1,j,j+1,...,k-1,\gamma_1)\rangle \\ \times\langle\mathcal{L}(\gamma_1,k,k+1,...,l-1,\delta_1)\rangle \times \langle\mathcal{L}(\delta_1,l,l+1,...,i-1,\alpha_1)\rangle \end{aligned}$$





Solution to the general, one-loop problem (any multiplicity, number of Wilson loops, MHV degree)

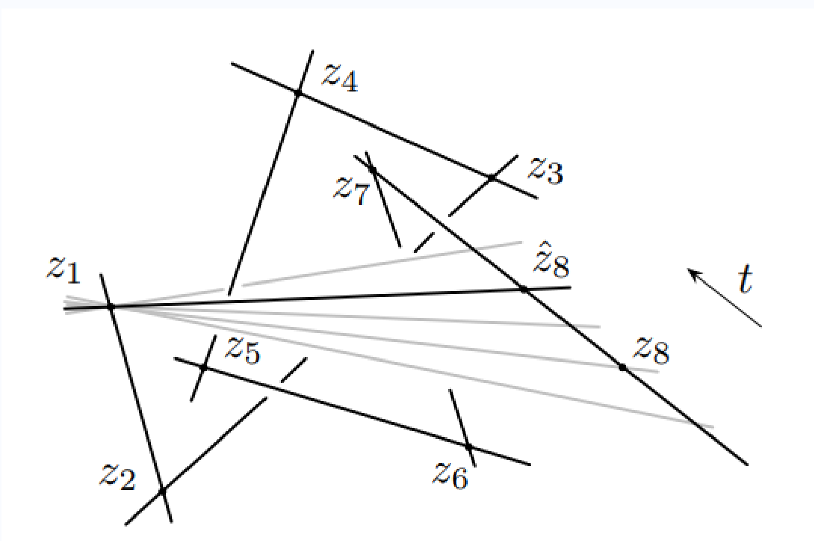
5. BCFW and Qbar equations

BCFW recursion relation

Tree-level BCFW recursion for a single WL [Bullimore, Skinner, 2011], derived via ‘holomorphic linking’; consider deforming the nodal curve defining the Wilson loop!

$$\langle \mathcal{L}[1,\ldots,n]\rangle = \langle \mathcal{L}[1,\ldots,n-1]\rangle + \sum_{j=3}^{n-2} [n-1,n,1,j-1,j]\langle \mathcal{L}[1,\ldots,j-1,I_j]\rangle\langle \mathcal{L}[I_j,j,\ldots,\hat{n}_j]\rangle$$

$$\mathcal{Z}_{I_j} = \mathcal{Z}_{j-1}\langle j\, n-1\, n\, 1\rangle - \mathcal{Z}_j\langle j-1\, n-1\, n\, 1\rangle$$
$$\hat{\mathcal{Z}}_n = \mathcal{Z}_{n-1}\langle n\, j-1\, j\, 1\rangle - \mathcal{Z}_n\langle n-1\, j-1\, j\, 1\rangle$$



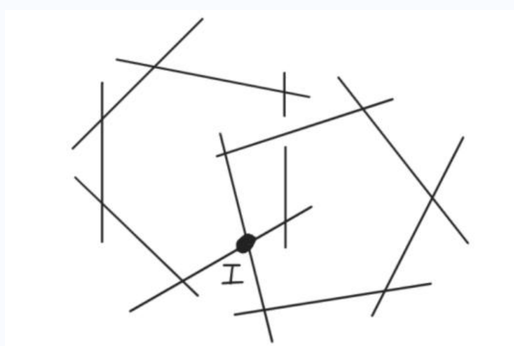
A similar equation exists for loop integrands; this provides a Wilson loop-derivation of the all-loop BCFW recursion [Arkani-Hamed et. al, 2010]

Recursively relates Wilson loops/amplitudes to simpler Wilson loops/amplitudes

BCFW recursion relation

Analogous result for multiple Wilson loops follows from a similar procedure [Drummond, Rochford, RW, to appear]:

$$
\begin{aligned}
&\langle \mathcal{L}[1,\ldots,n_1]\mathcal{L}[\tilde{1},\ldots,\tilde{n}_2]\rangle^{\mathrm{conn}} = \langle \mathcal{L}[1,\ldots,n_1-1]\mathcal{L}[\tilde{1},\ldots,\tilde{n}_2]\rangle^{\mathrm{conn}} \\
&+\sum_{j=3}^{n_1-2}[n_1-1,n_1,1,j-1,j]\Big[\langle \mathcal{L}[1,\ldots,j-1,I_j]\mathcal{L}[\tilde{1},\ldots,\tilde{n}_2]\rangle^{\mathrm{conn}}\langle \mathcal{L}[I_j,j,\ldots,n_1-1,\hat{n}_{1,j}]\rangle \\
&\qquad\qquad\qquad\qquad +\langle \mathcal{L}[1,\ldots,j-1,I_j]\rangle\langle \mathcal{L}[I_j,j,\ldots,n_1-1,\hat{n}_{1,j}]\mathcal{L}[\tilde{1},\ldots,\tilde{n}_2]\rangle^{\mathrm{conn}}\Big] \\
&+\sum_{\tilde{j}=\tilde{1}}^{\tilde{n}_2}[n_1-1,n_1,1,\tilde{j}-1,\tilde{j}]\Big[\langle \mathcal{L}[1,\ldots,\hat{n}_{1,\tilde{j}},I_{\tilde{j}},\tilde{j},\ldots,\tilde{j}-1,I_{\tilde{j}}]\rangle - c\langle \mathcal{L}[1,\ldots,\hat{n}_{1,\tilde{j}}]\rangle\langle \mathcal{L}[\tilde{1},\ldots,\tilde{n}_2]\rangle\Big]
\end{aligned}
$$



New, interesting term – a self-intersecting Wilson loop!

BCFW-like recursion for objects which are **not** directly dual to amplitudes!

Q-bar equation

Q-bar equation for a single Wilson loop [Caron-Huot, He, 2012]; relates action of the dual superconformal generators

$$\overline{Q}_A^{A'} = \sum_i \chi_i^{A'} \frac{\partial}{\partial Z_i^A}$$

on one Wilson loop('s remainder function) to the integral of a Wilson loop with one extra leg, subjected to a collinear limit.

Single WL:

$$\bar{Q}_a^A R_{n,k} = a \ \mathrm{Res}_{\epsilon=0} \int_{\tau=0}^{\tau=\infty} \left(d^{2|3} \mathcal{Z}_{n+1}\right)_a^A \left[R_{n+1,k+1} - R_{n,k} R_{n+1,1}^{\mathrm{tree}}\right] + \mathrm{cyclic}$$

$$\mathcal{Z}_{n+1} = \mathcal{Z}_n - \epsilon \mathcal{Z}_{n-1} + C\epsilon\tau \mathcal{Z}_1 + C'\epsilon^2 \mathcal{Z}_2$$

$$a := \frac{1}{4}\Gamma_{\mathrm{cusp}} = g^2 - \frac{\pi^2}{3} g^4 + \frac{11\pi^4}{45} g^6 +$$

$$\int d^{2|3} \mathcal{Z}_{n_r+1} = C(n_r - 1 n 1)_a \mathrm{res}_{\epsilon=0} \int \epsilon d\epsilon \int_0^\infty d\tau (d^3 \chi_{n_r+1})^A$$

Allows e.g. one-loop NNMHV result to be related to two-loop NMHV and then three-loop MHV [Li, Zhang, 2021]!

Natural generalisation to multiple Wilson loops (checked in sophisticated cases) [Drummond, Rochford, RW, 2026]

$$\overline{Q}_A^{A'} \mathcal{R}^{(k)}_{n_1,\ldots,n_m} = \sum_r \left[\frac{1}{4}\Gamma_{\text{cusp}} \int \left[d^{2|3}\mathcal{Z}_{n_r+1}\right]_A^{A'} \left[\mathcal{R}^{(k+1)}_{n_1,\ldots,n_r+1,\ldots,n_m} - \mathcal{R}^{(k)}_{n_1,\ldots,n_m} \mathcal{R}^{(1),\text{tree}}_{n_r+1} \right] + \text{cyc}_r \right]$$

$$\int d^{2|3}\mathcal{Z}_{n_r+1} = C(n_r - 1 n1)_a \text{res}_{\epsilon=0} \int \epsilon d\epsilon \int_0^\infty d\tau (d^3\chi_{n_r+1})^A$$

Using this to obtain integrated NMHV two-loop and MHV three-loop answers: work in progress!

6. Wild speculations

Speculations…

- Is there a positive geometry underlying these objects (higher genus amplituhedron…multiwilsohedron?)

BCFW-like recursion may be a hint!

One possible route: correlation functions of gauge invariant local operators related to Wilson loop expectation values under sequential light-like limits [Alday, Eden, Korchemsky, Maldacena, Sokatchev, 2011]; generalises to multiple Wilson loops [Drummond, Heslop, Rochford, RW, to appear?].

e.g.

$$\langle \mathcal{O}(x_1)\mathcal{O}(x_2)...\mathcal{O}(x_8)\rangle|_{x_{12}^2,x_{23}^2,x_{34}^2,x_{41}^2,x_{56}^2,x_{67}^2,x_{78}^2,x_{85}^2\to 0} \approx \langle \mathcal{L}(1,2,3,4)\mathcal{L}(5,6,7,8)\rangle$$

Apply to the correlahedron? [Eden, Heslop, Mason, 2017]

Speculations…

- Is there a generalisation of the Grassmannian integral for a single Wilson loop? [Mason, Skinner, 2009] [Arkani-Hamed, Cachazo, Cheung, Kaplan 2009]. Itself fixed by Yangian symmetry! [Drummond, Ferro, 2010]

Very natural candidate… just replace the cycles in the denominator of the integration measure!

e.g. for two squares instead of an octagon at NNMHV…

$$\oint \frac{d^{2\times 8}t}{(12)(23)(34)(45)(56)(67)(78)(81)\mathrm{GL}(2)} \prod_{j=1}^{2} \bar{\delta}^{4|4}\Big(\sum_{i=1}^{8} t_{ji}\mathcal{Z}_i\Big)$$

$$\oint \frac{d^{2\times 8}t}{(12)(23)(34)(41)(56)(67)(78)(85)\mathrm{GL}(2)} \prod_{j=1}^{2} \bar{\delta}^{4|4}\Big(\sum_{i=1}^{8} t_{ji}\mathcal{Z}_i\Big)$$

Needs to be studied!

Speculations…

- Do these correlators contain the Wilson loop with Lagrangian operator insertion as a limit?
- In principle should be related to the limit where one Wilson loop shrinks to become very small.
- What happens to our generalised Q-bar equation in this limit? Note stringent ‘final entry conditions’ on symbol of pentagonal Wilson loop at two [Chicherin, Henn, 2022] and now three [Chicherin, Henn, Xu, Zhang, Zhang, 2026] loops! Potentially very powerful for bootstrap.

Speculations…

- Is there a link to cluster algebras? Connection to possible branch cut singularities (and which can be appear consecutively) in case of single Wilson loop!
- Even outside planar N=4, links to cluster algebras recently suggested for general five-point and six-point massless scattering [Bossinger, Drummond, Glew, Gürdoğan, RW, 2025] and even cosmological correlators [Capuano, Ferro, Lukowski, Palazio, 2025] [Mazloumi, Xu, 2025]. Why not for these objects too?!
- Still kinematically e.g. Gr(4,8) for correlator of two squares, but usual notion of 'positive region' disrupted by cyclic symmetry… another relevant region of the Grassmannian?
- Two loop data will help!

7. Summary

Summary

- Multiple Wilson loop correlators are a new and interesting class of observable.
- Although they are quite complicated kinematically even in the simplest cases, they enjoy a rich mathematical structure similar to single Wilson loops and thus complicated calculations are tractable; general one-loop problem solved!
- Many interesting properties of single Wilson loops, e.g. BCFW recursion and Q-bar equations, admit natural generalisations.
- Might be of broader interest through e.g. relation to spectrum of two-point functions etc. – needs investigation!
- Many open questions to study about these objects!

Thank you!