Optimal Inapproximability of Promise Equations over Finite Groups

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Systems of equations
3LIN(p)

$$
\begin{array}{ccccccc}
x_{11} & - & x_{12} & + & x_{13} & = & 7 \mod p \\
x_{21} & + & x_{22} & + & x_{23} & = & 2 \mod p \\
x_{31} & + & x_{32} & - & x_{33} & = & 0 \mod p \\
\vdots & & & & \vdots & & \vdots \\
x_{m1} & - & x_{m2} & + & x_{m3} & = & 5 \mod p
\end{array}
$$

Systems of equations
3LIN(G)

$$\begin{array}{ccccccccc}
x_{11} & \cdot & x_{12}^{-1} & \cdot & x_{13} & = & g_1 \\
x_{21} & \cdot & x_{22} & \cdot & x_{23} & = & g_2 \\
x_{31} & \cdot & x_{32} & \cdot & x_{33}^{-1} & = & g_3 \\
\vdots & & & & \vdots & & \vdots \\
x_{m1} & \cdot & x_{m2}^{-1} & \cdot & x_{m3} & = & g_m
\end{array}$$

Systems of equations
Decision

Theorem (Goldman, Russell 2002)

Finding a solution to a satisfiable system of equations over a finite group G *is*

- *solvable in PTIME if* G *is Abelian;*
- *NP-hard otherwise.*

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An algorithm gives an α-approximation for a maximization problem P if it outputs a solution of value at least $\alpha \cdot \mathrm{OPT}(\mathcal{I})$ on every instance $\mathcal{I}$ of P.

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For 3-LIN(G), the *random assignment* over G gives a 1/|G|-approximation.
Can we do any better?

Systems of equations
Approximation

Theorem (Håstad 2001)

Let G *be a finite* ***Abelian*** *group. It is NP-hard to approximate* ***almost****-satisfiable instances of* 3LIN(G) *better than the random assignment threshold over* G.

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Theorem (Bhangale, Khot 2021)

Let G *be a finite group. It is NP-hard to approximate satisfiable instances of* $3\mathrm{LIN}(G)$ *better than the random assignment threshold over* $[G, G]$ *(and this is optimal).*

Let $\mathbb{A}, \mathbb{B}$ be fixed finite relational structures.

CSP($\mathbb{A}$)

Given an input structure $\mathbb{X}$, decide if $\mathbb{X} \to \mathbb{A}$.

$\to$ e.g. $\mathrm{CSP}(K_3)$, $\mathrm{CSP}(3\mathrm{SAT})$, $\mathrm{CSP}(3\mathrm{LIN})$ ...

PCSP($\mathbb{A}, \mathbb{B}$)

Given an input structure $\mathbb{X}$ with a promise that $\mathbb{X} \to \mathbb{A}$, find a homomorphism $\mathbb{X} \to \mathbb{B}$.

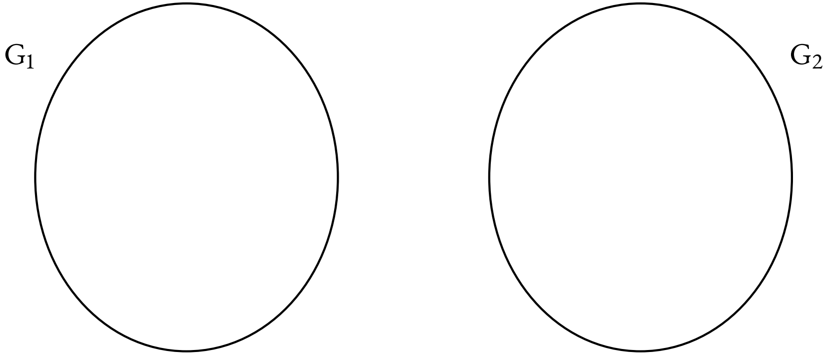
$\to$ e.g. $\mathrm{PCSP}(K_3, K_{100})$, $\mathrm{PCSP}(1\mathrm{in}3, \mathrm{NAE})$, ...

Systems of promise equations

Decision

3-LIN(G_1, G_2, φ)

Let $G_1 \to G_2$. Given a system of linear equations that is promised to be satisfiable in G_1, find a satisfying assignment in G_2.



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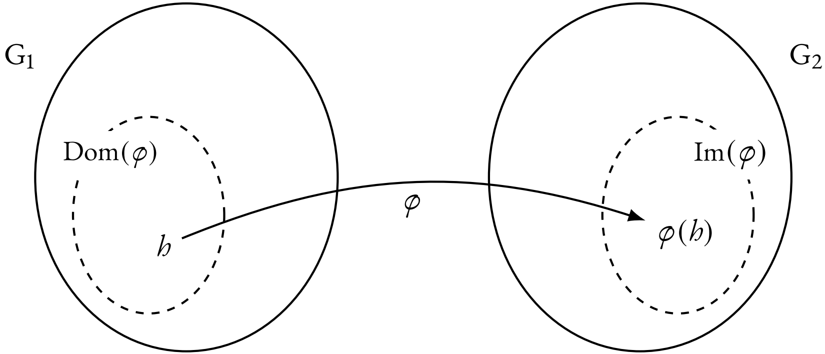

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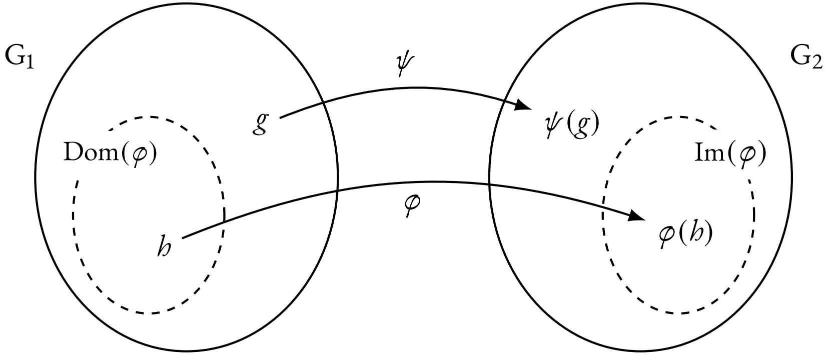

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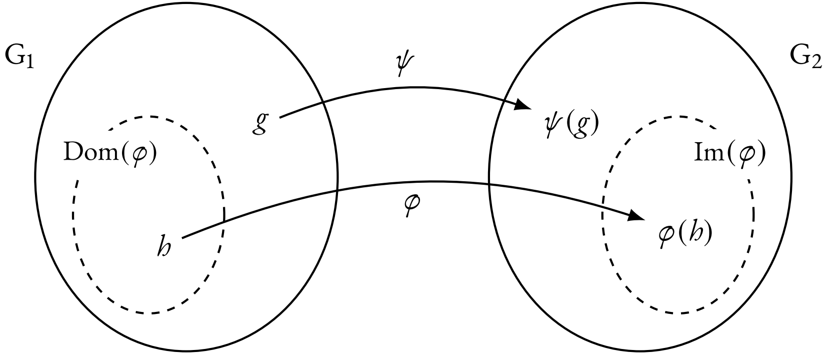
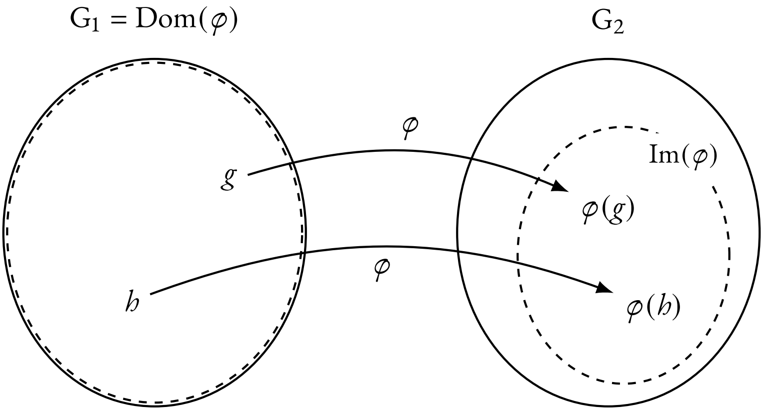


Equations: $x\,y\,z = h$ for $h \in \mathrm{Dom}(\varphi)$ interpreted via φ in G_2.

Systems of promise equations

Decision



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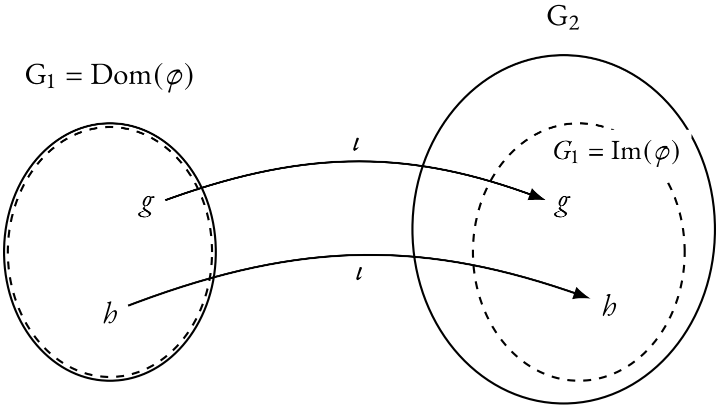

Theorem (Goldman, Russell 2002)

The problem of finding a solution to a satisfiable system of equations over a finite group G *is*

- *solvable in* PTIME *if* G *is Abelian;*
- *NP-hard otherwise.*

Theorem (Larrauri, Živný 2024)

3-LIN(G_1, G_2, φ) *is*

- *solvable in* PTIME *if* φ *has an Abelian extension;*
- *NP-hard otherwise.*

Approximation of promise equations

3-LIN$(G_1, G_2, \varphi, c, s)$

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Let G_1, G_2 be finite groups, φ a homomorphism between subgroups that extends to a full homomorphism, and $s \leq c$. Given a system of equations with constants in $\mathrm{Dom}(\varphi)$ that is c-satisfiable in G_1, find an s-satisfying assignment in G_2.

c

$$
\begin{array}{ccccccccc}
x_{11} & \cdot & x_{12}^{-1} & \cdot & x_{13} & = & h_1 \\
x_{21} & \cdot & x_{22} & \cdot & x_{23} & = & h_2 \\
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x_{m1} & \cdot & x_{m2}^{-1} & \cdot & x_{m3} & = & h_m
\end{array}
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x_{11} & \cdot & x_{12}^{-1} & \cdot & x_{13} & = & \varphi(h_1) \\
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x_{m1} & \cdot & x_{m2}^{-1} & \cdot & x_{m3} & = & \varphi(h_m)
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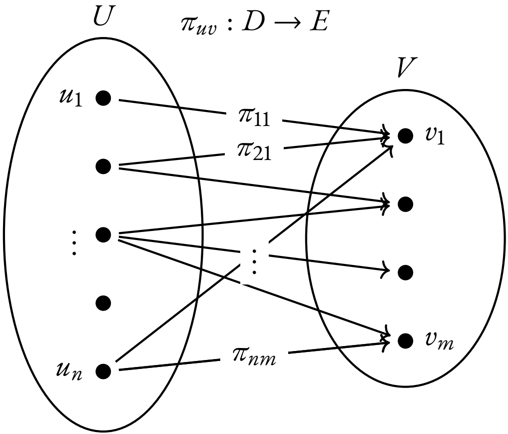
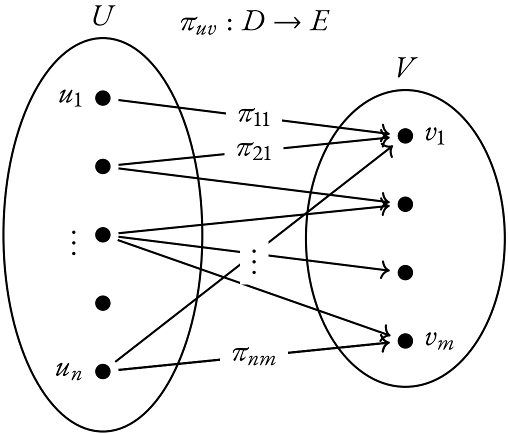
Theorem (B., Larrauri, Živný 2025)

Let G_1, G_2 be finite groups and φ a homomorphism between subgroups that extends to a full homomorphism. Given a system of equations with constants in $\mathrm{Dom}(\varphi)$ that is almost-satisfiable in G_1, it is NP-hard to find an assignment in G_2 that satisfies a $1/|\mathrm{Im}(\varphi)| + \delta$ fraction of the equations.

PCP Theorem

Hardness of Gap Label Cover





$\text{GLC}_{D,E}(c, s)$

Distinguish c-satisfiable from not s-satisfiable label cover instances.

PCP Theorem

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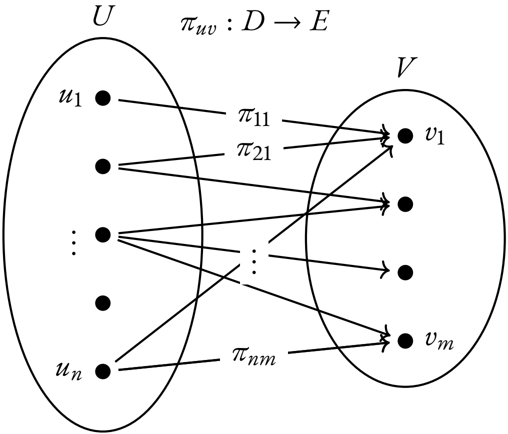

$\mathrm{GLC}_{D,E}(c, s)$

Distinguish c-satisfiable from not s-satisfiable label cover instances.

Theorem (PCP + Parallel repetition)

For every $\alpha > 0$ *there exists finite sets* D, E *such that* $\mathrm{GLC}_{D,E}(1, \alpha)$ *is* NP-*hard.*

The reduction
Dictatorship test

Test if functions $\begin{cases} A : G_1^E \to G_2 \\ B : G_1^D \to G_2 \end{cases}$ are close to dictators.

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(1) Sample an edge $\{u, v\}$ uniformly at random.

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(2) Sample elements $\mathbf{a} \in \mathrm{G}_1^E$, $\mathbf{b} \in \mathrm{G}_1^D$ uniformly at random.

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(3) Sample $\eta \in G_1^D$ so that for each $d \in D$, independently,
$\eta(d) = 1_{G_1}$ with probability $1 - \epsilon$, and
$\eta(d)$ selected uniformly at random from G_1 with probability ϵ.
Set $\mathbf{c} = \mathbf{b}^{-1}(\mathbf{a} \circ \pi_{uv})^{-1}\eta \in G_1^D$.

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Set $\mathbf{c} = \mathbf{b}^{-1}(\mathbf{a} \circ \pi_{uv})^{-1}\eta \in G_1^D$.

(4) Test $A_\varphi(\mathbf{a})B(\mathbf{b})B(\mathbf{c}) = 1_{G_2}$.

Folding

$A : \mathrm{G}^E \to \mathrm{G}$ is *folded* if $A(g \cdot \mathbf{a}) = g \cdot A(\mathbf{a})$ for all $g \in \mathrm{G}$.

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$A : \mathrm{G}_1^E \to \mathrm{G}_2$ is *folded over* φ if $A(h \cdot \mathbf{a}) = \varphi(h)A(\mathbf{a})$ for all $h \in \mathrm{Dom}(\varphi)$.

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$A : G_1^E \to G_2$ is *folded over* φ if $A(h \cdot \mathbf{a}) = \varphi(h)A(\mathbf{a})$ for all $h \in \mathrm{Dom}(\varphi)$.

Folding over φ changes equations

$$x \cdot y \cdot z = 1 \mapsto x' \cdot y \cdot z = h$$

such that

$$A(x')B(y)B(z) = h \iff A_\varphi(x)B(y)B(z) = 1_{G_2}$$

and $h \in \mathrm{Dom}(\varphi)$!

Soundness & Completeness

- It is easy to show that **projections** pass the test with probability $1 - \epsilon$.

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If the GLC instance is perfectly satisfiable, there is a map $f : U \to D,\ V \to E$ such that $\pi_{uv}(f(u)) = f(v)$ for each edge π_{uv}.

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$$\begin{aligned} A_v(\mathbf{a})B_u(\mathbf{b})B_u(\mathbf{c}) &= \mathbf{a}(f(v))\mathbf{b}(f(u))\mathbf{b}^{-1}(f(u))(\mathbf{a}\circ\pi_{uv})^{-1}(f(u))\eta(f(u)) \\ &= \mathbf{a}(\pi_{uv}(f(u)))(\mathbf{a}(\pi_{uv}(f(u))))^{-1}\eta(f(u)) \\ &= \eta(f(u)) = 1_{G_1} \text{ with probability } \geq 1-\epsilon. \end{aligned}$$

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Soundness & Completeness

- It is easy to show that **projections** pass the test with probability $1 - \epsilon$.
- We use **Fourier analysis** over non-Abelian groups + **representation theory** for the soundness analysis.
  - Given better-than-random assignments $A : G_1^E \to G_2,\ B : G_1^D \to G_2$ ...
  - We analyse complex-valued functions

$$\mathcal{A} := \omega \circ A : G_1^E \to \mathbb{C}^{d_\omega \times d_\omega}, \quad \mathcal{B} := \omega \circ B : G_1^D \to \mathbb{C}^{d_\omega \times d_\omega}$$

where ω is a 'nice' representation of G_2.

Soundness

$$\operatorname{tr} \mathbb{E}\left[\mathcal{A}(\mathbf{a})(\mathcal{B} * \mathcal{B})((\mathbf{a}\pi)^{-1}\mu)\right]$$

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& \operatorname{tr} \mathbb{E}\left[\mathcal{A}(\mathbf{a})(\mathcal{B} * \mathcal{B})((\mathbf{a}\pi)^{-1}\mu)\right] \\
= \quad & \operatorname{tr} \mathbb{E}\left[\Big(\sum_{\tau \in \widehat{\mathrm{G}_1^E}} \dim_\tau \widehat{\mathcal{A}}_\tau \tau(\mathbf{a})\Big)\Big(\sum_{\rho \in \widehat{\mathrm{G}_1^D}} \dim_\rho \widehat{(\mathcal{B} * \mathcal{B})}_\rho \rho((\mathbf{a}\pi)^{-1}\mu)\Big)\right]
\end{aligned}
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Soundness

$$\operatorname{tr}\mathbb{E}\Big[\mathcal{A}(\mathbf{a})(\mathcal{B}*\mathcal{B})((\mathbf{a}\pi)^{-1}\mu)\Big]$$

$$= \operatorname{tr}\mathbb{E}\Big[\Big(\sum_{\tau\in\widehat{G_1^E}}\dim_\tau \widehat{\mathcal{A}}_\tau\tau(\mathbf{a})\Big)\Big(\sum_{\rho\in\widehat{G_1^D}}\dim_\rho \widehat{(\mathcal{B}*\mathcal{B})}_\rho\rho((\mathbf{a}\pi)^{-1}\mu)\Big)\Big]$$

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$$\mathrm{tr}\,\mathbb{E}\Big[\mathcal{A}(\mathbf{a})(\mathcal{B}*\mathcal{B})((\mathbf{a}\pi)^{-1}\mu)\Big]$$

$$= \mathrm{tr}\,\mathbb{E}\Big[\Big(\sum_{\tau\in\widehat{G_1^E}} \dim_\tau \widehat{\mathcal{A}}_\tau \tau(\mathbf{a})\Big)\Big(\sum_{\rho\in\widehat{G_1^D}} \dim_\rho \widehat{(\mathcal{B}*\mathcal{B})}_\rho \rho((\mathbf{a}\pi)^{-1}\mu)\Big)\Big]$$

$$= \mathrm{tr}\,\mathbb{E}\Big[\widehat{\mathcal{A}}_1\Big(\sum_{\rho\in\widehat{G_1^D}} \dim_\rho \widehat{(\mathcal{B}*\mathcal{B})}_\rho \rho((\mathbf{a}\pi)^{-1}\mu)\Big)\Big] \qquad \leq\langle 1,\chi_\omega|_{\mathrm{Im}(\varphi)}\rangle$$

$$+ \mathrm{tr}\,\mathbb{E}\Big[\Big(\sum_{\tau\neq 1} \dim_\tau \widehat{\mathcal{A}}_\tau \tau(\mathbf{a})\Big)\Big(\sum_{|\rho|\text{ large}} \dim_\rho \widehat{(\mathcal{B}*\mathcal{B})}_\rho \rho((\mathbf{a}\pi)^{-1}\mu)\Big)\Big] \leq \dim_\omega \delta/2$$

$$+ \mathrm{tr}\,\mathbb{E}\Big[\Big(\sum_{\tau\neq 1} \dim_\tau \widehat{\mathcal{A}}_\tau \tau(\mathbf{a})\Big)\Big(\sum_{|\rho|\text{ small}} \dim_\rho \widehat{(\mathcal{B}*\mathcal{B})}_\rho \rho((\mathbf{a}\pi)^{-1}\mu)\Big)\Big] \quad \text{‘large’}$$

→ Allows us to devise a good sampling strategy for GLC:

- Sample reps of small degree according to Fourier coefficients;
- Sample u.a.r. from non-trivial subrepresentations.

The big picture

	3-LIN(G)	3-LIN(G_1, G_2, φ)
Abelian, almost-sat	$1/\|G\|$ [Håstad'01]	$1/\|\mathrm{Im}(\varphi)\|$ [this work]
non Abelian, almost-sat	$1/\|G\|$ [EHR'04]	$1/\|\mathrm{Im}(\varphi)\|$ [this work]
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3-LIN(G_1, G_2, φ, c, s) is a special case of a **Valued Promise CSP**.

Promise Valued CSPs

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PCSP($\mathbb{A}, \mathbb{B}, c, s$)

Given instance $\mathbb{X}$ that is c-satisfiable in $\mathbb{A}$, find an s-satisfying assignment in $\mathbb{B}$.

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Valued minion homomorphisms give reductions between Valued Promise CSPs.

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Theorem (B., Larrauri, Živný 2025)

There is a valued minion homomorphism

$$\mathrm{Plu}(\text{3-LIN}(G_1, G_2, \varphi, 1-\varepsilon, 1/|\mathrm{Im}(\varphi)| + \delta)) \rightsquigarrow \mathrm{Plu}(\mathrm{GLC}(1, \alpha)).$$

Thank You!