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Positive Geometry for Stringy Scalar Amplitudes

Jonah Stalknecht

Charles University

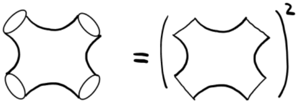
Based on: **Phys.Rev.Lett.** 136 (2026) 1, 011601
with C. Bartsch, K. Kampf, D. Podivín

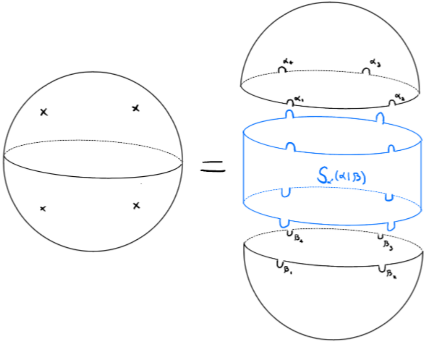
February 4, 2026, University of Hertfordshire

FORTE

Introduction: KLT Double Copy

Kawai-Lewellen-Tay (1986): closed string = (open string)^2





$$A^{\text{closed}} = \sum_{\alpha,\beta} A^{\text{open}}(\alpha)\, \underbrace{S_{\alpha'}(\alpha|\beta)}_{\text{KLT Kernel}}\, A^{\text{open}}(\beta)$$

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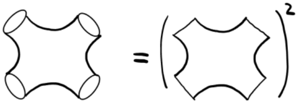

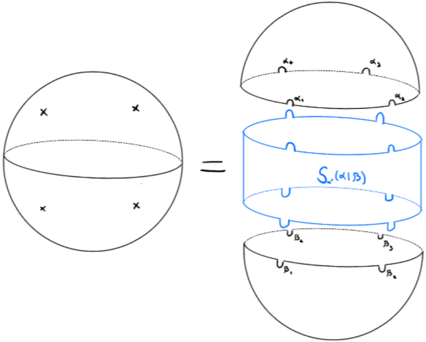

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Introduction: Bi-adjoint ϕ^3

Taking $\alpha' \to 0$: $\mathrm{GR} = (\text{Yang-Mills})^2$

$$A^{\mathrm{GR}} = \sum_{\alpha,\beta} A^{\mathrm{YM}}(\alpha) S(\alpha|\beta) A^{\mathrm{YM}}(\beta)$$

What is the kernel $S(\alpha|\beta)$? Does it have a physical interpretation?
Answer:

$$S(\alpha|\beta)^{-1} = m_n(\alpha|\beta)$$

Scattering amplitudes in *bi-adjoint* ϕ^3
Diagonal elements: $m_n \equiv m_n(\alpha|\alpha)$ are $\mathrm{Tr}(\phi^3)$ amplitudes

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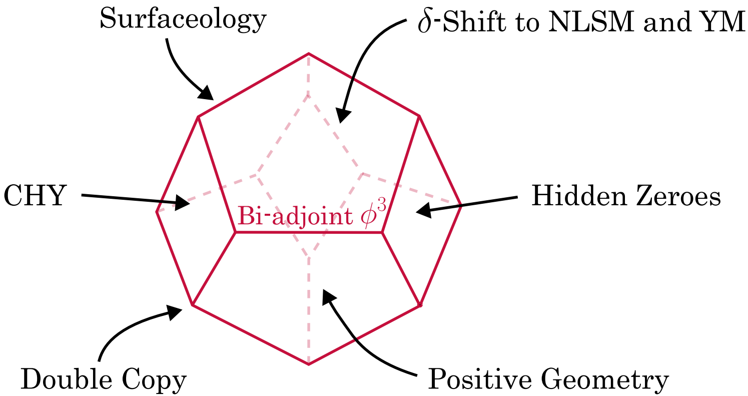
Bi-adjoint ϕ^3: A theory with many facets



Introduction: Inverse KLT Kernel

What about the inverse *string theory* KLT kernel?

$$S_{\alpha'}(\alpha|\beta)^{-1} = m_n^{\alpha'}(\alpha|\beta)$$

A lot less is known, studied by Mizera (2017)
Some 'stringy' version of bi-adjoint ϕ^3

$$\lim_{\alpha' \to 0} m_n^{\alpha'}(\alpha|\beta) = m_n(\alpha|\beta)$$

Looks like a string amplitude, but is *much simpler*

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Outline

- Review: $\mathrm{Tr}(\phi^3)$ Amplitudes
- Review: The ABHY Associahedron
- Geometry of Stringy $\mathrm{Tr}(\phi^3)$
- Pions and Mixed Amplitudes
- Off-Diagonal Amplitudes

$\mathrm{Tr}\,(\phi^3)$ **Amplitudes**

Cubic scalar theory, only planar Feynman diagrams

Propagators in tree-level $\mathrm{Tr}\,(\phi^3) \longleftrightarrow$ Planar Mandelstam variables

$$X_{ij} \coloneqq (p_i + p_{i+1} + \ldots + p_{j-1})^2$$

Masslessness: $X_{ii+1} = X_{1n} = 0$

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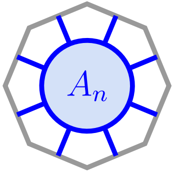

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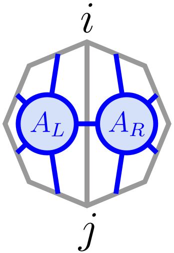
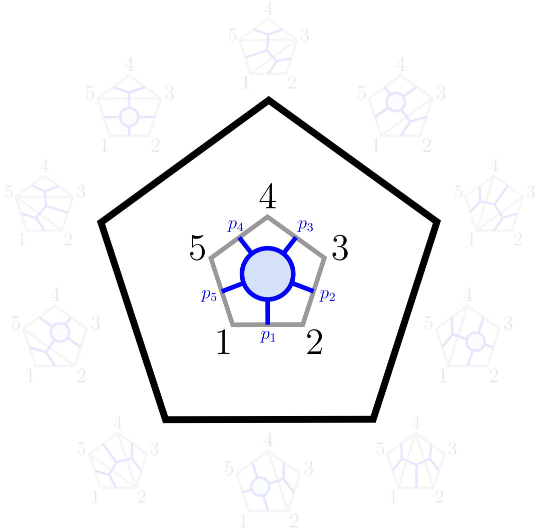


$\mathrm{Tr}\,(\phi^3)$ Amplitudes: m_5 Singularities



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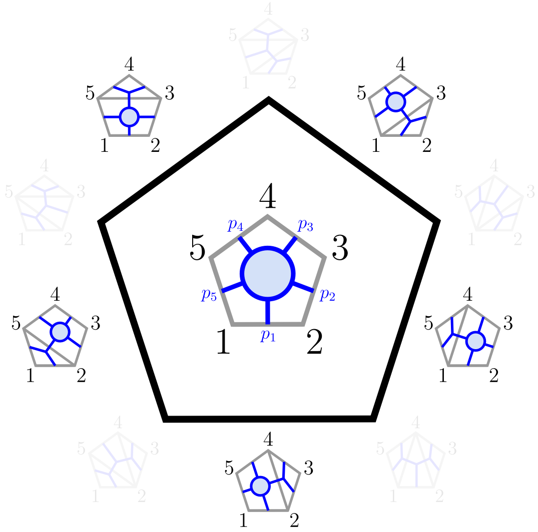

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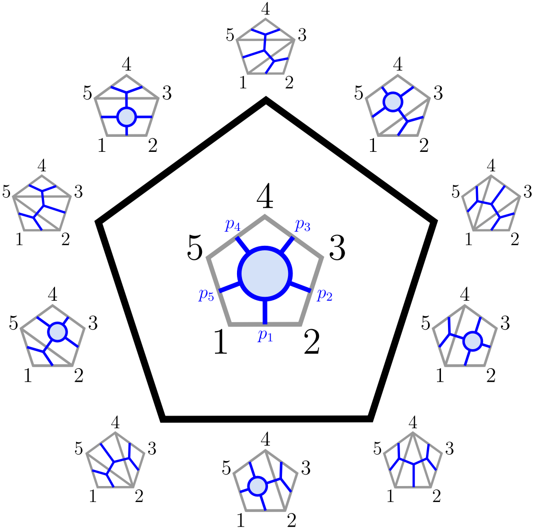
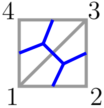
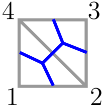


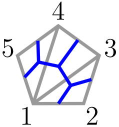
$\mathrm{Tr}\left(\phi^3\right)$ Amplitudes: Examples

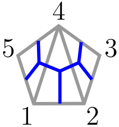
$$m_3 = 1$$

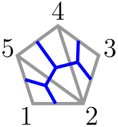
$m_4 =$ 

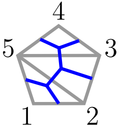
$+$ 

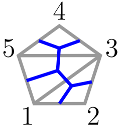
$$= \frac{1}{X_{13}} + \frac{1}{X_{24}}$$

$m_5 =$ 

$+$ 

$+$ 

$+$ 

$+$ 

$$= \frac{1}{X_{13}X_{14}} + \frac{1}{X_{14}X_{24}} + \frac{1}{X_{24}X_{25}} + \frac{1}{X_{25}X_{35}} + \frac{1}{X_{35}X_{13}}$$

The ABHY Associahedron

Positive Geometry:

- Geometry $\mathcal{A}$ with boundaries $\longleftrightarrow$ poles of amplitude
- With a unique canonical form $\Omega(\mathcal{A})$ ("=" scattering amplitude)
- $\Omega(\mathcal{A})$ has log singularities *at and only at* boundaries of $\mathcal{A}$

For $\mathrm{Tr}\,(\phi^3)$:

- Boundary structure: triangulations of an n-gon $\implies$ *Associahedron*
- Boundaries at $X_{ij} = 0$
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ABHY Associahedron $\mathcal{A}_n$ [Arkani-Hamed, Bai, He, Yuan]:

- All $X_{ij} \geq 0 \implies$ correct facets
- Not full X_{ij} space: keep $(n-3)$ X's and $(n-2)(n-3)/2$ constraints

$$c_{ij} = X_{ij} + X_{i+1j+1} - X_{ij+1} - X_{i+1j} = -(p_i + p_j)^2 = \text{ positive const.}$$

For $n = 4$: X_{13}, $X_{24} = c_{13} - X_{13}$

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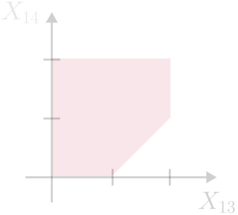

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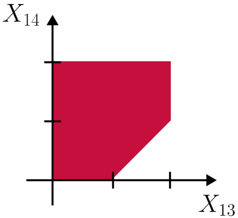

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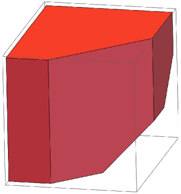
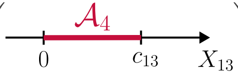


$\mathcal{A}_6$

The ABHY Associahedron: Canonical Form

Canonical form:

$\Omega\Bigg($  $\Bigg) = \dfrac{\mathrm{d}X_{13}}{X_{13}} + \dfrac{\mathrm{d}X_{13}}{c_{13} - X_{13}} = \left(\dfrac{1}{X_{13}} + \dfrac{1}{X_{24}}\right)\mathrm{d}X_{13}$

For $n \geq 5$, we can use *chambers* and *fibers*

$$\mathcal{A}_5 = \mathfrak{c}_1 \times f_1 \cup \mathfrak{c}_2 \times f_2$$

The ABHY Associahedron: Canonical Form

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$$\Omega\left(\underset{0 \qquad\qquad c_{13} \qquad X_{13}}{\overset{\mathcal{A}_4}{\longrightarrow}} \right) = \frac{\mathrm{d}X_{13}}{X_{13}} + \frac{\mathrm{d}X_{13}}{c_{13} - X_{13}} = \left(\frac{1}{X_{13}} + \frac{1}{X_{24}} \right) \mathrm{d}X_{13}$$

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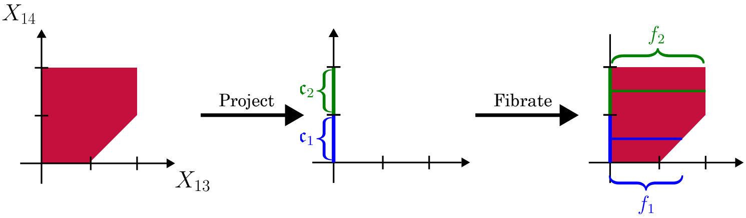

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The ABHY Associahedron: Canonical Form

In general:

- Chambers = product of lower-point associahedra
- Fibers = line-segment ($= \mathcal{A}_4$)

Geometric Recursion

$$m_n = \sum_{i=4}^{n} m(1,2,3,i)m(2,3,\ldots,i)m(i,i+1,\ldots,n,1,2)|_{X_{2j}\to X_{2j}-X_{2i}}$$

Example:

$$m_5 = \left(\frac{1}{X_{13}} + \frac{1}{X_{24}}\right)\left(\frac{1}{X_{14}} + \frac{1}{X_{25}-X_{24}}\right) + \left(\frac{1}{X_{13}} + \frac{1}{X_{25}}\right)\left(\frac{1}{X_{35}} + \frac{1}{X_{24}-X_{25}}\right)$$

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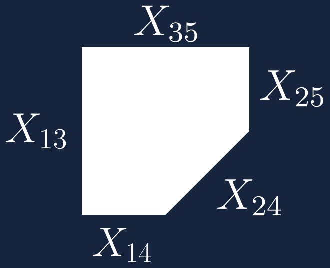
Summary

- $\mathrm{Tr}(\phi^3)$ Amplitudes m_n
  - Diagonal elements of $S(\alpha|\beta)^{-1} = m(\alpha|\beta)$
  - Poles as $X_{ij} = 0$
  - Factorizes on poles
- The ABHY Associahedron $\mathcal{A}_n$
  - Polytope in kinematic space
  - Boundaries at $X_{ij} = 0$
  - $\Omega(\mathcal{A}_n) = m_n$
  - Once we know $\Omega(\mathcal{A}_4)$, we can find m_n through geometric recursion



Stringy $\mathrm{Tr}(\phi^3)$

Recall:

$$S_{\alpha'}^{-1} = m_n^{\alpha'}(\alpha|\beta)$$

"Stringy" $\mathrm{Tr}(\phi^3)$: $m_n^{\alpha'} \equiv m_n^{\alpha'}(\alpha|\alpha)$

$$m_3^{\alpha'} = 1$$

$$m_4^{\alpha'} = \frac{1}{\tan(\pi\alpha' X_{13})} + \frac{1}{\tan(\pi\alpha' X_{24})}$$

$$m_5^{\alpha'} = 1 + \frac{1}{\tan(\pi\alpha' X_{13})\tan(\pi\alpha' X_{14})} + \frac{1}{\tan(\pi\alpha' X_{14})\tan(\pi\alpha' X_{24})}$$

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Geometry of Stringy $\mathrm{Tr}\,(\phi^3)$

$m_n^{\alpha'}$ is ‘stringy’:

- Infinite resonance structure: Poles at $X_{ij} = k/\alpha',\ k \in \mathbb{Z}$
- Factorizes on the poles
- Satisfies monodromy relations
- Reduces to m_n as $\alpha' \to 0$

For stringy $\mathrm{Tr}\,(\phi^3)$ amplitudes the geometric recursion *still holds*:

$$m_n^{\alpha'} = \sum_{i=4}^{n} m^{\alpha'}(1,2,3,i)m^{\alpha'}(2,3,\ldots,i)m^{\alpha'}(i,i+1,\ldots,n,1,2)\Big|_{X_{2j}\to X_{2j}-X_{2i}}$$

We can recurse down to products of 4-point $m_4^{\alpha'}$
Thus: A geometry for $m_4^{\alpha'} \implies$ a geometry for $m_n^{\alpha'}$!

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Geometry of Stringy $\mathrm{Tr}\,(\phi^3)$

Want: a one-dimensional geometry with canonical form

$$\omega_4^{\alpha'} = \mathrm{d}\log\frac{\sin(\pi\alpha' X_{13})}{\sin(\pi\alpha'(c_{13}-X_{13}))} = \left(\frac{1}{\tan(\pi\alpha' X_{13})}+\frac{1}{\tan(\pi\alpha' X_{24})}\right)\mathrm{d}X_{13}$$

Using Euler's infinite product formula for $\sin(\pi x)$:

$$\omega_4^{\alpha'} = \sum_{k\in\mathbb{Z}} \mathrm{d}\log\frac{X_{13}+k/\alpha'}{X_{13}-c_{13}+k/\alpha'}$$

Which is the canonical form of an infinite sum of line segments!

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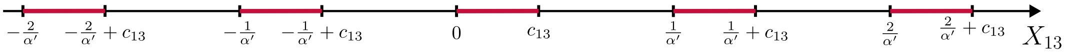
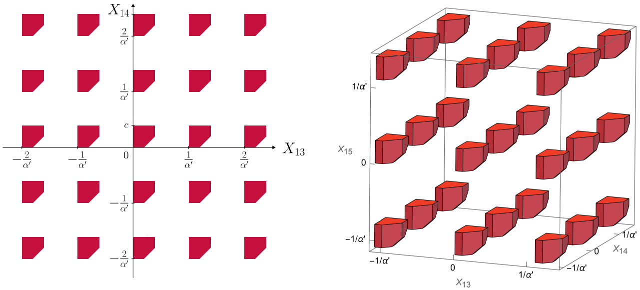


Geometry of Stringy $\mathrm{Tr}\,(\phi^3)$

For higher n: Infinite grid of ABHY associahedra!



- Correct poles as $X_{ij} = k/\alpha'$
- Factorises on poles
- $\Omega(\mathcal{A}_n^{\alpha'}) = m_n^{\alpha'}$

Pions and Mixed Amplitudes

$m_n^{\alpha'}$ contains $\mathrm{Tr}\left(\phi^3\right)$ as $\alpha' \to 0$

Surprisingly, it also contains pions in the NLSM! [Bartsch, Kampf, Novotný, Trnka]

- Rescale $\alpha' \to \alpha'/2$
- Shift certain $X_{ij} \to X_{ij} \pm 1/\alpha'$
- NLSM amplitudes as $\alpha' \to 0$

$$\frac{1}{\tan(\pi\alpha' X_{ij})} \to \frac{1}{\tan(\pi\alpha' X_{ij}/2 + \pi/2)} = -\tan(\pi\alpha' X_{ij}/2)$$

Example:

$$m_4^{\alpha'} = \frac{1}{\tan(\pi\alpha' X_{13})} + \frac{1}{\tan(\pi\alpha' X_{24})} \to -\tan(\pi\alpha' X_{13}/2) - \tan(\pi\alpha' X_{24}/2)$$

As $\alpha' \to 0$ we recover $A_4^{\mathrm{NLSM}} = -X_{13} - X_{24}$

Pions and Mixed Amplitudes

$m_n^{\alpha'}$ contains $\mathrm{Tr}\left(\phi^3\right)$ as $\alpha' \to 0$

Surprisingly, it also contains pions in the NLSM! [Bartsch, Kampf, Novotný, Trnka]

- Rescale $\alpha' \to \alpha'/2$
- Shift certain $X_{ij} \to X_{ij} \pm 1/\alpha'$
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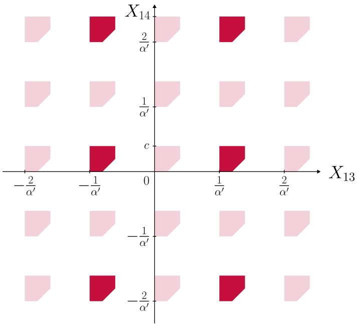
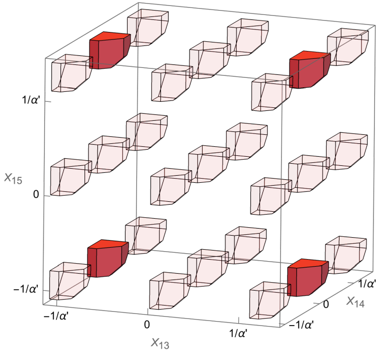
Geometrically: this isolates an infinite subgrid of associahedra



$m_5^{\alpha'}(\pi\pi\phi\phi\phi)$



$m_6^{\alpha'}(\pi\pi\pi\pi\pi\pi)$

Pions and Mixed Amplitudes

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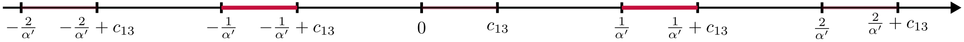
$\lim_{\alpha' \to 0}$ 

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This is the same as the δ-shift if we equate $\delta = 1/\alpha'$!

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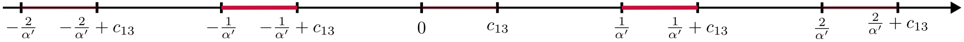
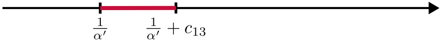
$\lim_{\alpha' \to 0}$ 

$= \lim_{\alpha' \to 0}$ 

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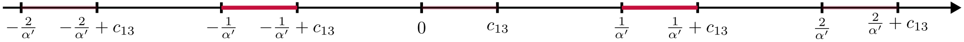
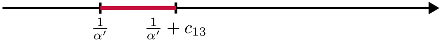
$\lim_{\alpha' \to 0}$ 

$= \lim_{\alpha' \to 0}$ 

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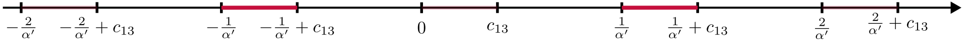

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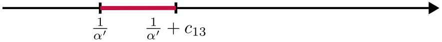

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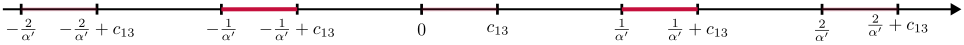

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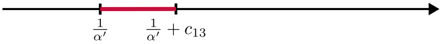

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Off-Diagonal Amplitudes

So far: stringy $\mathrm{Tr}\,(\phi^3) \leftarrow$ Diagonal $m_{\alpha'}(\mathbb{I}|\mathbb{I})$
How about stringy $m_{\alpha'}(\alpha|\beta)$?
They can be written as some $m_n^{\alpha'}$s times products of $1/\sin(\pi\alpha' X_{ij})$. Example:

$$m_{\alpha'}(\mathbb{I}|13245) = \frac{1}{\sin(\pi\alpha' X_{13})}\left(\frac{1}{\tan(\pi\alpha' X_{14})} + \frac{1}{\tan(\pi\alpha' X_{35})}\right)$$

Geometry for $1/\sin(\pi\alpha' X_{ij}) \implies$ Geometry for all off-diagonal amplitudes
Canonical form:

$$\mathrm{d}\log\tan(\pi\alpha' X_{ij}/2) = \frac{\mathrm{d}X_{ij}}{\sin(\pi\alpha' X_{ij})}$$

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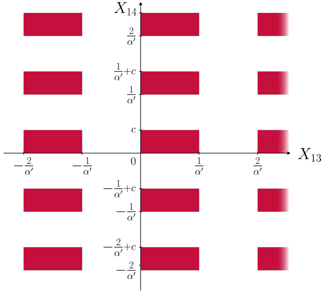
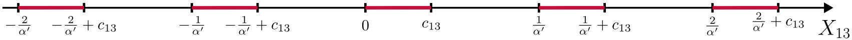


Treated on the same footing as stringy $\mathrm{Tr}\,(\phi^3)$

Summary

- "Stringy bi-adjoint ϕ^3 amplitudes" $m_n^{\alpha'}(\alpha|\beta)$
  - — Central in KLT double copy
  - — Natural stringification of the well-studied $\mathrm{Tr}\left(\phi^3\right)$
- Geometric description in terms of the *Associahedral grid*
- Geometry contains all NLSM and mixed π/ϕ amplitudes
- Positive geometry containing stringy features, beyond rational functions



Q&A

Thank you for listening!

Bonus Slides

$$
\begin{aligned}
m_5^{\alpha'} = &\left(\frac{1}{\tan(\pi\alpha' X_{13})} + \frac{1}{\tan(\pi\alpha' X_{24})}\right)\left(\frac{1}{\tan(\pi\alpha' X_{14})} + \frac{1}{\tan(\pi\alpha'(X_{25} - X_{24}))}\right) \\
&+ \left(\frac{1}{\tan(\pi\alpha' X_{13})} + \frac{1}{\tan(\pi\alpha' X_{25})}\right)\left(\frac{1}{\tan(\pi\alpha' X_{35})} + \frac{1}{\tan(\pi\alpha'(X_{24} - X_{25}))}\right)
\end{aligned}
$$

Bonus Slides

The ’spurious boundaries’ give all the correct contact terms:

$$\frac{1}{\tan(\pi\alpha'(X_a - X_b))} = \frac{1 + \tan(\pi\alpha' X_a)\tan(\pi\alpha' X_b)}{\tan(\pi\alpha' X_a) - \tan(\pi\alpha' X_b)}$$

Which leads to the correct

$$\begin{aligned} m_5^{\alpha'} = 1 &+ \frac{1}{\tan(\pi\alpha' X_{13})\tan(\pi\alpha' X_{14})} + \frac{1}{\tan(\pi\alpha' X_{14})\tan(\pi\alpha' X_{24})} \\ &+ \frac{1}{\tan(\pi\alpha' X_{24})\tan(\pi\alpha' X_{25})} + \frac{1}{\tan(\pi\alpha' X_{25})\tan(\pi\alpha' X_{35})} + \frac{1}{\tan(\pi\alpha' X_{13})\tan(\pi\alpha' X_{35})} \end{aligned}$$