Supersymmetric Quantum Mechanics, Line Defects and Quantum K-theory

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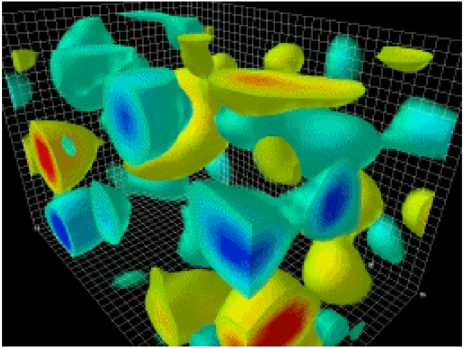
based on:
arXiv:2512.02984 with James Wynne,
2512.19802 & 2601.18881 with Wei Gu, Osama Khlaif, Eric Sharpe, Hao Zhang, Hao Zou

Line defects in QFT

Quantum Field Theory (QFT) is all about **fields**.

- At weak coupling: Particle excitations.
- **Delocalised excitations** are possible.
- In particular, **string-like excitations** carry a lot of interesting physics. (E.g. about confinement.)



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- Local operator $\mathcal{O}(x)$: creates particles.
- Line operator $\mathscr{L}(\mathcal{C})$: creates strings.

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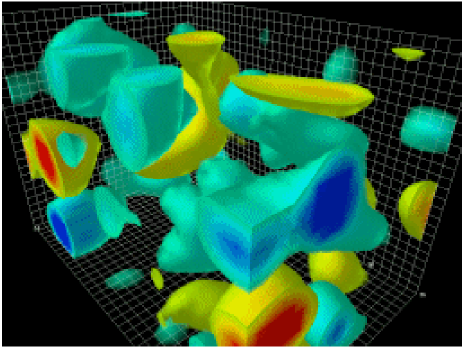

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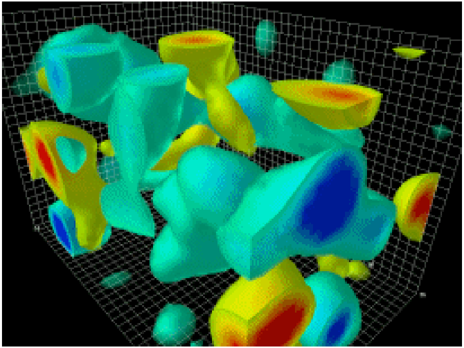

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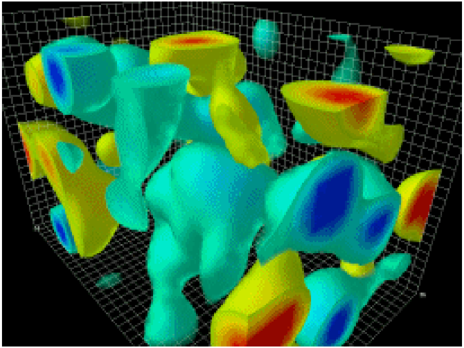

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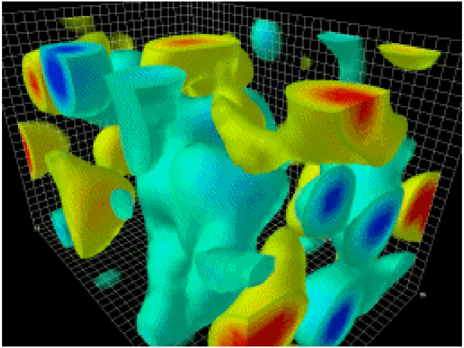

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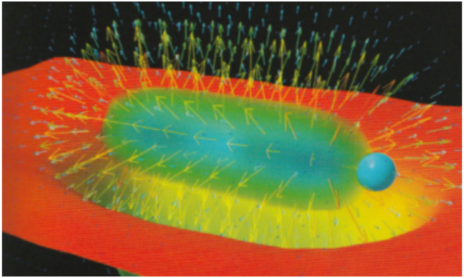

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Line defects in QFT

We will work in Euclidean signature, and in the path integral formulation.

'Line operators = line defect.'

Examples of a line operator in a gauge theory with gauge field A_μ:

- Wilson line (in any dimension):

$$W_R(\mathcal{C}) = \mathrm{Tr}_R \left(P \exp \int_{\mathcal{C}} A_\mu dx^\mu \right)$$

- 't Hooft line (in $d = 4$): Prescribed flux on any sphere that surrounds $\mathcal{C}$:

$$\int_{S^2} dA = 2\pi \mathfrak{m} \ .$$

  This is an example of a 'disorder operator'.

In general, a line defect is defined by choices of boundary conditions on the fundamental fields in the path integral, along a line $\mathcal{C}$ in space-time.

Hard to compute with, in general. We will focus on supersymmetric gauge theories, in which case we can do a lot of exact computations.

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From line defects in 3d GLSM to quantum K-theory

3d $\mathcal{N} = 2$ gauge theory

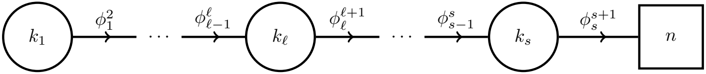
Consider 3d $\mathcal{N} = 2$ supersymmetric gauge theories. They consist of:

- Vector multiplet $\mathcal{V}$ for some gauge group G;
- Chiral multiplets Φ_i in representations $\mathfrak{R}_i$ of G.

We focus on unitary gauge theories,

$$G = \prod_{\ell=1}^{s} U(k_\ell)$$

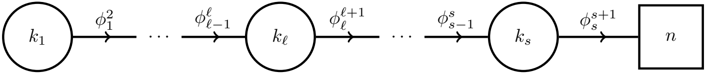
and matter in bifundamental representations. They are best represented as quivers:



The 3d vacua come in various forms: [Intriligator, Seiberg, 2013; CC, Khlaif, 2023]

- Higgs branch (G fully Higgsed);
- Topological vacua (Coulomb branch);
- Hybrid vacua.

3d $\mathcal{N} = 2$ gauge theory and flag manifolds



For some choice of Chern–Simons levels and Fayet–Ilipoulos (FI) parameters, the theory above only has Higgs vacua. The Higgs branch is the partial flag manifold

$$X = \mathrm{Fl}(\boldsymbol{k}; n) = \mathrm{Fl}(k_1, \cdots, k_s; n)$$

This is the space of all partial flag in $\mathbb{C}^n$:

$$\mathrm{Fl}(\boldsymbol{k}; n) := \{V_\bullet = (0 \subset V_1 \subset V_2 \subset \cdots \subset V_s \subset \mathbb{C}^n) \mid \dim(V_\ell) = k_\ell\}$$

The VEVs of the chiral multiplets $\phi_\ell^{\ell+1}$ are $k_\ell \times k_{\ell+1}$ matrices of maximal rank, parameterising $V_\ell \cong C^{k_\ell}$ modulo basis change — which are the $U(k_\ell)_\mathbb{C} \cong GL(k_\ell)$ gauge transformations.

The 3d GLSM and quantum K-theory

Consider the theory on

$$\mathcal{M}_3 = \mathbb{R}^2 \times S^1_\beta$$

This can be viewed as a 2d $\mathcal{N} = (2,2)$ gauge theory of Kaluza-Klein (KK) type:

$$\mathcal{V}_{3\mathrm{d}} \to \mathcal{V}_0 \bigoplus \oplus_{n\neq 0} \mathcal{V}_n \ , \qquad \Phi_{3\mathrm{d}} \to \oplus_{n\in\mathbb{Z}} \Phi_n$$

The **3d GLSM** is this 2d KK theory viewed as a GLSM into X. [Witten, 1992]

We often consider the topological twist of this 2d theory, which is a partial twist on

$$\mathcal{M}_3 = \Sigma \times S^1_\beta$$

This gives us a 2d TQFT on Σ in the cohomological sense (A-twist [Witten, 1988]).

The observables of this A-model captures the twisted chiral ring physics. In 2d GLSMs, this ring captures the quantum cohomology of the target space:

$$\mathcal{R}_{2\mathrm{d}} \cong \mathrm{QH}^\bullet(X)$$

Similarly, the 3d GLSM captures the **quantum K-theory** of the target:

$$\mathcal{R}_{3\mathrm{d}} \cong \mathrm{QK}(X)$$

[Kapustin, Willett, 2013; Jockers *et al.*, 2018]

The 3d GLSM and quantum K-theory

Twisted chiral operators in 3d are half-BPS line operators:

$$[\mathcal{Q}_-, \mathscr{L}] = 0 \ , \qquad [\bar{\mathcal{Q}}_+, \mathscr{L}]$$

In the 3d GLSM, we wrap the line $\mathscr{L}$ along S^1_β, which gives a local operator in the 2d KK theory on Σ. Any half-BPS will give us a quantum K-theory class

$$[\mathscr{L}] \in \mathrm{QK}(X)$$

The key physics question:

line operators defined in the UV $\quad \rightarrow \quad$ QK class in the IR ?

Example: The half-BPS Wilson loop

$$W_R \;=\; \mathrm{Tr}_R \; P \exp\left(-i \int_{S^1_\beta} (A - i\sigma d\tau)\right)$$

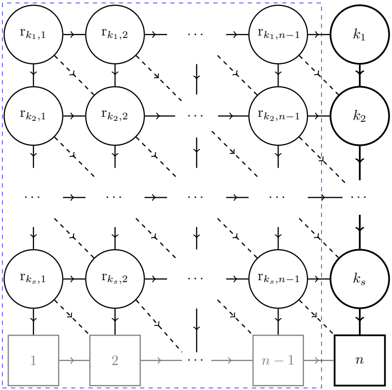
corresponds to the homogeneous vector bundle:

$$\mathcal{E}_R \;\equiv\; (\mathfrak{R} \times R)/G_{\mathbb{C}} \;\longrightarrow\; X$$

Line defects: coupling 1d $\mathcal{N} = 2$ gauge theories on the worldline

A very general way to construct half-BPS line defect is to couple a 1d $\mathcal{N} = 2$ supersymmetric quantum mechanics (SQM) to the 3d fields.

For the GLSM to partial flag manifolds, we can do this elegantly as a 1d/3d quiver:



Before studying such line defects in 3d, we need to understand the 1d $\mathcal{N} = 2$ SQM quivers better.

Generalities on 1d $\mathcal{N} = 2$ SQM

Gauged $\mathcal{N} = 2$ SQM

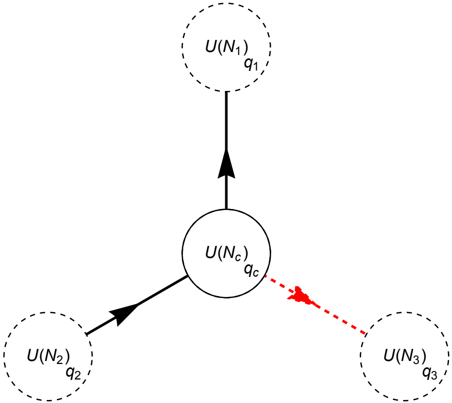
Consider now a **gauged $\mathcal{N} = 2$ supersymmetric quantum mechanics**.
In general, it is defined 'in the UV' by the following data:

- Vector multiplet $\mathcal{V} = (A_t, \sigma, \lambda, \bar{\lambda}, D)$;
- Chiral multiplets $\Phi = (\phi, \psi)$;
- Fermi multiplets $\Lambda = (\Lambda, \mathcal{G})$
- 1d $\mathcal{N} = 2$ superpotentials $E(\Phi)$ and $J(\Phi)$ for each Λ.

Considering unitary gauge groups, this allows us to construct (representations of) **generalised quivers.**

Note that, for each $U(N)$ gauge group, we have the freedom to turn on some "1d Chern–Simons term" – a background gauge charge:

$$L = Q \int dt \, \mathrm{Tr}(A_t) \ .$$



Gauged $\mathcal{N} = 2$ SQM

The 1d CS term

$$L = Q \int dt \, \mathrm{Tr}(A_t) \ , \qquad Q \in \mathbb{Z}$$

is the crucial new ingredient compared to extended supersymmetry (e.g. 1d $\mathcal{N} = 4$).

The physics of the CS level is easy to understand:

- In the SQM, it assigns a gauge charge to the vacuum and thus shifts the Gauss law constraint:

$$\langle \frac{\delta S}{\delta A_t} \rangle = \langle \mathbf{J}_{\mathrm{matter}} + Q \rangle = 0$$

  This often gives us many more ground states that if we had $Q = 0$.
- Relatedly, the $\mathcal{N} = 2$ SQM can have a parity anomaly / charge conjugation anomaly! [Elitzur, Frishman, Rabinovici, Schwimmer, 1986; CC, Dumitrescu, Festuccia, Komargodski, Seiberg, 2012]

Aside: Quantising free fields (correctly)

Let's display the parity anomaly explicitly for free fields. For instance, let us quantise the free chiral of electric charge q:

$$\begin{aligned} \phi &= \frac{1}{\sqrt{2\omega}}(a^\dagger + b)\ , \qquad & \bar{\phi} &= \frac{1}{\sqrt{2\omega}}(b^\dagger + a)\ , \\ \pi_\phi &= i\sqrt{\frac{\omega}{2}}(a - b^\dagger)\ , \qquad & \bar{\pi}_{\bar{\phi}} &= i\sqrt{\frac{\omega}{2}}(b - a^\dagger)\ , \end{aligned} \tag{1}$$

with the commutators

$$[a, a^\dagger] = 1\ , \qquad [b, b^\dagger] = 1\ , \qquad \{\psi, \bar{\psi}\} = 1\ .$$

For the real mass $\sigma \neq 0$, we have the frequency $\omega = |q\sigma|$. We have the supercharges

$$\mathcal{Q} = \begin{cases} 2i\sqrt{\omega}\, b^\dagger \psi & \text{if } q\sigma > 0\ , \\ -2i\sqrt{\omega}\, a\psi & \text{if } q\sigma < 0\ , \end{cases} \qquad \bar{\mathcal{Q}} = \begin{cases} -2i\sqrt{\omega}\, b\bar{\psi} & \text{if } q\sigma > 0\ , \\ 2i\sqrt{\omega}\, a^\dagger \bar{\psi} & \text{if } q\sigma < 0\ , \end{cases}$$

and the Hamiltonian and $U(1)$ charge

$$\mathbf{H} = |q\sigma|(a^\dagger a + b^\dagger b + 1) - q\sigma\psi\bar{\psi}\ , \qquad \mathbf{J} = q(a^\dagger a - b^\dagger b - \bar{\psi}\psi)\ .$$

so that

$$\{\mathcal{Q}, \bar{\mathcal{Q}}\} = 2\left(\mathbf{H} - \sigma\mathbf{J}\right)$$

Aside: Quantising free fields (correctly)

We define the 'vacuum' for σ positive or negative as:

$$a|0\rangle_\pm = 0 \ , \qquad b|0\rangle_\pm = 0 \ , \qquad \begin{cases} \psi|0\rangle_+ = 0 & \text{if } q\sigma > 0 \ , \\ \bar{\psi}|0\rangle_- = 0 & \text{if } q\sigma < 0 \ , \end{cases}$$

The supersymmetric ground states of the chiral multiplet take the form:

$$|\Psi_+; n\rangle = (a^\dagger)^n |0\rangle_+ \ , \qquad |\Psi_-; n\rangle = (b^\dagger)^n |0\rangle_- \ , \qquad \text{with } n \geq 0 \ ,$$

for $q\sigma > 0$ and $q\sigma < 0$, respectively. Here, the **parity anomaly** is the statement that:

$$\mathbf{J}|\Psi_+; n\rangle = nq|\Psi_+; n\rangle \ , \qquad \mathbf{J}|\Psi_-; n\rangle = (-n-1)q|\Psi_-; n\rangle \ ,$$

$$(-1)^{\mathrm{F}}|\Psi_+; n\rangle = |\Psi_+; n\rangle \ , \qquad (-1)^{\mathrm{F}}|\Psi_-; n\rangle = -|\Psi_-; n\rangle \ .$$

At the level of the index:

$$\mathrm{I_W}^{(q\sigma>0)} = 1 + x^q + x^{2q} + x^{3q} + \cdots \ ,$$

$$\mathrm{I_W}^{(q\sigma<0)} = -x^{-q}\left(1 + x^{-q} + x^{-2q} + x^{-3q} + \cdots\right) \ ,$$

and of course $\mathrm{I_W}^{(q\sigma>0)} = \mathrm{I_W}^{(q\sigma<0)}$. Adding a bare CS level $Q \in \mathbb{Z}$ for $U(1)$:

$$\mathrm{I_W} = x^Q \, \frac{1}{1 - x^q}$$

Dualities and trialities in $\mathcal{N} = 2$ SQM

Gauged $\mathcal{N} = 2$ SQM: SQCD

We consider unitary $\mathcal{N} = 2$ SQCD:

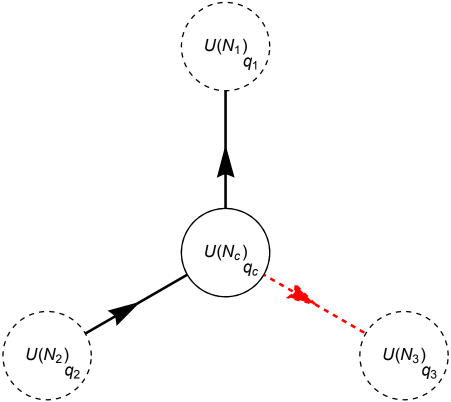
$$\text{1d } \mathcal{N} = 2 \ \ U(N_c)_{q_c} \quad \text{with} \quad N_1 \ \Phi \oplus N_2 \ \tilde{\Phi} \oplus N_3 \ \Lambda$$

Here we have the bare CS level:

$$Q_c \equiv q_c + \frac{1}{2}(N_1 - N_2 - N_3) \in \mathbb{Z}$$

This theory has no normalisable ground states. This is due to the non-compact target space:

$$\tilde{\Phi}\Phi \in \mathbb{C}^{N_1 N_2}$$



Gauged $\mathcal{N} = 2$ SQM: Γ-SQCD

Instead, let us focus on what we call Γ-SQCD:

$$1\text{d } \mathcal{N} = 2 \; U(N_c)_q \quad \text{with} \quad N_1 \; \Phi \oplus N_2 \; \tilde{\Phi} \oplus N_3 \; \Lambda \oplus N_1 N_2 \; \Gamma$$

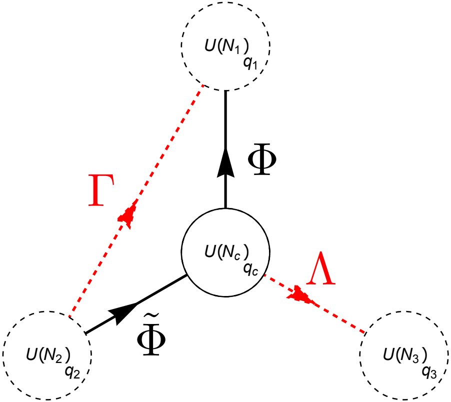
We introduced the Γ fields, which are gauge-singlets "fermionic mesons" with a E-term:

$$E_\Gamma = \tilde{\Phi}\Phi$$

We also have the FI term:

$$L_{\text{FI}} = \zeta \int dt D$$

For $\zeta \neq 0$, we expect **a finite number** of supersymmetric ground states.



The Witten index of Γ-SQCD

This theory has a flavour symmetry $G_F = (U(N_1) \times U(N_2) \times U(N_3))/U(1)$.

The flavoured Witten index

$$\mathrm{I}_W(y;\zeta) = \mathrm{Tr}((-1)^{\mathrm{F}} y^{Q_F}) \in \mathbb{Z}(y)$$

can be computed by supersymmetric localisation on the 1d Coulomb branch:

$$\mathrm{I}_W(y;\zeta) = \frac{1}{N_c!} \oint_{\mathrm{JK}(\zeta)} \prod_{a=1}^{N_c} \left[-\frac{dx_a}{2\pi i x_a}\, x_a^{Q_c} \frac{\prod_{k=1}^{N_3}(1-\frac{x_a}{y_{3,k}})}{\prod_{i=1}^{N_1}(1-\frac{x_a}{y_{1,i}}) \prod_{j=1}^{N_2}(1-\frac{y_{2,j}}{x_a})} \right] \times \prod_{a\neq b}^{N_c}(1-\frac{x_a}{x_b}) \times \prod_{i=1}^{N_1}\prod_{j=1}^{N_2}(1-\frac{y_{2,j}}{y_{1,i}})$$

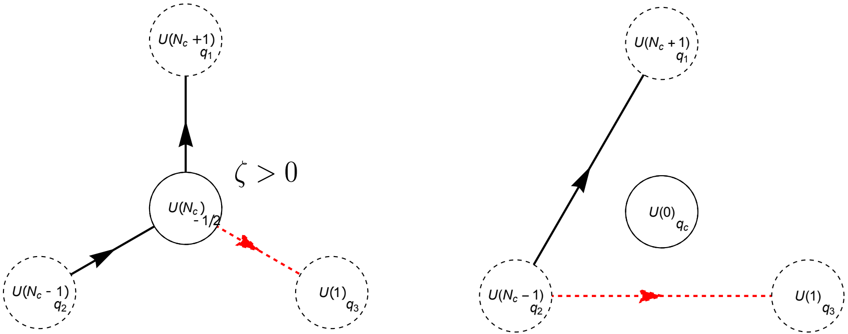
Here the JK residue depends on the sign of the FI parameter ζ. [Kim, Yi, Hori, 2014]
We can have **non-trivial wall-crossing,** with distinct ground states for $\zeta > 0$ or $\zeta < 0$:

$$\Delta\mathrm{I}_W(y) \equiv \mathrm{I}_W(y;\zeta>0) - \mathrm{I}_W(y;\zeta<0) \neq 0$$

Confinement of gauged $\mathcal{N} = 2$ SQM

Definition: A gauged SQM “confines” if the ground state can be described by free fields.

Example: We claim that the following two descriptions are dual:



$U(N_c)_{-1/2}$, $(N_1, N_2, N_3) = (N_c+1, N_c-1, 1)$.

Free chirals and fermis.

This was found “experimentally” looking at 1d-3d coupled systems. [CC, Khlaif, 2023]

Question: How can we determine whether a 1d theory confines (ideally without doing any computation)?

Seiberg-like dualities of $\mathcal{N} = 2$ gauged SQM

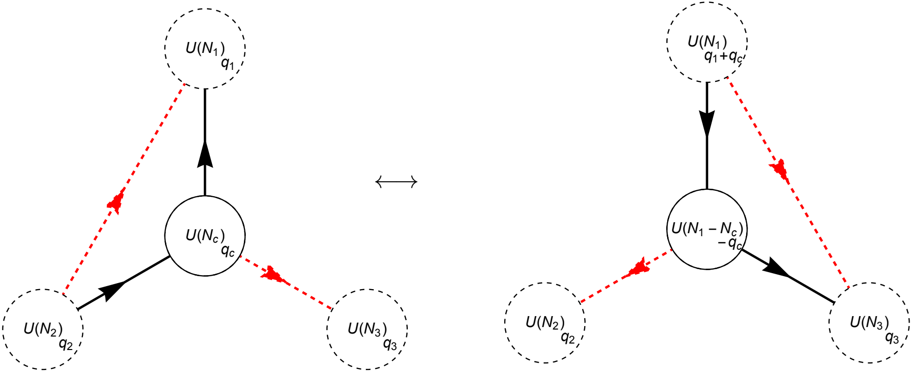
[CC, Wynne, 2025]

To answer this question, we derived new Seiberg-like dualities of $U(N_c)$ $\mathcal{N} = 2$ SQM.

- These are **infrared dualities:** in quantum mechanics, this simply means that the supersymmetric ground states are isomorphic.
- The dualities, like the ground states, depend on the sign of the FI term.
- There are two basic dualities: the one for $\zeta > 0$ and for $\zeta < 0$, respectively, which we may call right-mutation and left-mutation. They are inverse of each other.
- As an operation on generalised quiver, these are known generalised mutation operations (see *e.g.* [Franco, Lee, Seong, 2016]). New crucial element: transformations of the 1d CS levels.

Seiberg-like dualities of $\mathcal{N} = 2$ gauged SQM

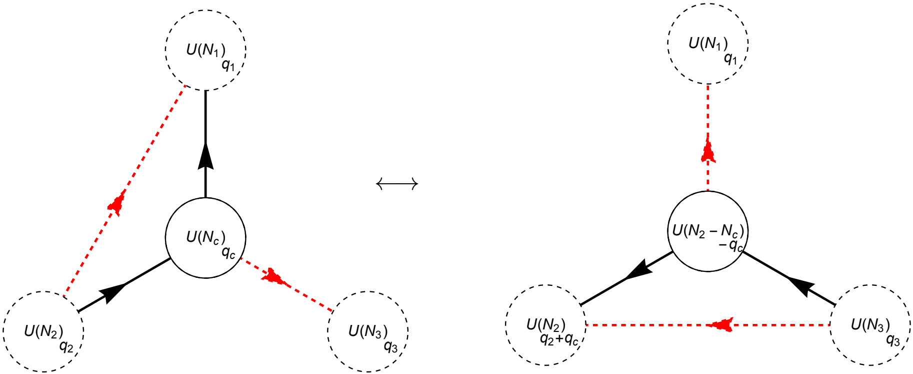
Right-mutation: duality for $U(N_c)_{q_c}$ with $\zeta > 0$.



$U(N_c)_{q_c}$, $\zeta > 0$ $\longleftrightarrow$ $U(N_1 - N_c)_{-q_c}$, $\zeta^D < 0$

Seiberg-like dualities of $\mathcal{N}=2$ gauged SQM

Left-mutation: duality for $U(N_c)_{q_c}$ with $\zeta < 0$.



$U(N_c)_{q_c}$, $\zeta < 0$ $\qquad\qquad$ $U(N_2 - N_c)_{-q_c}$, $\zeta^D > 0$

Note the shift of the 1d CS term of the adjacent 'flavour' node, which becomes relevant when we embed this into a larger 1d $\mathcal{N}=2$ quiver.

Duality + trivial wall-crossing (TWC) = triality

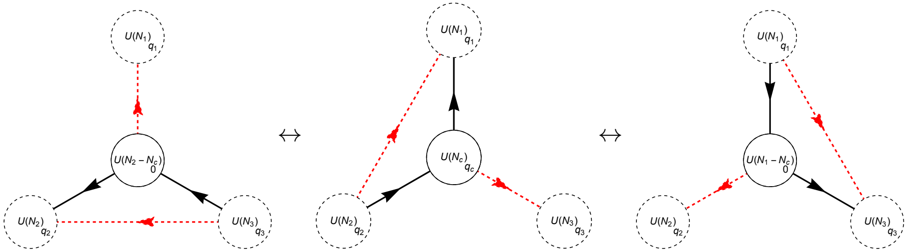
Sometimes wall-crossing is trivial (TWC). We can consider this as an infrared duality as well:

$$U(N_c)_{q_c} \ , \ \zeta > 0 \qquad \longleftrightarrow \qquad U(N_c)_{q_1} \ , \ \zeta < 0$$

A sufficient set of conditions to have trivial wall-crossing is

$$-\mathcal{A} - 1 < q_c < \mathcal{A} + 1 \ , \qquad \mathcal{A} \equiv \frac{1}{2}(N_1 + N_2 - N_3) - N_c \ .$$

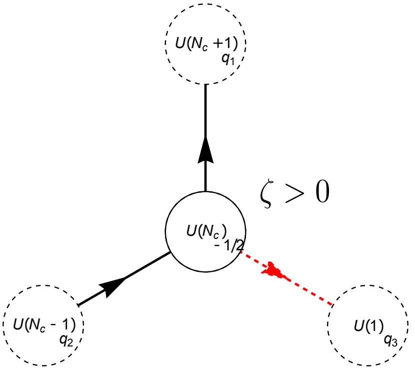
Combining TWC with the mutation dualities, we obtain various types of trialities.
A very special case is $\mathcal{A} = 0$ and $q_c = 0$:



This is the 1d reduction of the 2d $\mathcal{N} = (0, 2)$ triality of [Gadde, Gukov, Putrov, 2013].

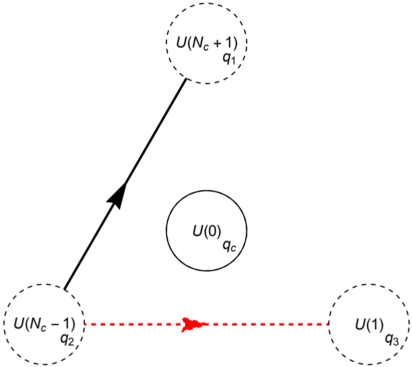
Duality + trivial wall-crossing (TWC) = triality

This explains our earlier example of confinement, which is part of a triality of this type:



$U(N_c)_{-1/2}$, $(N_1, N_2, N_3) = (N_c{+}1, N_c{-}1, 1)$.

$\mathcal{A} = -\frac{1}{2}$ and $q_c = -\frac{1}{2}$ and non-trivial WC.



Free chirals and fermis?

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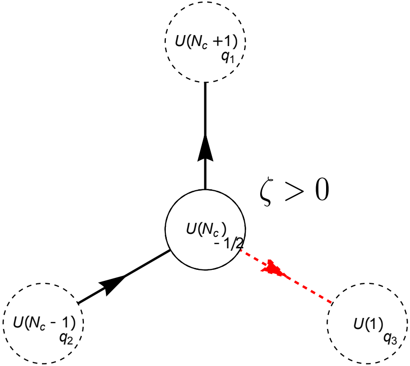

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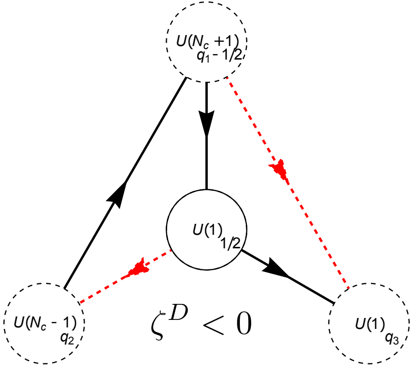

Abelian dual with $\zeta^D < 0$.

$\mathcal{A}^D = \frac{1}{2}$ and $q_c^D = -\frac{1}{2}$: TWC!

Duality + trivial wall-crossing (TWC) = triality

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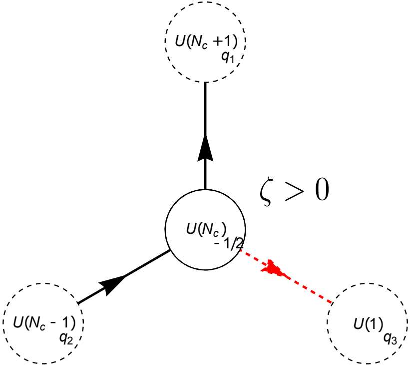

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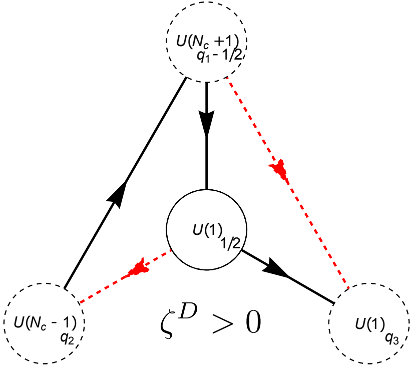
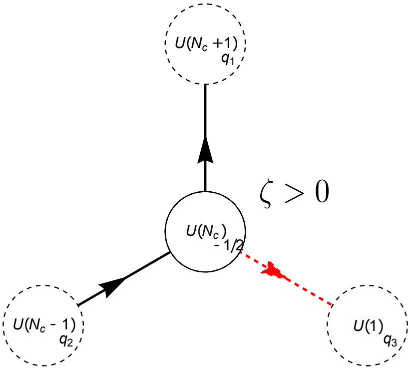


Abelian dual with $\zeta^D > 0$.

$\mathcal{A}^D = \frac{1}{2}$ and $q_c^D = -\frac{1}{2}$: TWC!

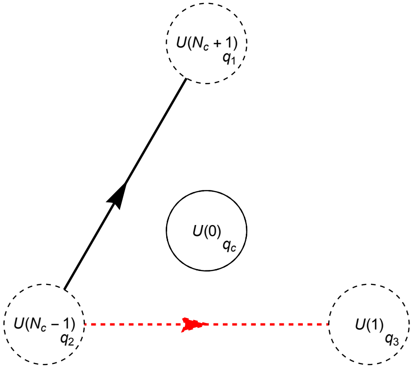
Duality + trivial wall-crossing (TWC) = triality

This explains our earlier example of confinement, which is part of a triality of this type:



$U(N_c)_{-1/2}$, $(N_1, N_2, N_3) = (N_c{+}1, N_c{-}1, 1)$.

$\mathcal{A} = -\frac{1}{2}$ and $q_c = -\frac{1}{2}$ and non-trivial WC.



Free chirals and fermis!

Ground states of Γ-SQCD: Higgs branch perspective

NLSM model phase of Γ-SQCD

Consider $n_1 \geq N_c$. For $\zeta \gg 0$, the 1d GLSM flows to a 1d NLSM onto

$$\mathcal{E}_+ \longrightarrow X_+ \ , \qquad\qquad X_+ = \mathrm{Gr}(N_c, n_1) \ ,$$

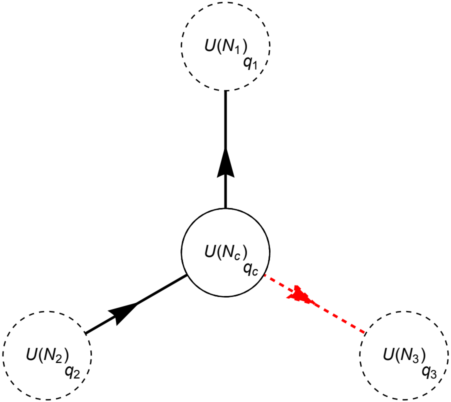
The base is the Grassmannian

$$X_+ \cong \mathbb{C}^{N_c n_1} /\!/_{\zeta} U(N_c) \ .$$

corresponding to VEVs for the fundamental chiral multiplet of maximal rank:

$$\mathrm{rank}(\phi_i^a) = N_c$$

What kind of vector bundle $\mathcal{E}_+$ do we obtain from the remaining fields in the GLSM?



NLSM model phase of Γ-SQCD

First, let us review some basics about vector bundles on Grassmannians. We have the tautological bundle $\mathcal{S}$ and the quotient bundle $\mathcal{Q}$, with the Euler sequence

$$0 \longrightarrow \mathcal{S} \longrightarrow \mathbb{C}^{n_1} \longrightarrow \mathcal{Q} \longrightarrow 0$$

One finds that, for Γ-SQCD:

$$\mathcal{E}_+ = \det(\mathcal{S})^{Q_c} \otimes \Lambda^\bullet((\mathcal{Q}^*)^{\oplus n_2}) \otimes \Lambda^\bullet(\mathcal{S}^{\oplus n_3})$$

- Modified Gauss law $\quad\leftrightarrow\quad$ line bundle $\det(\mathcal{S})^{Q_c}$.
- Fundamental fermi multiplets Λ^a are valued in $\mathcal{S} \rightarrow$ full fermionic Fock space.
- The antifundamental chirals and the gauge-neutral fermis are coupled together by:

  $$W = \phi_i^a \bar{\Gamma}_j^i \tilde{\phi}_a^j$$

  On the Grassmannian, the condition

  $$(E_\Gamma)_i^j = \tilde{\phi}_a^j \phi_i^a = 0$$

  sets $\tilde{\phi} = 0$, and gives mass to all $\tilde{\phi}$'s and most Γ's. The Γ's that survive are orthogonal to $\bar{\phi} \rightarrow$ valued in $\mathcal{Q}^*$.

Ground states from geometry

The ground states of Γ-SQCD are isomorphic to the cohomology of $\mathcal{E}_+$:

$$|\Psi\rangle \qquad \longleftrightarrow \qquad [\Psi] \in H^\bullet(X_+, \mathcal{E}_+)$$

Luckily, this can be computed very explicitly.

Given a vector bundle E of rank n and a SL_n highest weight λ, we have the Schur functor bundle:

$$S_\lambda(E) \qquad \text{with fibers} \quad S_\lambda(\mathbb{C}^n) \cong \mathbb{C}^{\dim(\mathfrak{R}_\lambda)} \ .$$

We have that

$$\Lambda^\bullet(E^{\oplus m}) = \Lambda^\bullet(E \otimes \mathbb{C}^m) = \bigoplus_{\lambda \in Y(n,m)} S_\lambda(E) \otimes S_{\lambda^T}(\mathbb{C}^m) \ ,$$

with $Y(n,m)$ the Young tableaux that fit into a $n \times m$ rectangle.

The bundle (or rather, K-theory class) $\mathcal{E}_+$ can then be written as

$$\mathcal{E}_+ = \bigoplus_{\substack{\lambda \in Y(n_1 - N_c, n_2) \\ \mu \in Y(N_c, n_3)}} (-1)^{|\lambda|+|\mu|} \, \mathcal{F}_{\lambda,\mu}(-Q_c) \otimes S_{\lambda^T}(\mathbb{C}^{n_2}) \otimes S_{\mu^T}(\mathbb{C}^{n_3})$$

with

$$\mathcal{F}_{\lambda,\mu}(-Q_c) \equiv \det(\mathcal{S})^{Q_c} \otimes S_\lambda(\mathcal{Q}^*) \otimes S_\mu(\mathcal{S})$$

Ground states from Borel-Weil-Bott

The Borel-Weil-Bott allows us to compute the cohomology explicitly. For each choice of λ and μ, define the SL_{n_1} weight

$$\omega = [-Q_c - \mu_{N_c}, -Q_c - \mu_{N_c-1}, \cdots, -Q_c - \mu_1, \lambda_1, \lambda_2, \cdots, \lambda_{n_1-N_c}] \ .$$

Let $\rho = (n_1 - 1, \cdots, 1, 0)$ be the Weyl vector. The weight ω is called *regular* if $\omega + \rho$ has no repeated entries, and then there is a unique permutation $\sigma \in S_{n_1}$ to make it a highest weight. Let $l(\sigma)$ be the length of σ. We then have

$$H^k(X_+, \mathcal{F}_{\lambda,\mu}(-Q_c)) = \begin{cases} S_\nu(\mathbb{C}^{n_1}) & \text{if } \omega \text{ is regular and } k = \ell(\sigma) \ , \\ 0 & \text{otherwise} \end{cases}$$

We then find **all the supersymmetric ground states:**

$$H^\bullet(X_+, \mathcal{E}_+) = \bigoplus_{\substack{\lambda \in Y(n_1 - N_c, n_2) \\ \mu \in Y(N_c, n_3) \\ \omega \text{ regular}}} (-1)^{|\lambda|+|\mu|} S_\nu(\mathbb{C}^{n_1}) \otimes S_{\lambda^T}(\mathbb{C}^{n_2}) \otimes S_{\mu^T}(\mathbb{C}^{n_3})$$

Witten index from Borel-Weil-Bott

This directly gives us a formula for the flavoured Witten index

$$\mathrm{I}_W[N_c, q_c, \boldsymbol{n}^F]_+^\Gamma = \chi_T(X_+, \mathcal{E}_+) \equiv \sum_{k=0}^{\dim_T(X_+)} (-1)^k \dim_T H^k(X_+, \mathcal{E}_+) \ .$$

The 'equivariant dimension' of a Schur functor-vector space is the Schur polynomial

$$\dim_T(S_\lambda(\mathbb{C}^{n_I})) \equiv \mathfrak{s}_\lambda(y_I)$$

Therefore:

$$\mathrm{I}_W^+ = \sum_{\substack{\lambda \in Y(n_1 - N_c, n_2) \\ \mu \in Y(N_c, n_3) \\ \omega \text{ regular}}} (-1)^{|\lambda| + |\mu| + \ell(\sigma)} \, \mathfrak{s}_\nu(y_1^{-1}) \mathfrak{s}_{\lambda^T}(y_2) \mathfrak{s}_{\mu^T}(y_3^{-1})$$

Unlike the JK-residue formula, this is an explicit (finite) polynomial!

The other NLSM model phase of Γ-SQCD

Now, consider $\zeta < 0$ in the Γ-SQCD theory.

the target geometry becomes:

$$\mathcal{E}_- \longrightarrow X_- \ , \qquad\qquad X_- = \mathrm{Gr}(N_c, n_2)$$

with the bundle:

$$\mathcal{E}_- = \det(\mathcal{S}^*)^{Q_c} \otimes \Lambda^\bullet((\mathcal{Q}^*)^{\oplus n_1}) \otimes \Lambda^\bullet((\mathcal{S}^*)^{\oplus n_3})$$

This can be rewriten as

$$\mathcal{E}_- = (-1)^{n_3 N_c} (\det y_3)^{-N_c} \det(\mathcal{S})^{-(Q_c+n_3)} \otimes \Lambda^\bullet((\mathcal{Q}^*)^{\oplus n_1}) \otimes \Lambda^\bullet(\mathcal{S}^{\oplus n_3}) \ .$$

Mutations from Grassmannian duality

We can now understand our mutation dualities from geometry, following [Jia, Sharpe, Wu, 2014]. Consider the right mutation for definiteness.

There is a classic Grassmannian duality, which is the isomorphism:

$$\varphi \ : \ X = \mathrm{Gr}(N_c, n_1) \ \longrightarrow \ X_D = \mathrm{Gr}(n_1 - N_c, n_1) \ .$$

At the level of the bundles:

$$\varphi^*(\mathcal{S}_D) = \mathcal{Q}^* \ , \qquad \qquad \varphi^*(\mathcal{Q}_D) = \mathcal{S}^* \ ,$$

Under the mapping of parameters for our SQCD duality, we precisely find

$$\varphi^*(\mathcal{E}_-^D) = \mathcal{E}_+ \ .$$

Hence the ground states are indeed isomorphic.

Importantly, this proves the infrared duality beyond matching the indices!

Perturbative ground states of Γ-SQCD: Coulomb branch perspective

Ground states on the Coulomb branch

Consider a different scaling of the GLSM parameters: [Hori, Kim, Yi, 2014]

$$|\zeta| \to 0 \;, \qquad g^2 \to \infty \;, \qquad M_C \equiv g^2 \zeta^2 \text{ fixed.}$$

At energies below the Coulomb-branch scale M_C, we have a Coulomb-branch effective theory. The analysis of the perturbative ground states in this phase is done in three steps:

1. At $\sigma \neq 0$, consider the free theory of chiral and fermi mutliplets, then impose Gauss law to obtain a subspace of the free-theory Fock space:

$$\mathcal{F}^0 \quad \longrightarrow \quad \mathcal{F}^0_{\text{phys}} = \left\{ |\Psi\rangle \in \mathcal{F}^0 \mid \mathbf{J}|\Psi\rangle = 0 \right\} \;.$$

2. For each would be ground state $|\Psi\rangle \in \mathcal{F}^0_{\text{phys}}$, analyse the effective theory of σ and the gaugino partners:

$$L_{\text{eff}} = \frac{1}{2}\dot{\sigma}^2 - i\bar{\lambda}\dot{\lambda} - \frac{1}{2}(h'(\sigma))^2 + h''(\sigma)\bar{\lambda}\lambda \;,$$

   the classic model analysed in [Witten, 1981].

3. Treat the E-term interactions in perturbation theory (first order is enough).

Ground states on the Coulomb branch

The effective theory superpotential is given by

$$h(\sigma) = -\tilde{\zeta}\sigma + \frac{1}{2}\sum_i N_{\phi_i} q_i \,\mathrm{sign}(q_i\sigma + m_i) \log|q_i\sigma + m_i| \ ,$$

from integrating out massive chiral multiplets.

Consider the wavefunctions

$$\Sigma(\sigma) = e^{h(\sigma)} \ , \qquad \tilde{\Sigma}(\sigma) = e^{-h(\sigma)} \ .$$

We find perturbative ground states near each critical point of $h(\sigma)$:

$$|\Sigma\rangle \ \text{ is normalisable} \quad \Leftrightarrow \quad h''(\sigma_*) < 0 \ ,$$
$$\bar{\lambda}|\tilde{\Sigma}\rangle \ \text{ is normalisable} \quad \Leftrightarrow \quad h''(\sigma_*) > 0 \ .$$

In this way, we find all perturbative ground states of the SQCD theory in its Coulomb phase, together with contributions from free Γ fields. The E-term interaction lifts an infinite number of boson-fermion pairs (in general).

Ground states on the Coulomb branch

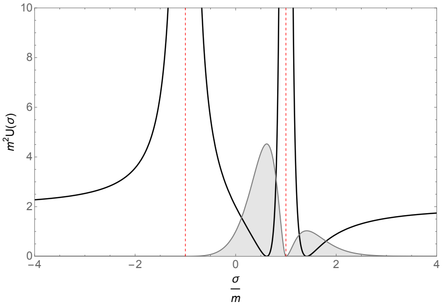
We end up with **a finite number of perturbative ground states** localised near critical points

$$\sigma = \sigma_*^{(\alpha)} \ , \qquad h'(\sigma_*^{(\beta)}) = 0 \ .$$

In general, the true ground states are fewer in number. Typically, we find some fermion-boson pairs of perturbative ground states

$$|\Sigma; \sigma_*^{(\alpha)}\rangle \ , \qquad \bar{\lambda}|\Sigma; \sigma_*^{(\beta)}\rangle \ , \qquad \alpha \neq \beta$$

which have all the same quantum numbers otherwise. (They cancel out from the index.)



We note that *if we (unphysically) smoothen out the potential*

$$V(\sigma) = (h'(\sigma))^2$$

near locations of massless chirals by "capping" at around $V = M_C$, then there are instantons that lift the paired states. [Witten, 1981]

In this way, we can exactly match the Higgs branch description! (For $N_c = 1$ at least...)

A simple example

Let us illustrate the above with a concrete example.

Consider the $U(1)_{q_c}$ theory with $(n_1, n_2, 0) = (2, 1, 0)$, so $Q_c = q_c + \frac{1}{2} \in \mathbb{Z}$.
The geometry for $\zeta > 0$ is simply

$$X_+ = \mathbb{P}^1 \ ,$$

and since $\mathcal{S} = \mathcal{O}(-1) \cong \mathcal{Q}^*$ in this case, we have

$$\begin{aligned} \mathcal{E}_+ \quad &= \mathcal{O}(-Q_c) \otimes \Lambda^\bullet(\mathcal{O}(-1)) \\ &= \mathcal{O}(-Q_c) - \mathcal{O}(-Q_c - 1) \end{aligned}$$

Note that

$$H^\bullet(\mathbb{P}^1, \mathcal{O}(l)) \cong \begin{cases} H^0(\mathbb{P}^1, \mathcal{O}(l)) \cong \mathrm{Sym}^l(\mathbb{C}^2) & \text{if } l \geq 0 \ , \\ 0 & \text{if } l = -1 \ , \\ H^1(\mathbb{P}^1, \mathcal{O}(l)) \cong \mathrm{Sym}^{-l-2}(\mathbb{C}^2) & \text{if } l \leq -2 \ . \end{cases}$$

We thus directly read off the ground states and index from the geometry.

A simple example

Let us consider the Coulomb branch perspective in the specific case $Q_c = 2$:

$$\mathcal{E}_+ = \mathcal{O}(-2) - \mathcal{O}(-3)$$

giving us one fermionic state and two bosonic states:

$$H^\bullet(\mathbb{P}^1, \mathcal{E}_+) \cong -H^1(\mathbb{P}^1, \mathcal{O}(-2)) + H^1(\mathbb{P}^1, \mathcal{O}(-3)) \cong -\mathbb{C} + \mathbb{C}^2$$

with the index

$$\mathrm{I}_{W+} = -y_{1,1}y_{1,2} + (y_{1,1} + y_{1,2})y_{2,1}$$

On the Coulomb branch, we need to introduce a mass term for the fundamental and antifundamental, so that:

$$V(\sigma) = \left(-\zeta + \frac{N_\phi}{2|\sigma - m|} - + \frac{N_{\tilde{\phi}}}{2|\sigma + m|}\right)^2$$

After perturbation by the E-term, we find 5 perturbative states on the Coulomb branch:

$$\mathrm{I}_{W+}^{(\sigma<-m)} = -y_{1,1}y_{1,2} + (y_{1,1} + y_{1,2})y_{2,1} - y_{2,1}^2$$
$$\mathrm{I}_{W+}^{(-m<\sigma<m)} = y_{2,1}^2$$

The pair that cancels out in the index is conjecturally lifted by SQM instantons.

A simple example and comment on the JK residues

In this example, the JK residue formula reads:

$$\mathrm{I}_{W+} = -(1 - y_{2,1} y_{1,1}^{-1})(1 - y_{2,1} y_{1,2}^{-1}) \oint_{\mathrm{JK}} \frac{dx}{2\pi i} \frac{x}{(1 - x y_{1,1}^{-1})(1 - x y_{1,2}^{-1})(1 - x^{-1} y_{2,1})}$$

where we pick the poles at $x = y_{1,i}$.

We can find the perturbative states by deforming the contour to pick the other residues:

- Residue at $x = \infty$ ($\sigma \to -\infty$): we find $\mathrm{I}_{W+}^{(\sigma < -m)}$;
- Residue at $x = y_{2,1}$: we find $\mathrm{I}_{W+}^{(-m < \sigma < m)}$;
- Residue at $x = 0$ ($\sigma \to -\infty$) vanishes.

This lesson generalises: Perturbative states can be obtained by taking residues corresponding to the location $\sigma = \sigma_*$ of the perturbative ground state!

- Not a coincidence: the 'other' residues are the naive ones we would have chosen simply to impose Gauss' law on perturbative states;
- In the sense, the 'mysterious' JK residue knows exactly about what appears to us as non-perturbative corrections to the Coulomb-branch description.

Another example: $U(1)_0$ with $(n_1, n_2, n_3) = (2,2,2)$: a triality

In this example, we have $Q_c = -1$, no wall-crossing, and the states:

(q,p) $\nu = [\nu_1, \nu_2]$	Higgs phase	Coulomb phase
$(0,0)$ $\nu = [1,0]$	$H^0_T(\mathbb{P}^1, \mathcal{O}(1)) \cong \mathbb{C}^2_{y_1^{-1}}$	$a^\dagger_{1,i}\|0\rangle$
$(1,0)$ $\nu = [1,1]$	$\left(H^0_T(\mathbb{P}^1, \mathcal{O}) \cong \Lambda^2(\mathbb{C}^2_{y_1^{-1}})\right) \otimes \mathbb{C}^2_{y_2}$	$\left(\Gamma^j_1 a^\dagger_{1,2} - \Gamma^j_2 a^\dagger_{1,1}\right)\|0\rangle$
$(0,1)$ $\nu = [0,0]$	$\left(H^0_T(\mathbb{P}^1, \mathcal{O}) \cong 1\right) \otimes \mathbb{C}^2_{y_3^{-1}}$	$\eta_{3,k}\|0\rangle$
$(2,1)$ $\nu = [1,1]$	$\left(H^1_T(\mathbb{P}^1, \mathcal{O}(-2)) \cong \Lambda^2(\mathbb{C}^2_{y_1^{-1}})\right)$ $\otimes \Lambda^2(\mathbb{C}^2_{y_2}) \otimes \mathbb{C}^2_{y_3^{-1}}$	$\left(\Gamma^2_2 a^\dagger_{1,1} a^\dagger_{2,1} - \Gamma^1_2 a^\dagger_{1,1} a^\dagger_{2,2}\right.$ $\left. - \Gamma^2_1 a^\dagger_{1,2} a^\dagger_{2,1} + \Gamma^1_1 a^\dagger_{1,2} a^\dagger_{2,2}\right)\eta_{3,k}\|0\rangle$
$(1,2)$ $\nu = [0,0]$	$\left(H^1_T(\mathbb{P}^1, \mathcal{O}(-2)) \cong 1\right) \otimes \mathbb{C}^2_{y_2} \otimes \Lambda^2(\mathbb{C}^2_{y_3^{-1}})$	$a^\dagger_{2,j}\eta_{3,1}\eta_{3,2}\|0\rangle$
$(2,2)$ $\nu = [1,0]$	$\left(H^1_T(\mathbb{P}^1, \mathcal{O}(-3)) \cong \mathbb{C}^2_{y_1^{-1}}\right)$ $\otimes \Lambda^2(\mathbb{C}^2_{y_2}) \otimes \Lambda^2(\mathbb{C}^2_{y_3^{-1}})$	$\left(\Gamma^2_i a^\dagger_{2,1} - \Gamma^1_i a^\dagger_{2,2}\right)\eta_{3,1}\eta_{3,2}\|0\rangle$

Schubert line defects

Schubert line defects

Coming back to our 3d $\mathcal{N} = 2$ quiver for partial flag manifolds:

We are interested in engineering a 1d $\mathcal{N} = 2$ quiver coupled to the 3d quiver that corresponds to Schubert classes over the partial flag manifold $X = \mathrm{Fl}(\boldsymbol{k}; n)$:

$$\mathscr{L}_w \qquad \to \qquad [\mathscr{L}_w] = [\mathcal{O}_w] \in \mathrm{QK}(X)$$

Recall the definition

$$\mathrm{Fl}(\boldsymbol{k}; n) := \{V_\bullet = (0 \subset V_1 \subset V_2 \subset \cdots \subset V_s \subset \mathbb{C}^n) \mid \dim(V_\ell) = k_\ell\}$$

The flag manifold admits a decomposition in disjoint open sets, the Schubert cells:

$$\mathrm{Fl}(\mathbf{k}; n) = \bigsqcup_{w \in \mathrm{W}^{(\mathbf{k};n)}} X_w^\circ$$

Their closure are the Schubert varieties X_w. The **Schubert class** is the K-theory class corresponding to the structure sheaf

$$\mathcal{O}_w := \mathcal{O}_{X_w}$$

This is a coherent sheaf which is not locally free (*i.e.* not a vector bundle). So it cannot be obtained as a Wilson loop. Schubert classes are the natural objects in many K-theory computation, just like in cohomology (Schubert calculus...).

Schubert line defects

An alternative definition of the partial flag manifold is as a quotient of complex groups:

$$X \cong GL(n)/P\ , \qquad P \supset U(k_1) \times U(k_2 - k_1) \times \cdots \times U(k_s - k_{s-1}) \times U(n - k_s)$$

Let W_P be the Weyl group of the parabolic subgroup P. The Schubert varieties in the partial flag manifold are defined as follows.

- Fix a reference complete flag

$$E_\bullet = (E_1 \subset E_2 \subset \cdots \subset E_n \equiv \mathbb{C}^n)$$

- Choose some 'maximal length' permutation of n elements:

$$w \in \mathrm{W}^P := S_n/\mathrm{W}_P$$

- Define the matrix of integers

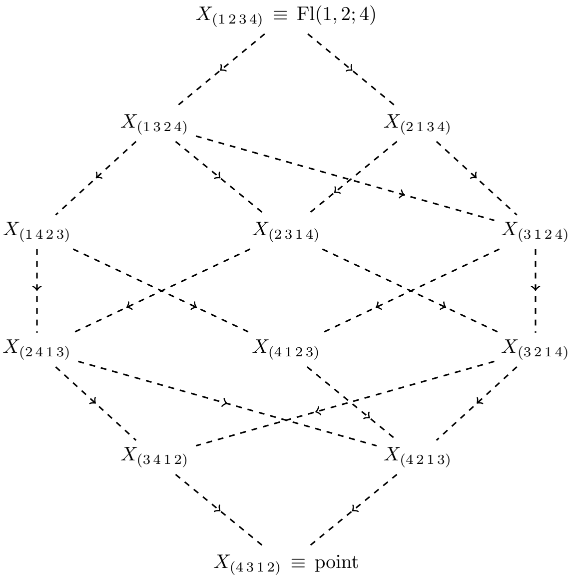
$$\mathsf{r}_{k_\ell,j} := \#\{\, l \leq k_\ell \,|\, n + 1 - w(l) \leq j \,\}$$

- The Schubert variety is then defined by the conditions:

$$X_w := \{\, F_\bullet \in \mathrm{Fl}(\boldsymbol{k}; n) \,|\, \dim(F_{k_\ell} \cap E_j) \geq \mathrm{r}_{k_\ell,j}\ , \quad \forall 1 \leq \ell \leq s\,, \forall 1 \leq j \leq n \,\}$$

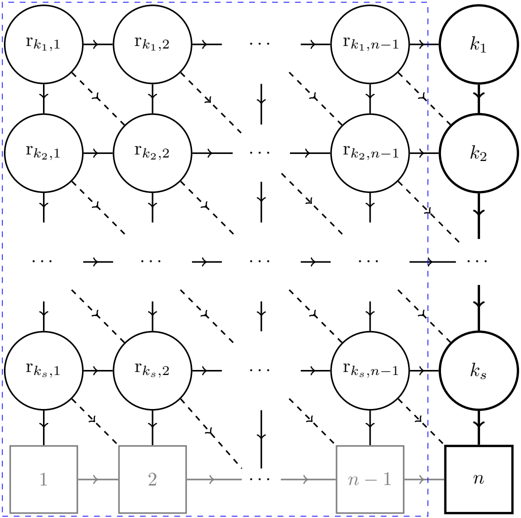
Schubert line defects

Example: $X = \mathrm{Fl}(1,2;4)$ has the Schubert varieties:



Schubert line defects

The Schubert line defect $\mathscr{L}_w$ is defined in terms of w and of the data defining X_w:



Schubert line defects

By a direct analysis of the 1d semi-classical vacuum equation, one can show that the insertion of $\mathscr{L}_w$ at a point $x \in \Sigma$ restricts the GLSM (quasi)map (that is the 3d chiral multiplet VEVs):

$$\phi \ : \Sigma \ \longrightarrow \ X$$

to the Schubert variety:

$$\phi|_x \ : x \ \longrightarrow \ X_w$$

Moreover, the defect actually engineers an algebraic resolution of the Schubert variety:

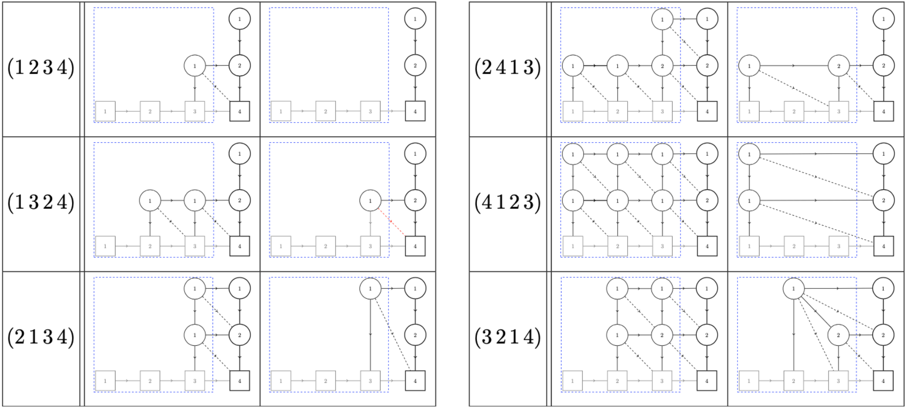
$$\pi \ : \ \tilde{X}_w \to X_w$$

This connects nicely to recent mathematical results on such resolutions [Cibotaru, 2018; Iezzi, 2025].

$$
\begin{array}{ccccccccc}
V_{1,1} & \longrightarrow & V_{1,2} & \longrightarrow & \cdots & \longrightarrow & V_{1,n-1} & \longrightarrow & F_{k_1} \\
\downarrow & \circlearrowleft & \downarrow & & & & \downarrow & \circlearrowleft & \downarrow \\
V_{2,1} & \longrightarrow & V_{2,2} & \longrightarrow & \cdots & \longrightarrow & V_{2,n-1} & \longrightarrow & F_{k_2} \\
\downarrow & & \downarrow & & & & \downarrow & & \downarrow \\
\vdots & & \vdots & & \ddots & & \vdots & & \vdots \\
\downarrow & & \downarrow & & & & \downarrow & & \downarrow \\
V_{s,1} & \longrightarrow & V_{s,2} & \longrightarrow & \cdots & \longrightarrow & V_{s,n-1} & \longrightarrow & F_{k_s} \\
\downarrow & \circlearrowleft & \downarrow & & & & \downarrow & \circlearrowleft & \downarrow \\
E_1 & \longrightarrow & E_2 & \longrightarrow & \cdots & \longrightarrow & E_{n-1} & \longrightarrow & \mathbb{C}^n
\end{array}
$$

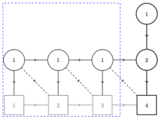
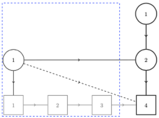
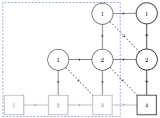
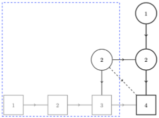
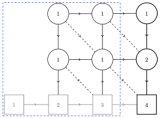
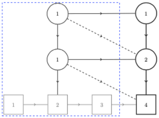
Schubert line defects

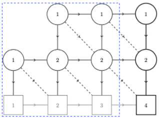
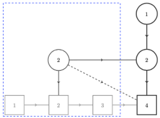
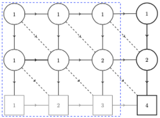
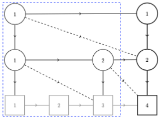
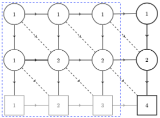
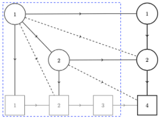
Many of the 1d quivers can simplified using the 1d mutation dualities:



Schubert line defects

Many of the 1d quivers can simplified using the 1d mutation dualities:

(1 4 2 3)		
(2 3 1 4)		
(3 1 2 4)		

(3 4 1 2)		
(4 2 1 3)		
(4 3 1 2)		

Parabolic quantum polynomials for partial flag manifolds

Twisted chiral ring and Whitney presentation

The twisted chiral ring of the 2d KK theory can be computed explicitly from a one-loop exact effective twisted superpotential on the 2d Coulomb branch:

$$\exp \frac{\partial \mathcal{W}}{\partial \log x^{(\ell)}_{a_\ell}} = 1 \ , \qquad \ell = 1, \cdots , s \ , \qquad a_\ell = 1, \cdots , k_\ell \ ,$$

where $x^{(\ell)}_{a_\ell}$ denote the K-theoretic Chern roots of the tautological vector bundle $\mathcal{S}_\ell$. Let the variable x_i to denote the Chern roots of the quotient bundles $\mathcal{Q}_\ell$:

$$0 \longrightarrow \mathcal{S}_\ell \longrightarrow \mathcal{S}_{\ell+1} \longrightarrow \mathcal{Q}_{\ell+1} \longrightarrow 0$$

More precisely, we have the Chern roots $\widehat{x}^{(\ell)} \equiv \{ x_{k_{\ell-1}+1} , \cdots , x_{k_\ell} \}$. Then the ring equations are

$$\sum_{a+b=c} e_a(x^{(\ell)})\, e_b(\widehat{x}^{(\ell+1)}) = e_c(x^{(\ell+1)}) - \frac{q_\ell}{1-q_\ell}\, x_{k_\ell+1} \cdots x_{k_{\ell+1}} \left(e_{c-k_{\ell+1}+k_\ell}(x^{(\ell)}) - e_{c-k_{\ell+1}+k_\ell}(x^{(\ell-1)}) \right) \ ,$$

where q_ℓ are the 'quantum parameters' — the FI parameters for $U(k_\ell)$, $q_\ell \sim e^{-\xi_\ell}$.

Twisted chiral ring and Whitney presentation

Define the field of parameters

$$\mathbb{K} \equiv \mathbb{Z}(y_1, \cdots, y_n, q_1, \cdots, q_s) \ .$$

Here, y_j correspond to $GL(n)$ symmetry complexified fugacities (exponentiated twisted masses) in the 3d gauge theory. Mathematically, they are equivariant deformation parameters.

The Whitney presentation of the equivariant quantum K-theory ring take the form:

[Gu, Mihalcea, Sharpe, Xu, Zhang, Zou, 2023]

$$\mathrm{QK}_T(\mathrm{Fl}(\boldsymbol{k}; n)) \cong \mathbb{K}[x^{(\bullet)}, x_1, \cdots, x_n]^{\mathrm{W}_G \times \mathrm{W}_P} / (\mathrm{QI})$$

with (QI) the ideal corresponding to the chiral ring relations.

The key mathematical physics question for us:

How do we represent $[\mathcal{O}_w]$ as a polynomial in the Whitney presentation?

Parabolic Whitney polynomials

Mathematical answer: There exists polynomials that do the job, defined very recently in [Amini, Huq-Kuruvilla, Mihalcea, Orr, Xu, 2025]. We dubbed them the parabolic Whitney polynomials:

$$\mathfrak{W}_{w}^{(\boldsymbol{k};n)}(x^{(\bullet)},y) := \pi^{(y)}_{w^{-1}w_0^{\boldsymbol{k}}}\,\mathfrak{W}_{w_0^{\boldsymbol{k}}}^{(\boldsymbol{k};n)}(x^{(\bullet)},y) \cong \mathrm{ch}(\mathcal{O}_w)$$

They are defined recursively starting from the point class $[X_{w_0^{\boldsymbol{k}}}]$:

$$\mathfrak{W}_{w_0^{\boldsymbol{k}}}^{(\boldsymbol{k};n)}(x^{(\bullet)},y) = \prod_{\ell=1}^{s}\prod_{j=k_\ell}^{k_{\ell+1}-1}\prod_{a_\ell=1}^{k_\ell}\left(1-\frac{x^{(\ell)}_{a_\ell}}{y_{n-j}}\right),$$

Example: For $X=\mathrm{Fl}(1,2;4)$, some examples of Whitney polynomials:

$$\mathfrak{W}^{(1,2;4)}_{(1234)} = 1\,,$$

$$\mathfrak{W}^{(1,2;4)}_{(1324))} = 1-\frac{x_1^{(2)}x_2^{(2)}}{y_1y_2}\,,$$

$$\mathfrak{W}^{(1,2;4)}_{(2134)} = 1-\frac{x_1^{(1)}}{y_1}\,,$$

$$\mathfrak{W}^{(1,2;4)}_{(1423)} = 1-\frac{x_2^{(2)}x_1^{(2)}}{y_1y_2}-\frac{x_1^{(2)}x_2^{(2)}}{y_1y_3}-\frac{x_1^{(2)}x_2^{(2)}}{y_2y_3}+\frac{x_1^{(2)}x_2^{(2)\,2}}{y_1y_2y_3}+\frac{x_1^{(2)\,2}x_2^{(2)}}{y_1y_2y_3}\,.$$

Parabolic Whitney polynomials are Witten indices

Physics answer: The natural polynomial in $x^{(\bullet)}$ that should represent the Schubert class is the 1d $\mathcal{N}=2$ flavoured Witten index of the 1d defect theory:

$$\mathcal{I}_w^{(1\mathrm{d})}(x^{(\bullet)}, y) = \mathrm{Tr}\left((-1)^{\mathrm{F}}\, x^{Q_G} y^{Q_F}\right)$$

This can be computed by supersymmetric localisation: [Hori, Kim, Yi, 2014]

$$\mathcal{I}_w^{(1\mathrm{d})}(x^{(\bullet)}, y) \;=\; \oint_{\mathrm{JK}} (\mathrm{dM})\, \mathrm{Z}_{\mathrm{chiral}}^{\mathrm{ver}}\, \mathrm{Z}_{\mathrm{chiral}}^{\mathrm{hor}}\, \mathrm{Z}_{\mathrm{Fermi}}^{\mathrm{black}}\, \mathrm{Z}_{\mathrm{Fermi}}^{\mathrm{red}}$$

$$\begin{aligned}
\mathrm{Z}_{\mathrm{chiral}}^{\mathrm{ver}} &:= \prod_{\ell=1}^{s-1}\prod_{i=1}^{n-1}\prod_{\alpha=1}^{\mathrm{r}_{k_\ell,i}}\prod_{\beta=1}^{\mathrm{r}_{k_{\ell+1},i}} \left(1 - \frac{z_\alpha^{(k_\ell,i)}}{z_\beta^{(k_{\ell+1},i)}}\right)^{-1}, \\
\mathrm{Z}_{\mathrm{chiral}}^{\mathrm{hor}} &:= \prod_{\ell=1}^{s}\left[\prod_{\alpha=1}^{\mathrm{r}_{k_\ell,n-1}}\prod_{a=1}^{k_\ell}\left(1 - \frac{z_\alpha^{(k_\ell,n-1)}}{x_a^{(\ell)}}\right)^{-1}\prod_{i=1}^{n-2}\prod_{\beta=1}^{\mathrm{r}_{k_\ell,i}}\prod_{\gamma=1}^{\mathrm{r}_{k_\ell,i+1}}\left(1 - \frac{z_\beta^{(k_\ell,i)}}{z_\gamma^{(k_\ell,i+1)}}\right)^{-1}\right], \\
\mathrm{Z}_{\mathrm{Fermi}}^{\mathrm{black}} &:= \prod_{\ell=1}^{s-1}\left[\prod_{\gamma=1}^{\mathrm{r}_{k_\ell,n-1}}\prod_{a=1}^{k_{\ell+1}}\left(1 - \frac{z_\gamma^{(k_\ell,n-1)}}{x_a^{(\ell+1)}}\right)\prod_{i=1}^{n-1}\prod_{\alpha=1}^{\mathrm{r}_{k_\ell,i}}\prod_{\beta=1}^{\mathrm{r}_{k_{\ell+1},i+1}}\left(1 - \frac{z_\alpha^{(k_\ell,i)}}{z_\beta^{(k_{\ell+1},i+1)}}\right)\right], \\
\mathrm{Z}_{\mathrm{Fermi}}^{\mathrm{red}} &:= \prod_{i=1}^{n-1}\prod_{\alpha=1}^{\mathrm{r}_{k_s,i}}\left(1 - \frac{z_\alpha^{(k_s,i)}}{y_{n-i}}\right),
\end{aligned}$$

(2)

Parabolic Whitney polynomials are Witten indices

Our key conjecture is that **Witten = Whitney** — that is,

$$\mathcal{I}^{(1\mathrm{d})}[\mathscr{L}_w] = \mathfrak{W}_w^{(\boldsymbol{k};n)}$$

We checked it explicitly in all examples with $n \leq 4$, and in some other cases as well.

- The Whitney polynomials do not depend on q_ℓ. They represent classes in both $\mathrm{K}_T(X)$ and $\mathrm{QK}_T(X)$. The quantum parameters appear in the ring relations.
- We can use the ring relations to solve for $x^{(\bullet)}$ in terms of the n variables x_i. Substitution into the Whitney polynomials gives us new representatives in the so-called Toda presentation of the QK ring

$$\mathrm{QK}_T(\mathrm{Fl}(\boldsymbol{k};n)) \cong \mathbb{K}[x_1, \cdots, x_n]^{\mathrm{W}_P}/(\tilde{\mathrm{Q}}\mathrm{I}) \ .$$

These new polynomials **depend on q_ℓ explicitly.** We called them the parabolic quantum Grothendieck polynomials. They are new mathematical objects, which subsumes many previously known cases (for complete flags and Grassmannians). [CC, Gu, Khlaif, Sharpe, Zhang, Zou, 2026]

Parabolic Whitney polynomials are Witten indics

Example: Again with $X = \mathrm{Fl}(1,2;4)$:

$(1\,2\,4\,3)$	$1 - \frac{x_1^{(2)} x_2^{(2)} x_3^{(2)}}{y_1 y_2 y_3}$	$1 - \frac{x_1 x_2 x_3}{(1-q_1)\, y_1 y_2 y_3}$
$(1\,3\,2\,4)$	$1 - \frac{x_1^{(1)} x_2^{(1)}}{y_1 y_2}$	$1 - \frac{x_1 x_2}{y_1 y_2}$
$(1\,3\,4\,2)$	$1 - \frac{x_1^{(2)} x_2^{(2)}}{y_1 y_2} - \frac{x_2^{(2)} x_3^{(2)}}{y_1 y_2} - \frac{x_1^{(2)} x_3^{(2)}}{y_1 y_2} + \frac{x_1^{(2)} x_2^{(2)} x_3^{(2)}}{y_1^2 y_2} + \frac{x_1^{(2)} x_2^{(2)} x_3^{(2)}}{y_1 y_2^2}$	$1 - \frac{x_1 x_2}{y_1 y_2} - \frac{x_2 x_3}{(1-q_1)\, y_1 y_2} - \frac{x_1 x_3}{(1-q_1)\, y_1 y_2} + \frac{x_1 x_2 x_3}{(1-q_1)\, y_1^2 y_2} + \frac{x_1 x_2 x_3}{(1-q_1)\, y_1 y_2^2}$
$(1\,4\,2\,3)$	$1 - \frac{x_1^{(1)} x_2^{(1)}}{y_1 y_2} - \frac{x_1^{(1)} x_2^{(1)}}{y_1 y_3} - \frac{x_1^{(1)} x_2^{(1)}}{y_2 y_3} + \frac{x_1^{(1)} x_2^{(1)} x_1^{(2)}}{y_1 y_2 y_3} + \frac{x_1^{(1)} x_2^{(1)} x_2^{(2)}}{y_1 y_2 y_3} + \frac{x_1^{(1)} x_2^{(1)} x_3^{(2)}}{y_1 y_2 y_3} - \frac{x_1^{(2)} x_2^{(2)} x_3^{(2)}}{y_1 y_2 y_3}$	$1 - \frac{x_1 x_2}{y_1 y_2} - \frac{x_1 x_2}{y_1 y_3} - \frac{x_1 x_2}{y_2 y_3} + \frac{x_1^2 x_2}{y_1 y_2 y_3} + \frac{x_1 x_2^2}{y_1 y_2 y_3} - \frac{q_1 x_1 x_2 x_3}{(1-q_1) y_1 y_2 y_3}$
$(2\,3\,1\,4)$	$1 - \frac{x_1^{(1)}}{y_1} - \frac{x_2^{(1)}}{y_1} + \frac{x_1^{(1)} x_2^{(1)}}{y_1^2}$	$1 - \frac{x_1}{y_1} - \frac{x_2}{y_1} + \frac{x_1 x_2}{y_1^2}$
$(1\,4\,3\,2)$	$1 - \frac{x_1^{(1)} x_2^{(1)}}{y_1 y_3} - \frac{x_1^{(1)} x_2^{(1)}}{y_2 y_3} - \frac{x_1^{(2)} x_2^{(2)}}{y_1 y_2} - \frac{x_1^{(2)} x_3^{(2)}}{y_1 y_2} - \frac{x_2^{(2)} x_3^{(2)}}{y_1 y_2} + \frac{x_1^{(1)} x_2^{(1)} x_1^{(2)}}{y_1 y_2 y_3} + \frac{x_1^{(1)} x_2^{(1)} x_2^{(2)}}{y_1 y_2 y_3} + \frac{x_1^{(1)} x_2^{(1)} x_3^{(2)}}{y_1 y_2 y_3} + \frac{x_1^{(2)} x_2^{(2)} x_3^{(2)}}{y_1^2 y_2} + \frac{x_1^{(2)} x_2^{(2)} x_3^{(2)}}{y_1 y_2^2} - \frac{x_1^{(1)} x_2^{(1)} x_1^{(2)} x_2^{(2)} x_3^{(2)}}{y_1^2 y_2^2 y_3}$	$1 - \frac{x_1 x_2}{y_1 y_3} - \frac{x_1 x_2}{y_2 y_3} - \frac{x_1 x_2}{y_1 y_2} - \frac{x_1 x_3}{(1-q_1)\, y_1 y_2} - \frac{x_2 x_3}{(1-q_1)\, y_1 y_2} + \frac{x_1^2 x_2}{y_1 y_2 y_3} + \frac{x_1 x_2^2}{y_1 y_2 y_3} + \frac{x_1 x_2 x_3}{y_1 y_2 y_3} + \frac{x_1 x_2 x_3}{(1-q_1)\, y_1^2 y_2} + \frac{x_1 x_2 x_3}{(1-q_1)\, y_1 y_2^2} - \frac{x_1^2 x_2^2 x_3}{(1-q_1)\, y_1^2 y_2^2 y_3}$
$(2\,3\,4\,1)$	$1 - \frac{x_1^{(2)}}{y_1} - \frac{x_2^{(2)}}{y_1} - \frac{x_3^{(2)}}{y_1} + \frac{x_1^{(2)} x_2^{(2)}}{y_1^2} + \frac{x_1^{(2)} x_3^{(2)}}{y_1^2} + \frac{x_2^{(2)} x_3^{(2)}}{y_1^2} - \frac{x_1^{(2)} x_2^{(2)} x_3^{(2)}}{y_1^3}$	$1 - \frac{x_1}{y_1} - \frac{x_2}{y_1} - \frac{x_3}{y_1} + \frac{x_1 x_2}{y_1^2} + \frac{x_1 x_3 + x_2 x_3}{(1-q_1) y_1^2} - \frac{x_1 x_2 x_3}{(1-q_1) y_1^3}$

Summary and outlook

Summary:

- $\mathcal{N} = 2$ gauged SQM remains under-studied, especially the 'chiral' effects such as ones from the 1d Chern–Simons levels. Clear motivation from enumerative geometry.
- We established the basic duality moves for 1d $\mathcal{N} = 2$ unitary gauge groups by a rigorous analysis of the SQM.
- We constructed line defects in 3d $\mathcal{N} = 2$ GLSM for partial flag varieties X, which have the interpretation as Schubert classes in the (quantum) K-theory of X.
- Our main result (conjecture), Witten=Whitney, allowed us to unify various mathematical results about K-theoretic Schubert calculus.

Outlook:

- Prove the conjecture? (Role of Demazure operators as difference operators acting on lines, in physics.)
- Many obvious generalisations of the 1d dualities to explore:
  - general quivers;
  - non-unitary gauge groups (Spin(n), Sp$(2n)$,...);
  - 1d duality from 3d duality?
- Further extensions of our Schubert line defects to engineer motivic Chern classes?
- Other flag varieties (not of type A)?
- Generalisation to elliptic cohomology? (4d A-model.)