Twisted D-branes and TQFTs valued in Calabi-Yau categories

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Based on WIP with Philsang Yoo

Twisted string theory

Conjecture ([Costello and Li, 2016])

There exist backgrounds for type II strings with the following properties:

- the supergravity approximation describes fluctuations around a *twisted supergravity* background.
- worldvolume theories of D-branes in such backgrounds are subjected to a *holomorphic-topological twist*.

In such backgrounds, type II strings are equivalent to certain topological strings.

Vast generalization of previously studied relations between type II and topological strings.

Motivation continued

The worldsheet theory of a topological string is a 2d fully-extended TQFT.

Theorem ([Costello, 2006, Lurie, 2009])

The following types of data are equivalent

- symmetric monoidal functors $Z : \mathrm{Bord}_2^{or} \to \mathrm{DGCat}$
- Calabi-Yau categories

Can we extract BPS subsectors of worldvolume theories of D-branes and supergravity from the data of a Calabi-Yau category?

PERTURBATIVE CLASSICAL FIELD THEORIES

Definition

A *perturbative Lagrangian classical field theory* on M is

- a graded vector bundle $E \to M$ with sheaf of sections $\mathcal{E}$
- a collection of polydifferential operators $\{\ell_n : \mathcal{E}^{\otimes n} \to \mathcal{E}[2-n]\}$ equipping $\mathcal{E}$ with the structure of an L_∞-algebra
- a nondegenerate invariant pairing $\langle -, - \rangle : E^{\otimes 2} \to \mathrm{Dens}_M$ of degree -3.

[Variants : ghost # anomaly ⇒ odd pairing; nonLagrangian ⇒ degenerate copairing]

Example

① Chern-Simons Theory :

let M be a cpt 3-mfld, $\mathfrak{g}$ - a simple Lie algebra

$\mathcal{E} = \Omega^\bullet_M \otimes \mathfrak{g}$

- dgla w/ $\ell_1 = d$, $\ell_2 = (-\wedge-) \otimes [-,-]$
- $\langle \alpha, \beta \rangle = \int_M \mathrm{Tr}(\alpha \wedge \beta)$

② Minimal Twist of 7d $\mathcal{N}=1$

let X be a complex 3-fold, $\mathfrak{g}$ - a simple Lie algebra

$\mathcal{E} = \Omega^\bullet_{\mathbb{R}} \otimes \Omega^{0,\bullet}_X(\mathcal{O}_X \oplus K_X[-1]) \otimes \mathfrak{g}$

- dgla w/ $\ell_1 = \bar\partial$, $\ell_2 = (-\wedge-) \otimes [-,-]$
- $\langle \alpha, \beta \rangle = \int_M \mathrm{Tr}(\alpha \wedge \beta)$ Need to 2-periodize

③ Kodaira-Spencer gravity

let (X, Ω) be a CY3

$\mathcal{E} = PV^{1,\bullet}_X \xrightarrow{\partial_\Omega} PV^{0,\bullet}_X$

- dgla w/ $\ell_1 = \bar\partial + \partial_\Omega$, $\ell_2 = (-\wedge-) \otimes \{-,-\}$
- degenerate copairing

Nonperturbative classical field theories

Definition

A *Lagrangian classical field theory* on M is a (-1)-shifted symplectic stack

[Variants: ghost # anomaly ⇒ 2-periodize
nonLagrangian ⇒ shifted Poisson]

Example

① Chern-Simons Theory

$\mathrm{Loc}_G(M) = \mathrm{Maps}(M_B, BG)$ is (-1)-shifted symplectic

$\mathcal{T}_0[-1]\,\mathrm{Loc}_G(M) = \Omega^\bullet_M \otimes \mathfrak{g}$

② Minimal Twist of 7d $\mathcal{N}=1$

$\mathrm{Maps}(M_B, T^*[-2]\,\mathrm{Bun}_G X)$ is (-3)-shifted symplectic

$\mathcal{T}_0[-1]\,\mathrm{Maps}(M_B, T^*[-2]\,\mathrm{Bun}_G X) = \Omega^\bullet_M \otimes \Omega^{0,\bullet}_X(\mathcal{O}_X \oplus K_X[-1]) \otimes \mathfrak{g}$

③ Kodaira-Spencer gravity

Moduli of CY3s is (-1)-shifted Poisson ...

Lagrangian topological field theories

Examples describing topological field theories can be placed in the context of the Atiyah-Segal axioms

Proposition ([Calaque et al., 2022])

There is a symmetric monoidal (∞, n)-category Lag_n^s with

- objects: s-shifted symplectic stacks
- i-morphisms: i-fold Lagrangian correspondences

Moreover every object is fully dualizable.

Definition

An n-dimensional *Lagrangian topological field theory* is a symmetric monoidal functor

$$Z : \mathrm{Bord}_n^{or} \to \mathrm{Lag}_n^s$$

Proposition ([Calaque et al., 2022])

For any s-shifted symplectic stack Y, the functor $\mathrm{Map}((-)_B, Y)$ defines a Lagrangian TFT.

String field theories from Calabi-Yau categories

The following field theories are associated to a Calabi-Yau d-category.

	Open	Closed
Formal	$\mathcal{F} \in \mathrm{Ob}(\mathcal{C}) \rightsquigarrow \mathrm{End}_{\mathcal{C}}(\mathcal{F})$ controls deformations of $\mathcal{F}$	$\mathrm{HC}^{-}(\mathcal{C})[1-d]$ controls deformations of $\mathcal{C}$ [Brav and Rozenblyum, 2018]
Global	*moduli of objects* $\mathcal{M}_{\mathcal{C}}$ $(2-d)$-shifted symplectic [Brav and Dyckerhoff, 2018]	*moduli of CYd categories* $(5-2d)$-shifted Poisson [Coates, 2008] [Barannikov and Kontsevich, 1998 [Costello and Li, 2012]

Open-closed coupling encoded in a Lie map
$\mathrm{HC}^{-}(\mathcal{C})[1-d] \to \mathrm{SVect}(\mathcal{M}_{\mathcal{C}})$ [Brav and Rozenblyum, 2018]

TWISTED SUPER YANG-MILLS THEORIES

Perturbative twists of super Yang-Mills theories are classified [Elliott et al., 2020]:

d	$\mathcal{N}$	Twist	Description	Invariant Directions
10	(1,0)	Rank (1,0)	Holomorphic Chern–Simons Theory $\mathrm{Map}(\mathbb{C}^5, B\mathfrak{g})$	5 (holomorphic)
9	1	Rank 1	Generalized Chern–Simons Theory $\mathrm{Map}(\mathbb{C}^4 \times \mathbb{R}_{\mathrm{dR}}, B\mathfrak{g})$	5 (minimal)
8	1	Rank (1,0) pure	Holomorphic BF Theory $T^*[-1]\mathrm{Map}(\mathbb{C}^4, B\mathfrak{g})$	4 (holomorphic)
		Rank (1,1)	Generalized Chern–Simons Theory $\mathrm{Map}(\mathbb{C}^3 \times \mathbb{R}^2_{\mathrm{dR}}, B\mathfrak{g})$	5
		Rank (1,0) impure	Perturbatively trivial (Spin(7) Instanton) $\mathrm{Map}(\mathbb{C}^4, B\mathfrak{g})_{\mathrm{dR}}$	8 (topological)
7	1	Rank 1 pure	Generalized BF Theory $T^*[-1]\mathrm{Map}(\mathbb{C}^3 \times \mathbb{R}_{\mathrm{dR}}, B\mathfrak{g})$	4 (minimal)
		Rank 2	Generalized Chern–Simons Theory $\mathrm{Map}(\mathbb{C}^2 \times \mathbb{R}^3_{\mathrm{dR}}, B\mathfrak{g})$	5
		Rank 1 impure	Perturbatively trivial (G_2 Monopole) $\mathrm{Map}(\mathbb{C}^3 \times \mathbb{R}_{\mathrm{dR}}, B\mathfrak{g})_{\mathrm{dR}}$	7 (topological)
6	(1,1)	Rank (1,0)	Holomorphic BF Theory $T^*[-1]\mathrm{Map}(\mathbb{C}^3, \mathfrak{g}/\mathfrak{g})$	3 (holomorphic)
		Rank (1,1) special	Generalized BF Theory $T^*[-1]\mathrm{Map}(\mathbb{C}^2 \times \mathbb{R}^2_{\mathrm{dR}}, B\mathfrak{g})$	4
		Rank (2,2)	Generalized Chern–Simons Theory $\mathrm{Map}(\mathbb{C} \times \mathbb{R}^4_{\mathrm{dR}}, B\mathfrak{g})$	5
		Rank (1,1) generic	Perturbatively trivial $\mathrm{Map}(\mathbb{C}^2 \times \mathbb{R}^2_{\mathrm{dR}}, B\mathfrak{g})_{\mathrm{dR}}$	6 (topological)
5	2	Rank 1	Generalized BF Theory $T^*[-1]\mathrm{Map}(\mathbb{C}^2 \times \mathbb{R}_{\mathrm{dR}}, \mathfrak{g}/\mathfrak{g})$	3 (minimal)
		Rank 2 special	Generalized BF Theory $T^*[-1]\mathrm{Map}(\mathbb{C} \times \mathbb{R}^3_{\mathrm{dR}}, B\mathfrak{g})$	4
		Rank 4	5d Chern–Simons Theory $\mathrm{Map}(\mathbb{R}^5_{\mathrm{dR}}, B\mathfrak{g})$	5 (topological)
		Rank 2 generic	Perturbatively trivial $\mathrm{Map}(\mathbb{C} \times \mathbb{R}^3_{\mathrm{dR}}, B\mathfrak{g})_{\mathrm{dR}}$	5 (topological)
4	4	Rank (1,0)	Holomorphic BF Theory $T^*[-1]\mathrm{Map}(\mathbb{C}^2_{\mathrm{Dol}}, B\mathfrak{g})$	2 (holomorphic)
		Rank (1,1)	Generalized BF Theory $T^*[-1]\mathrm{Map}(\mathbb{C}_{\mathrm{Dol}} \times \mathbb{R}^2_{\mathrm{dR}}, B\mathfrak{g})$	3
		Rank (2,2) special	BF Theory $T^*[-1]\mathrm{Map}(\mathbb{R}^4_{\mathrm{dR}}, B\mathfrak{g})$	4 (topological)
		Rank (2,1)	Perturbatively trivial $\mathrm{Map}(\mathbb{C}_{\mathrm{Dol}} \times \mathbb{R}^2_{\mathrm{dR}}, B\mathfrak{g})_{\mathrm{dR}}$	4 (topological)
		Rank (2,0)	Perturbatively trivial $\mathrm{Map}(\mathbb{C}^2_{\mathrm{Dol}}, B\mathfrak{g})_{\mathrm{dR}}$	4 (topological)
		Rank (2,2) generic	Perturbatively trivial $\mathrm{Map}(\mathbb{R}^4_{\mathrm{dR}}, B\mathfrak{g})_{\mathrm{dR}}$	4 (topological)
3	8	Rank 1	Generalized BF Theory $T^*[-1]\mathrm{Map}(\mathbb{C}_{\mathrm{Dol}} \times \mathbb{R}_{\mathrm{dR}}, \mathfrak{g}/\mathfrak{g})$	2 (minimal)
		Rank 2 (B)	BF Theory $T^*[-1]\mathrm{Map}(\mathbb{R}^3_{\mathrm{dR}}, \mathfrak{g}/\mathfrak{g})$	3 (topological)
		Rank 2 (A)	Perturbatively trivial $\mathrm{Map}(\mathbb{R}^3_{\mathrm{dR}}, \mathfrak{g}/\mathfrak{g})_{\mathrm{dR}}$	3 (topological)

Table 1: Twists of Maximally Supersymmetric Pure Yang–Mills Theories with Lie algebra $\mathfrak{g}$ (16 supercharges).

d	$\mathcal{N}$	Twist	Description	Invariant Directions
6	(1,0)	Rank (1,0)	Holomorphic BF Theory coupled to a holomorphic symplectic boson $\mathrm{Sect}(\mathbb{C}^3, (U \otimes K^{1/2}_{\mathbb{C}^3})/\!\!/\mathfrak{g})$	3 (holomorphic)
5	1	Rank 1	Generalized BF Theory coupled to a generalized symplectic boson $\mathrm{Sect}(\mathbb{C}^2 \times \mathbb{R}_{\mathrm{dR}}, (U \otimes K^{1/2}_{\mathbb{C}^2})/\!\!/\mathfrak{g})$	3 (minimal)
4	2	Rank (1,0)	Holomorphic BF Theory $T^*[-1]\mathrm{Sect}(\mathbb{C}^2, (U \otimes K^{1/2}_{\mathbb{C}^2})/\!\!/\mathfrak{g})$	2 (holomorphic)
		Rank (1,1)	Generalized BF Theory coupled to a generalized symplectic boson $\mathrm{Sect}(\mathbb{C} \times \mathbb{R}^2_{\mathrm{dR}}, (U \otimes K^{1/2}_{\mathbb{C}})/\!\!/\mathfrak{g})$	3
		Rank (2,0)	Perturbatively trivial $\mathrm{Sect}(\mathbb{C}^2, (U \otimes K^{1/2}_{\mathbb{C}^2})/\!\!/\mathfrak{g})_{\mathrm{dR}}$	4 (topological)
3	4	Rank 1	Generalized BF Theory coupled to a generalized symplectic boson $T^*[-1]\mathrm{Sect}(\mathbb{C} \times \mathbb{R}_{\mathrm{dR}}, (U \otimes K^{1/2}_{\mathbb{C}})/\!\!/\mathfrak{g})$	2 (minimal)
		Rank 2 (B)	BF Theory coupled to a symplectic boson $\mathrm{Map}(\mathbb{R}^3_{\mathrm{dR}}, U/\!\!/\mathfrak{g})$	3 (topological)
		Rank 2 (A)	Perturbatively trivial $\mathrm{Sect}(\mathbb{C} \times \mathbb{R}_{\mathrm{dR}}, (U \otimes K^{1/2}_{\mathbb{C}})/\!\!/\mathfrak{g})_{\mathrm{dR}}$	3 (topological)

Table 2: Twists of Supersymmetric Yang–Mills Theories with gauge Lie algebra $\mathfrak{g}$ with a hypermultiplet valued in a symplectic representation U (8 supercharges).

d	$\mathcal{N}$	Twist	Description	Invariant Directions
4	1	Rank (1,0)	Holomorphic BF Theory coupled to R-matter $T^*[-1]\mathrm{Map}(\mathbb{C}^2, R/\mathfrak{g})$	2 (holomorphic)
3	2	Rank 1	Generalized BF Theory coupled to R-matter $T^*[-1]\mathrm{Map}(\mathbb{C} \times \mathbb{R}_{\mathrm{dR}}, R/\mathfrak{g})$	2 (minimal)

Table 3: Twists of Supersymmetric Yang–Mills Theories with gauge Lie algebra $\mathfrak{g}$ with a chiral multiplet valued in a representation R (4 supercharges).

$\mathcal{N}$	Twist	Description	Invariant Directions
(4,4)	Rank (1,0)	Holomorphic BF theory coupled to a holomorphic symplectic boson $T^*[-1]\mathrm{Sect}(\mathbb{C}, T[1]((U \otimes K^{1/2}_{\mathbb{C}})/\!\!/\mathfrak{g}))$	1 (holomorphic)
	Rank (1,1) (B)	Topological BF theory coupled to a holomorphic symplectic boson $T^*[-1]\mathrm{Map}(\mathbb{R}^2_{\mathrm{dR}}, U/\!\!/\mathfrak{g})$	2 (topological)
	Rank (1,1) (A)	Perturbatively trivial (A-model) $\mathrm{Map}(\mathbb{R}^2_{\mathrm{dR}}, (U/\!\!/\mathfrak{g})_{\mathrm{dR}})$	2 (topological)
(2,2)	Rank (1,0)	Holomorphic BF theory coupled to R matter $T^*[-1]\mathrm{Map}(\mathbb{C}, T[1](R/\mathfrak{g}))$	1 (holomorphic)
	Rank (1,1) (B)	Topological BF theory coupled to R matter $T^*[-1]\mathrm{Map}(\mathbb{R}^2_{\mathrm{dR}}, R/\mathfrak{g})$	2 (topological)
	Rank (1,1) (A)	Perturbatively trivial (A-model) $T^*[-1]\mathrm{Map}(\mathbb{C}, (R/\mathfrak{g})_{\mathrm{dR}})$	2 (topological)
$(\mathcal{N}_+, 0)$	Rank (1,0)	Holomorphic BF theory coupled to $\mathcal{N}_+ - 2$ free fermions $T^*[-1]\mathrm{Sect}(\mathbb{C}, (\mathfrak{g}^{\mathcal{N}_+ - 2} \otimes K^{1/2}_{\mathbb{C}})/\mathfrak{g})$	1 (holomorphic)
(4,0)	Rank (1,0)	Holomorphic BF theory coupled to a holomorphic symplectic boson $T^*[-1]\mathrm{Sect}(\mathbb{C}, (U \otimes K^{1/2}_{\mathbb{C}})/\!\!/\mathfrak{g})$	1 (holomorphic)
(2,0)	Rank (1,0)	Holomorphic BF theory coupled to R matter $T^*[-1]\mathrm{Map}(\mathbb{C}, R/\mathfrak{g})$	1 (holomorphic)

Table 4: Twists of Supersymmetric Yang–Mills Theories in two dimensions with gauge group G. When $\mathcal{N} = (0,2)$ and $(2,2)$ the theory includes a chiral multiplet valued in a representation R. When $\mathcal{N} = (0,4)$ and $(4,4)$ the theory includes a hypermultiplet valued in a symplectic representation U. We can promote the supersymmetry to $\mathcal{N} = (8,8)$ when $U = T^*\mathfrak{g}$, but no new twists occur.

Twisted super Yang-Mills theories as open string theories

Theorem ([Yoo, 2025])

For every perturbative Lagrangian classical field theory $\mathcal{E}$ describing a twist of pure $\mathrm{GL}(N)$ super Yang-Mills theory, there is a pair

- a Calabi-Yau category $\mathcal{C}$
- an object $\mathcal{F} \in \mathrm{Ob}\,\mathcal{C}$

such that $\mathcal{E} \cong \mathrm{End}_{\mathcal{C}}(\mathcal{F})$.

Example

Holomorphic-Topological Twist of 4d $\mathcal{N}=4$

M - a real surface,

Σ - a proper curve

$\mathfrak{g}$ - a simple Lie alg.

$\rightsquigarrow \mathcal{E} = \Omega^{\bullet}_M \otimes \Omega^{0,\bullet}_{\Sigma}[\varepsilon_1, \varepsilon_2] \otimes \mathfrak{g}$, $|\varepsilon_i|$ odd

Let $\mathcal{C} = \mathrm{Loc}\, M \otimes \mathrm{Perf}(T^*\Sigma \times \mathbb{C})$

$\mathcal{F} = \underline{\mathbb{C}}_M \otimes \mathcal{O}_{\Sigma}^{\oplus N}$

$\rightsquigarrow \mathrm{End}_{\mathcal{C}}(\mathcal{F}) = \Omega^{\bullet}_M \otimes \Omega^{0,\bullet}_{\Sigma}(\wedge^{\bullet} N) \otimes \mathfrak{gl}_N$

Twisted super Yang-Mills theories as open string theories

Theorem ([Yoo, 2025])

For every perturbative Lagrangian classical field theory $\mathcal{E}$ describing a twist of pure $GL(N)$ super Yang-Mills theory, there is a pair

- a Calabi-Yau category $\mathcal{C}$
- an object $\mathcal{F} \in \operatorname{Ob} \mathcal{C}$

such that $\mathcal{E} \cong \operatorname{End}_{\mathcal{C}}(\mathcal{F})$.

The categories $\mathcal{C}$ that appear as such all have the form $\operatorname{Loc}(M) \otimes \operatorname{Perf}(X)$ where M is a compact n-manifold, and X is a Calabi-Yau d-fold, with $n + d = 5$.

- If n is odd, $\mathcal{C}$ describes a twist of type IIA
- If n is even, $\mathcal{C}$ describes a twist of type IIB

Caveat: When ghost # anomalies can be cancelled (eg enough residual R-symm.) can shear X to a graded CY, s.th the CY dim^n of $\mathcal{C}$ is 3.

Lagrangian TFTs from open string field theory

The global open-string field theories associated to such categories have an extra bit of structure.

Proposition ([Kinjo et al., 2024])

Let $\mathcal{C}$ be a smooth Calabi-Yau d-category and M a compact k-manifold. Then there is an equivalence of $(2-k-d)$-shifted sympelctic stacks

$$\mathcal{M}_{\mathrm{Loc}M\otimes\mathcal{C}} \cong \mathrm{Map}(M_B, \mathcal{M}_{\mathcal{C}}).$$

The RHS is the value of a Lagrangian TFT on M. We would like to say the LHS is as well.

Noncommutative AKSZ theories

Proposition ([Bozec et al., 2024])

There is a symmetric monoidal (∞, n)-category CY_n^d with:

- objects: Calabi-Yau d-categories
- i-morphisms: i-fold *Calabi-Yau cospans*

Every object is fully dualizable.

Definition

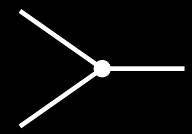
A *noncommutative AKSZ theory* is a symmetric monoidal functor $\mathrm{Bord}_n^{or} \to \mathrm{CY}_n^d$.

Theorem ([Bozec et al., 2024])

$$\begin{array}{ccc} \mathrm{Bord}_n^{or} & \xrightarrow{\mathrm{Loc}(-)\otimes\mathcal{C}} & \mathrm{CY}_n^d \\ & {\scriptstyle \mathrm{Map}((-)_B,\mathcal{M}_\mathcal{C})} \searrow & \downarrow {\scriptstyle \mathcal{M}_{(-)}} \\ & & \mathrm{Lag}_n^{2-d} \end{array}$$

Domain walls

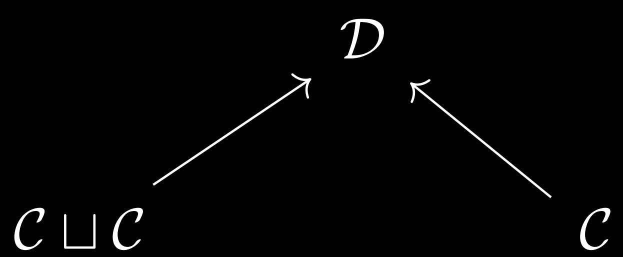
Suppose we want to evaluate a nc-AKSZ theory $\mathrm{Loc}(-) \otimes \mathcal{C}$ on

 $\times X \notin \mathrm{Bord}_n^{or}$

Cobordism hypothesis with singularities [Lurie, 2009], [Johnson-Freyd and Scheimbauer, 2017], [Stewart, 2024]: need to specify a natural transformation

$$\tau_{\leq n-1} \left(\mathrm{Loc}(-) \otimes \mathcal{C}\right)^{\otimes 2} \to \tau_{\leq n-1} \left(\mathrm{Loc}(-) \otimes \mathcal{C}\right)$$

Suffices to specify a Calabi-Yau cospan



Domain walls

Any Calabi-Yau category $\mathcal{C}$ has a canonical cospan

$$\mathcal{C} \otimes Sh_{\top}(\mathbb{R}) \cong \mathcal{C} \otimes \mathrm{Mod}(A_2)$$

$$\mathcal{C} \sqcup \mathcal{C} \longrightarrow \mathcal{C} \otimes Sh_{\top}(\mathbb{R}) \cong \mathcal{C} \otimes \mathrm{Mod}(A_2) \longleftarrow \mathcal{C}$$

$\rightsquigarrow \tau_{\leq n-1} \left(\mathrm{Loc}(-) \otimes \mathcal{C}\right)$ is an algebra object in $\mathrm{Fun}^{\otimes}(\mathrm{Bord}^{or}_{n-1}, \mathrm{CY}^{d}_{n-1})$

Goal: deform the outputs of such theories and then quantize

Quantization

Shifted geometric quantization [Safronov, 2020]:

- n-shifted symplectic stack X
- n-gerbe $\mathcal{G}$ with $c_1(\mathcal{G}) = \omega$
- n-shifted Lagrangian foliation $\mathcal{M} \to T_X$

$\rightsquigarrow$ n- category of sections $\Gamma_\nabla(X, \mathcal{G})$ flat along the leaves of $\mathcal{M}$

Hope

Shifted geometric quantization is functorial and monoidal

$$\begin{array}{ccc} \mathrm{Bord}_n^{or} & \longrightarrow & \mathrm{Lag}_n^{2-d,prequant} \\ & \searrow^{Z_C} & \downarrow \\ & & \mathrm{Cat}^{2-d} \end{array}$$

$SU(4)$-invariant twist of type IIA

Let X be a Calabi-Yau 3-fold
$\rightsquigarrow$ twist of IIA on $T^*M \times X \times \mathbb{C}$ where M is a compact one-manifold.
$\rightsquigarrow$ Calabi-Yau 2 category $\mathcal{C} = \mathrm{Perf}(X \times B^2\mathbb{G}_a)$.
$\rightsquigarrow$ 1d nc-AKSZ theory $\mathrm{Bord}_1^{or} \to CY_1^2$

worldline theory of D0-D2-D4-D6 bound states in IIA on $T^*S \times X \times \mathbb{C}$

Proposition

The commutative limit is given by $\mathrm{Map}\left((-)_B, T^*\mathcal{M}_{\mathrm{Perf}X}\right)$.

There is a Maurer-Cartan element in $\mathrm{HC}^-(\mathcal{C})[-1]$ whose image under the map $\mathrm{HC}^-(\mathcal{C})[-1] \to \mathrm{SVect}(\mathcal{M}_\mathcal{C})$ deforms $T^*\mathcal{M}_{\mathrm{Perf}X}$ to $(\mathcal{M}_{\mathrm{Perf}X})_{\mathrm{dR}}$.
$\rightsquigarrow$ family of ncAKSZ theories whose commutative limit is $\mathrm{Map}\left((-)_B, (\mathcal{M}_{\mathrm{Perf}X})_{\mathrm{Hod}}\right)$.
$\rightsquigarrow$ family of TQFTs $Z_{\mathcal{C}_\lambda}$ by geometric quantization.

$SU(4)$-invariant twist of type IIA

Want to describe $Z_{\mathcal{C}_{\lambda\neq 0}}(pt)$.

Conjecture

Let X be a (-1)-shifted symplectic stack admitting orientation data. Then $H^\bullet_{crit}(X)$ agrees with a geometric quantization of X_{dR} as a 0-shifted symplectic stack.

- Granting this, we see that $Z_{\mathcal{C}_{\lambda\neq 0}}(pt)$ should recover the cohomological DT invariant of Perf$X$.
- The algebra structure coming from that on the ncAKSZ theory is CoHA multiplication.

$U(1)$-invariant twist of type IIB

Let N be a compact 3-manifold
$\rightsquigarrow$ twist of IIB on $T^*(M \times N) \times \mathbb{C}$ where M is a compact one-manifold.
$\rightsquigarrow$ Calabi-Yau 2 category $\mathcal{C} = \mathrm{Loc}(N) \otimes \mathrm{Perf}(B^2\mathbb{G}_a)$.
$\rightsquigarrow$ 1d nc-AKSZ theory $\mathrm{Bord}_1^{or} \to CY_1^2$

$M \mapsto$ worldline theory of D3-D5 bound states in IIB on $T^*(M \times N) \times \mathbb{C}$
A B

Proposition

The commutative limit is given by $\mathrm{Map}((-)_B, T^*\mathcal{M}_{\mathrm{Loc}N})$.

- Same deformation as before gives a family over $\mathbb{A}^1$ - generic fiber is compactification of KW twist of 4d $\mathcal{N} = 4$ at $\Psi \neq 0, \infty$ on N.
- Recover the cohomological DT invariant of $\mathrm{Loc}(N)$ as the state space.

$SU(3)$-invariant twist of type IIB

Let Σ be a proper curve
$\rightsquigarrow$ twist of type IIB on $T^*(C \times \Sigma) \times \mathbb{C}$ where C is a proper curve.
$\rightsquigarrow$ Calabi-Yau 1-category $\mathcal{C} = \mathrm{Perf}(T^*\Sigma \times B^2\mathbb{G}_a)$.
$\rightsquigarrow$ 2d nc-AKSZ theory $\mathrm{Bord}_2^{or} \to CY_2^1$

Proposition

There is an equivalence of 1-shifted symplectic stacks $\mathcal{M}_{\mathcal{C}} = T^*[1]\mathrm{Higgs}\Sigma$.

- The 1-shifted geometric quantization is $\mathrm{IndCoh}(\mathrm{Higgs}\Sigma)$.
- The expected algebra structure is the categorified Hall structure of [Porta et al., 2023].

$SU(3)$-invariant twist of type IIB

Let Σ be a proper curve
$\rightsquigarrow$ twist of type IIB on $T^*(C \times \Sigma) \times \mathbb{C}$ where C is a proper curve.
$\rightsquigarrow$ Calabi-Yau 1-category $\mathcal{C} = \mathrm{Perf}(T^*\Sigma \times B^2\mathbb{G}_a)$.
$\rightsquigarrow$ 2d nc-AKSZ theory $\mathrm{Bord}_2^{or} \to CY_2^1$

The dg Lie algebra $\mathrm{HC}^-(\mathcal{C})$ carries an action by S-duality [Raghavendran and Yoo, 2019]. There are two deformations in the same orbit such that under the map to symplectic vector fields

- one deforms $T^*[1]\mathrm{Higgs}\Sigma$ to $T^*[1]\mathrm{Flat}\Sigma$.
- one deforms $T^*[1]\mathrm{Higgs}\Sigma$ to $T^*[1](\mathcal{M}_{\mathrm{Perf}\Sigma})_{\mathrm{dR}}$.

The 1-shifted geometric quantizations of these are $\mathrm{IndCoh}(\mathrm{Flat}\Sigma)$ and $\mathrm{IndCoh}((\mathcal{M}_{\mathrm{Perf}\Sigma})_{\mathrm{dR}})$ respectively.

QUESTIONS AND DESIDERATA

- Noncommutative counterparts to prequantizations and polarizations

	$\mathcal{C}$	$\mathcal{M}_{\mathcal{C}}$
Prequantization	??	n-gerbe $\mathcal{G}$ with $c_1(G) = \omega$
Polarization	??	$\mathcal{M} \to \mathcal{M}_{\mathcal{C}}$ Lagrangian foliation

- Noncommutative counterpart to action of polarization-compatible Hamiltonians

$$\begin{array}{ccccc} HC^{-}_{\bullet}(\mathcal{C})[1-d] & \longrightarrow & \mathrm{SVect}(\mathcal{M}_{\mathcal{C}}) & & \\ \uparrow & & \uparrow & & \\ HC_{\bullet}(\mathcal{C})[2-d] & \longrightarrow & \mathcal{O}(\mathcal{M}_{\mathcal{C}})[2-d] & & \\ \uparrow & & \uparrow & & \\ ?? & \longrightarrow & \mathcal{O}_{\nabla}(\mathcal{M}_{\mathcal{C}})[2-d] & & \curvearrowright \Gamma_{\nabla}(\mathcal{M}_{\mathcal{C}}, \mathcal{G}) \end{array}$$

- WIP: use cobordism hypothesis with singularities to codify brane constructions of boundary conditions and defects
- What role do t-structures play in this story?

Thanks!

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