Geometry of EFT positivity cones

joint with Q. Bonnefoy, V. Cortés, C. Grojean & K. Ritter von Merkl

Paula Naomi Pilatus

University of Hamburg

March 25, 2026

Overview

1. Positivity in EFTs
2. Convex cones & extremal representations
3. Extremal elements of the positivity cone
4. Summary and Outlook

Positivity in EFTs

Effective field theories

Effective field theories (EFTs) are a way to approximate the full theory in terms of the field content at lower energies.

Example: standard model EFT **(SMEFT)**:

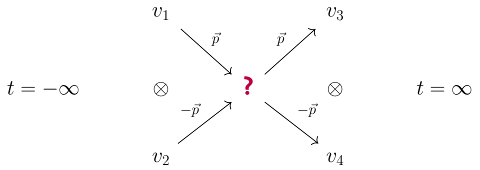
$$\mathcal{L}_{\text{full}} \approx \mathcal{L}_{\text{SM}} + \sum_{d>4} \sum_{i} f_i^{(d)} \mathcal{O}_i^{(d)},$$

where $\left\{ \mathcal{O}_i^{(d)} \right\}$ is a basis of operators (containing SM fields) of mass dimension d.

The coefficients $f_i^{(d)}$ are called **Wilson coefficients**.

Two-to-two forward scattering

Two particles of momentum $\vec{p}$ are colliding:



The v_i are elements in a finite dimensional $\mathbb{C}$-vector space V_c, corresponding to the irrep of the particle type.

We define the **S-matrix** $S\colon v_1 \otimes v_2 \mapsto v_3 \otimes v_4$ and the **forward scattering amplitude**:

$$\begin{aligned} A(s)\colon (V_c \otimes V_c) \times (V_c \otimes V_c) &\longrightarrow \mathbb{C} \\ (v \otimes w) \times (v' \otimes w') &\longmapsto \langle v' \otimes w', (S - \mathrm{Id})(v \otimes w) \rangle, \end{aligned}$$

where s is the center-of-mass energy.

Properties of the amplitude

The forward scattering amplitude

$$\begin{aligned} A(s)\colon (V_c \otimes V_c) \times (V_c \otimes V_c) &\longrightarrow \mathbb{C} \\ (v \otimes w) \times (v' \otimes w') &\longmapsto \langle v' \otimes w', (S - \mathrm{Id})(v \otimes w)\rangle \end{aligned}$$

has the following **properties**:

- **crossing symmetry:** $A(s)(v_1, v_2, v_3, v_4) = A(s - m^2)(v_1, \bar{v}_4, v_3, \bar{v}_2)$, with m^2 sum of the four masses squared
- in addition assume $A(s)(v_1, v_2, v_3, v_4) = A(s)(v_2, v_1, v_4, v_3)$ ($\rightsquigarrow$ parity conservation)

Unitarity, Causality & Locality

- **Unitarity**, i.e. S-matrix is unitary
  $\rightsquigarrow$ optical theorem
- **Causality**, i.e. observables commute at space-like distance
  $\rightsquigarrow$ $A(s)$ can be analytically continued to the complex plane
- **Locality**, i.e. Lagrangian consists of local operators ($+$ extra assumption)
  $\rightsquigarrow$ $A(s)$ is quadratically bounded for large s

Positivity bounds

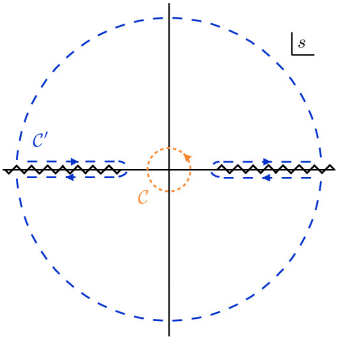
Assume a given EFT is the **low energy limit of a unitary, causal & local QFT**

$\implies$ **Positivity bounds** on Wilson coefficients!!

Schematically: For small s, write

$$A(s + m^2/2) = \sum_{k=0}^{\infty} A_{(k)}\, s^k$$

Then we obtain bounds on $A_{(k)}$ for $k \geq 2$.



Remmen&Rodd '19

The positivity cone

Restrict $A(s)$ to real subspace $V \subset V_c$ of self-conjugate vectors.

Then the positivity bounds can be expressed in terms of

$$M := \frac{\mathrm{d}^2}{\mathrm{d}s^2}\bigg|_{s=m^2/2} A(s) \quad + \text{c.c.}$$

as $\quad M \in \mathcal{C} := \text{conv}\{Q^{\otimes 2} + \tau Q^{\otimes 2} : Q \in \text{Sym}^2 V \text{ or } Q \in \Lambda^2 V\}$,

where $\tau S(v_1, v_2, v_3, v_4) = S(v_1, v_4, v_3, v_2)$.

Observe: $\mathcal{C}$ is a closed convex cone.

Convex cones & extremal representations

Convex cones and dual cones

Recall: $\mathcal{C} \subset \mathbb{R}^N$ is called

- a **cone** if for all $x \in \mathcal{C}$, $\lambda > 0$ we have $\lambda x \in \mathcal{C}$;
- **convex** if for all $x, y \in \mathcal{C}$, $t \in [0,1]$ we have $t\,x + (1-t)\,y \in \mathcal{C}$.

Definition: The **dual cone** $\mathcal{C}^*$ of a cone $\mathcal{C} \subset \mathbb{R}^N$ is defined as

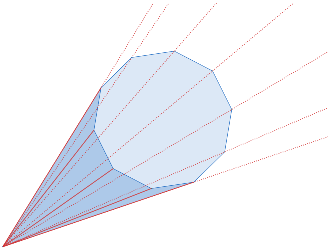
$$\mathcal{C}^* = \left\{ x \in \mathbb{R}^N : \langle x, y \rangle \geq 0 \;\forall y \in \mathcal{C} \right\}.$$

Example: Cone $\mathcal{P}_n$ of symmetric positive semi-definite $n \times n$ matrices satisfies $\mathcal{P}_n^* = \mathcal{P}_n$.

Faces and extremal rays

A **face** of a convex set $\mathcal{C}$ is a convex subset $\mathcal{C}' \subset \mathcal{C}$ such that every closed line segment in $\mathcal{C}$ with interior point in $\mathcal{C}'$ has both endpoints in $\mathcal{C}'$.

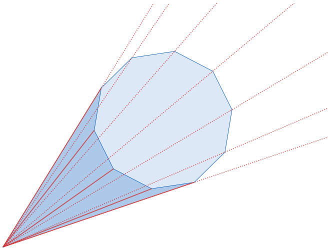
An **extremal ray** of a convex cone $\mathcal{C}$ is a face of the form $\mathbb{R}_{\geq 0}\, x \cap \mathcal{C}$ for some $x \in \mathcal{C}$. The element x is then called **extremal element**.



Faces and extremal rays

A **face** of a convex set $\mathcal{C}$ is a convex subset $\mathcal{C}' \subset \mathcal{C}$ such that every closed line segment in $\mathcal{C}$ with interior point in $\mathcal{C}'$ has both endpoints in $\mathcal{C}'$.

An **extremal ray** of a convex cone $\mathcal{C}$ is a face of the form $\mathbb{R}_{\geq 0}\, x \cap \mathcal{C}$ for some $x \in \mathcal{C}$. The element x is then called **extremal element**.



Theorem (Klee): A non-trivial closed convex cone $\mathcal{C}$ containing no lines is **completely determined by its extremal rays**:

Given a set $T \subset \mathcal{C}$ such that every extremal ray is determined by some $x \in T$, then every element in $\mathcal{C}$ can be written as a (finite) positive linear combination of elements in T.

Positivity cone

Recall:

Positivity bounds $\iff$ $M \in \mathcal{C} := \mathrm{conv}\{Q^{\otimes 2} + \tau Q^{\otimes 2} : Q \in \mathrm{Sym}^2 V \text{ or } Q \in \Lambda^2 V\}$,
where $\tau S(v_1, v_2, v_3, v_4) = S(v_1, v_4, v_3, v_2)$.

Observe: $\mathcal{C} \subset \left\{S \in \mathrm{Sym}^2(\mathrm{Sym}^2 V) \oplus \mathrm{Sym}^2(\Lambda^2 V) : \tau S = S\right\}$.

Define

$$W = \left\{S \in \mathrm{Sym}^2(\mathrm{Sym}^2 V^*) \oplus \mathrm{Sym}^2(\Lambda^2 V*) : \tau S = S\right\}$$

Then

$$\mathcal{C}^* = \mathcal{C}_W := \{S \in W : S \geq 0\}.$$

$\mathcal{C}_W$ is an example of a **linear spectrahedron**.

Faces of spectrahedra

A **linear spectrahedron** is an intersection of the cone of PSD matrices $\mathcal{P}_n$ with a linear subspace of $\mathrm{Sym}^2(\mathbb{R}^n)$.

Theorem (Ramana&Goldman): Let $\mathcal{C}$ be a linear spectrahedron. Then the minimal face of $x \in \mathcal{C}$ (i.e. the face of the smallest dimension containing x) is given by

$$F_{\mathcal{C}}(x) = \{y \in \mathcal{C} : \ker y \supset \ker x\}.$$

In particular: $x \in \mathcal{C} \setminus \{0\}$ is extremal iff $y \in \mathcal{C}$, $\ker y \supsetneq \ker x \implies y = 0$.

Extremal elements of the positivity cone

Properties of the space W (1/2)

Recall: $W = \{S \in \mathrm{Sym}^2(\mathrm{Sym}^2 V^*) \oplus \mathrm{Sym}^2(\Lambda^2 V^*) : \tau S = S\}$.

We consider the canonical projections/**restriction maps**

$$R : W \longrightarrow \mathrm{Sym}^2(\mathrm{Sym}^2 V^*)$$
$$R' : W \longrightarrow \mathrm{Sym}^2(\Lambda^2 V^*)$$

and the **total symmetrization map**

$$\mathrm{tot} : W \longrightarrow \mathrm{Sym}^4 V^*.$$

Denote by $K_W := \ker(\mathrm{tot})$. Then

$$W = \mathrm{Sym}^4 V^* \oplus K_W.$$

Properties of the space W (2/2)

The restriction maps have the following **properties**:

1. The map $R : W \to \mathrm{Sym}^2(\mathrm{Sym}^2 V^*)$ is an isomorphism;

Properties of the space W (2/2)

The restriction maps have the following **properties**:

1. The map $R : W \to \mathrm{Sym}^2(\mathrm{Sym}^2 V^*)$ is an isomorphism;
2. $R|_{\mathrm{Sym}^4 V^*} = \mathrm{Id}$;

Properties of the space W (2/2)

The restriction maps have the following **properties**:

1. The map $R : W \to \operatorname{Sym}^2(\operatorname{Sym}^2 V^*)$ is an isomorphism;
2. $R|_{\operatorname{Sym}^4 V^*} = \operatorname{Id}$;
3. $\ker R' = \operatorname{Sym}^4 V^*$;

Properties of the space W (2/2)

The restriction maps have the following **properties**:

1. The map $R : W \to \mathrm{Sym}^2(\mathrm{Sym}^2 V^*)$ is an isomorphism;
2. $R|_{\mathrm{Sym}^4 V^*} = \mathrm{Id}$;
3. $\ker R' = \mathrm{Sym}^4 V^*$;
4. $\mathrm{im}\, R' = K(V) := \{S \in \mathrm{Sym}^2(\Lambda^2 V^*) : \sum_{\mathrm{cycl}} S(v_1, v_2, v_3, \cdot) = 0 \quad \forall\, v_i \in V\}$;

Properties of the space W (2/2)

The restriction maps have the following **properties**:

1. The map $R : W \to \mathrm{Sym}^2(\mathrm{Sym}^2 V^*)$ is an isomorphism;
2. $R|_{\mathrm{Sym}^4 V^*} = \mathrm{Id}$;
3. $\ker R' = \mathrm{Sym}^4 V^*$;
4. $\operatorname{im} R' = K(V) := \{S \in \mathrm{Sym}^2(\Lambda^2 V^*) : \sum_{\mathrm{cycl}} S(v_1, v_2, v_3, \cdot) = 0 \quad \forall\, v_i \in V\}$;
5. $K_w \cong K(V)$ via R'.

Elastic extremal elements in $\mathcal{C}_W$ (1/3)

Recall: $\mathcal{C}_W = \{S \in \mathrm{Sym}^2(\mathrm{Sym}^2 V^*) \oplus \mathrm{Sym}^2(\Lambda^2 V^*) : \tau S,\ S \geq 0\}$

Observe:

- $S \in \mathcal{C}_W \iff R(S) \geq 0$ and $R'(S) \geq 0$.

Elastic extremal elements in $\mathcal{C}_W$ (1/3)

Recall: $\mathcal{C}_W = \{S \in \mathrm{Sym}^2(\mathrm{Sym}^2 V^*) \oplus \mathrm{Sym}^2(\Lambda^2 V^*) : \tau S,\ S \geq 0\}$

Observe:

- $S \in \mathcal{C}_W \iff R(S) \geq 0$ and $R'(S) \geq 0$.
- In particular $R \colon \mathcal{C}_W \longrightarrow \mathcal{C}_{\mathrm{Sym}} := \{S \in \mathrm{Sym}^2(\mathrm{Sym}^2) : S \geq 0\}$.

Elastic extremal elements in $\mathcal{C}_W$ (1/3)

Recall: $\mathcal{C}_W = \{S \in \mathrm{Sym}^2(\mathrm{Sym}^2 V^*) \oplus \mathrm{Sym}^2(\Lambda^2 V^*) : \tau S,\ S \geq 0\}$

Observe:

- $S \in \mathcal{C}_W \iff R(S) \geq 0$ and $R'(S) \geq 0$.
- In particular $R\colon \mathcal{C}_W \longrightarrow \mathcal{C}_{\mathrm{Sym}} := \{S \in \mathrm{Sym}^2(\mathrm{Sym}^2) : S \geq 0\}$.
- Elements in $\mathcal{C}_W$ projecting to extremal elements in $\mathcal{C}_{\mathrm{Sym}}$ are extremal in $\mathcal{C}_W$.

Elastic extremal elements in $\mathcal{C}_W$ (1/3)

Recall: $\mathcal{C}_W = \{S \in \mathrm{Sym}^2(\mathrm{Sym}^2 V^*) \oplus \mathrm{Sym}^2(\Lambda^2 V^*) : \tau S,\ S \geq 0\}$

Observe:

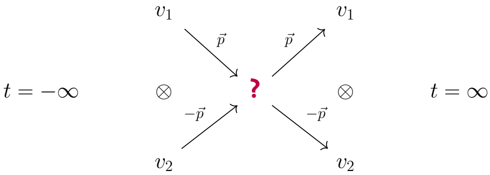
- $S \in \mathcal{C}_W \iff R(S) \geq 0$ and $R'(S) \geq 0$.
- In particular $R\colon \mathcal{C}_W \longrightarrow \mathcal{C}_{\mathrm{Sym}} := \{S \in \mathrm{Sym}^2(\mathrm{Sym}^2) : S \geq 0\}$.
- Elements in $\mathcal{C}_W$ projecting to extremal elements in $\mathcal{C}_{\mathrm{Sym}}$ are extremal in $\mathcal{C}_W$.
- These are elements $S \in \mathcal{C}_W$ such that $\operatorname{rank} R(S) = 1$.

Elastic extremal elements in $\mathcal{C}_W$ (2/3)

Theorem: Let $S \in \mathcal{C}_W$ such that $R(S)$ is extremal in $\mathcal{C}_{\mathrm{Sym}}$. Then

$$S = (\alpha \otimes \beta)^{\otimes 2} + (\beta \otimes \alpha)^{\otimes 2}, \quad \text{for some } \alpha, \beta \in V^* \setminus \{0\}.$$

These correspond to **elastic positivity bounds**:



Elastic extremal elements in $\mathcal{C}_W$ (3/3)

Proof:

- Let $S \in \mathcal{C}_W$ such that $R(S)$ is extremal in $\mathcal{C}_{\mathsf{Sym}}$.

Elastic extremal elements in $\mathcal{C}_W$ (3/3)

Proof:

- Let $S \in \mathcal{C}_W$ such that $R(S)$ is extremal in $\mathcal{C}_{\mathrm{Sym}}$.
- Then $R(S) = \gamma^2$ for some $\gamma \in \mathrm{Sym}^2 V^*$.

Elastic extremal elements in $\mathcal{C}_W$ (3/3)

Proof:

- Let $S \in \mathcal{C}_W$ such that $R(S)$ is extremal in $\mathcal{C}_{\mathrm{Sym}}$.
- Then $R(S) = \gamma^2$ for some $\gamma \in \mathrm{Sym}^2 V^*$.
- We have $R' \circ R^{-1}(\gamma^2)(x, y, z, w) = \gamma(x, w)\gamma(y, z) - \gamma(y, w)\gamma(x, z)$.

Elastic extremal elements in $\mathcal{C}_W$ (3/3)

Proof:

- Let $S \in \mathcal{C}_W$ such that $R(S)$ is extremal in $\mathcal{C}_{\mathrm{Sym}}$.
- Then $R(S) = \gamma^2$ for some $\gamma \in \mathrm{Sym}^2 V^*$.
- We have $R' \circ R^{-1}(\gamma^2)(x, y, z, w) = \gamma(x, w)\gamma(y, z) - \gamma(y, w)\gamma(x, z)$.
- Since $S \in \mathcal{C}_W$, for all $x, y \in V^*$ we must have

$$0 \leq R'(S)(x \wedge y, x \wedge y) = \gamma(x, y)^2 - \gamma(x, x)\gamma(y, y) = -\det(\gamma|_{\mathrm{span}\{x,y\}}).$$

Elastic extremal elements in $\mathcal{C}_W$ (3/3)

Proof:

- Let $S \in \mathcal{C}_W$ such that $R(S)$ is extremal in $\mathcal{C}_{\mathrm{Sym}}$.
- Then $R(S) = \gamma^2$ for some $\gamma \in \mathrm{Sym}^2 V^*$.
- We have $R' \circ R^{-1}(\gamma^2)(x, y, z, w) = \gamma(x, w)\gamma(y, z) - \gamma(y, w)\gamma(x, z)$.
- Since $S \in \mathcal{C}_W$, for all $x, y \in V^*$ we must have

$$0 \leq R'(S)(x \wedge y, x \wedge y) = \gamma(x, y)^2 - \gamma(x, x)\gamma(y, y) = -\det(\gamma|_{\mathrm{span}\{x,y\}}).$$

$\implies$ For all 2d subspaces E, $\gamma|_E$ is indefinite or degenerate.

Elastic extremal elements in $\mathcal{C}_W$ (3/3)

Proof:

- Let $S \in \mathcal{C}_W$ such that $R(S)$ is extremal in $\mathcal{C}_{\mathrm{Sym}}$.
- Then $R(S) = \gamma^2$ for some $\gamma \in \mathrm{Sym}^2 V^*$.
- We have $R' \circ R^{-1}(\gamma^2)(x, y, z, w) = \gamma(x, w)\gamma(y, z) - \gamma(y, w)\gamma(x, z)$.
- Since $S \in \mathcal{C}_W$, for all $x, y \in V^*$ we must have

$$0 \leq R'(S)(x \wedge y, x \wedge y) = \gamma(x, y)^2 - \gamma(x, x)\gamma(y, y) = -\det(\gamma|_{\mathrm{span}\{x,y\}}).$$

$\implies$ For all 2d subspaces E, $\gamma|_E$ is indefinite or degenerate.

$\implies$ $\gamma = \alpha \vee \beta$ for some $\alpha, \beta \in V^*$.

Classification of extremal elements for $\dim V = 2$

Theorem (Li, Xu, Yang, Zhang, Zhou '21): Suppose $\dim V = 2$.
Then $S \in \mathcal{C}_W$ is extremal iff

$$S = (\alpha \otimes \beta)^{\otimes 2} + (\beta \otimes \alpha)^{\otimes 2}, \quad \text{for some } \alpha, \beta \in V^* \setminus \{0\}.$$

Classification of extremal elements for $\dim V = 2$

Theorem (Li, Xu, Yang, Zhang, Zhou '21): Suppose $\dim V = 2$.
Then $S \in \mathcal{C}_W$ is extremal iff

$$S = (\alpha \otimes \beta)^{\otimes 2} + (\beta \otimes \alpha)^{\otimes 2}, \quad \text{for some } \alpha, \beta \in V^* \setminus \{0\}.$$

Proof: $(\dim V = 2 \implies \dim \mathrm{Sym}^2 V = 3, \dim \Lambda^2 V = 1)$

Classification of extremal elements for $\dim V = 2$

Theorem (Li, Xu, Yang, Zhang, Zhou '21): Suppose $\dim V = 2$.
Then $S \in \mathcal{C}_W$ is extremal iff

$$S = (\alpha \otimes \beta)^{\otimes 2} + (\beta \otimes \alpha)^{\otimes 2}, \quad \text{for some } \alpha, \beta \in V^* \setminus \{0\}.$$

Proof: $(\dim V = 2 \implies \dim \mathrm{Sym}^2 V = 3, \dim \Lambda^2 V = 1)$

- Suppose S is extremal in $\mathcal{C}_W$ with rank $R(S) > 1$.

Classification of extremal elements for $\dim V = 2$

Theorem (Li, Xu, Yang, Zhang, Zhou '21): Suppose $\dim V = 2$.
Then $S \in \mathcal{C}_W$ is extremal iff

$$S = (\alpha \otimes \beta)^{\otimes 2} + (\beta \otimes \alpha)^{\otimes 2}, \quad \text{for some } \alpha, \beta \in V^* \setminus \{0\}.$$

Proof: $(\dim V = 2 \implies \dim \operatorname{Sym}^2 V = 3, \dim \Lambda^2 V = 1)$

- Suppose S is extremal in $\mathcal{C}_W$ with $\operatorname{rank} R(S) > 1$.
- Then $\operatorname{rank} R(S) = 2$, $\operatorname{rank} R'(S) = 0$ ($\implies S \in Sym^4 V^*$)

Classification of extremal elements for $\dim V = 2$

<u>Theorem</u> (Li, Xu, Yang, Zhang, Zhou '21): Suppose $\dim V = 2$.
Then $S \in \mathcal{C}_W$ is extremal iff

$$S = (\alpha \otimes \beta)^{\otimes 2} + (\beta \otimes \alpha)^{\otimes 2}, \quad \text{for some } \alpha, \beta \in V^* \setminus \{0\}.$$

Proof: $(\dim V = 2 \implies \dim \mathrm{Sym}^2 V = 3, \dim \Lambda^2 V = 1)$

- Suppose S is extremal in $\mathcal{C}_W$ with rank $R(S) > 1$.
- Then rank $R(S) = 2$, rank $R'(S) = 0$ ($\implies S \in Sym^4 V^*$)
- For $S \in \mathcal{C}_W \cap \mathrm{Sym}^4 V^*$, every $T \in \ker S \cap \mathrm{Sym}^2(\mathrm{Sym}^2 V^*) \setminus \{0\}$ must be indefinite.

Classification of extremal elements for $\dim V = 2$

Theorem (Li, Xu, Yang, Zhang, Zhou '21): Suppose $\dim V = 2$.
Then $S \in \mathcal{C}_W$ is extremal iff

$$S = (\alpha \otimes \beta)^{\otimes 2} + (\beta \otimes \alpha)^{\otimes 2}, \quad \text{for some } \alpha, \beta \in V^* \setminus \{0\}.$$

Proof: $(\dim V = 2 \implies \dim \operatorname{Sym}^2 V = 3, \dim \Lambda^2 V = 1)$

- Suppose S is extremal in $\mathcal{C}_W$ with rank $R(S) > 1$.
- Then rank $R(S) = 2$, rank $R'(S) = 0$ ($\implies S \in Sym^4 V^*$)
- For $S \in \mathcal{C}_W \cap \operatorname{Sym}^4 V^*$, every $T \in \ker S \cap \operatorname{Sym}^2(\operatorname{Sym}^2 V^*) \setminus \{0\}$ must be indefinite.
- If $\ker S \cap \operatorname{Sym}^2(\operatorname{Sym}^2 V^*) = \lambda T$, every $\alpha \in V^*$ such that $T(\alpha, \alpha) = 0$ satisfies $\ker \alpha^4 \supsetneq \ker S$, a contradiction.

Classification of extremal elements for $\dim V = 3$

Theorem: Suppose $\dim V = 3$. Then $S \in \mathcal{C}_W$ is extremal iff it is of one of the following types:

1. $S = (\alpha \otimes \beta)^{\otimes 2} + (\beta \otimes \alpha)^{\otimes 2}$ for some $\alpha, \beta \in V^* \setminus \{0\}$,

Classification of extremal elements for $\dim V = 3$

Theorem: Suppose $\dim V = 3$. Then $S \in \mathcal{C}_W$ is extremal iff it is of one of the following types:

1. $S = (\alpha \otimes \beta)^{\otimes 2} + (\beta \otimes \alpha)^{\otimes 2}$ for some $\alpha, \beta \in V^* \setminus \{0\}$,
2. For a basis $\{\alpha_i\}_{i=1}^3$ of V^*, $g, d, h \in \mathbb{R}$ s.t. $g^2 > 1 - d^2 + dh$, we have

$$S = S_{\text{tot}} + 2\left(g^2 + d^2 - 1 - dh\right)\left[(\alpha_2 \otimes \alpha_3)^{\otimes 2} + (\alpha_3 \otimes \alpha_2)^{\otimes 2}\right],$$

where

$$S_{\text{tot}} = \alpha_1^4 + 6\,\alpha_1^2\alpha_2^2 + 6\,\alpha_1^2\alpha_3^2 + 12d\,\alpha_1\alpha_2^2\alpha_3 + 12g\,\alpha_1\alpha_2\alpha_3^2 + 4h\,\alpha_1\alpha_3^3 + (1 + d^2)\,\alpha_2^4 + 4dg\,\alpha_2^3\alpha_3 + 6(1 + dh)\,\alpha_2^2\alpha_3^2 + 4g(d + h)\,\alpha_2\alpha_3^3 + (1 + g^2 + h^2)\,\alpha_3^4.$$

Proof of the classification

Strategy:

Proof of the classification

Strategy:

- establish **bounds on rank**: for $S \in \mathcal{C}_W$ extremal,

$$\dim \ker S > 3, \quad \dim \ker R(S) \geq 1, \quad \dim \ker R'(S) \geq 1;$$

Proof of the classification

Strategy:

- establish **bounds on rank**: for $S \in \mathcal{C}_W$ extremal,

$$\dim \ker S > 3, \quad \dim \ker R(S) \geq 1, \quad \dim \ker R'(S) \geq 1;$$

- distinguish elements by the dimension of characteristic subspaces of their kernels;

Proof of the classification

Strategy:

- establish **bounds on rank**: for $S \in \mathcal{C}_W$ extremal,

$$\dim\ker S > 3, \quad \dim\ker R(S) \geq 1, \quad \dim\ker R'(S) \geq 1;$$

- distinguish elements by the dimension of characteristic subspaces of their kernels;
- a (non-elastic) element $S \in \mathcal{C}_W$ is extremal iff
  1. $\dim(\ker S \cap \operatorname{Sym}^2 V) = 3$,
  2. $\dim(\ker S \cap \Lambda^2 V) = 2 \quad (\implies \exists\, z \in V \setminus \{0\} : \ker S \cap \Lambda^2 V = z \wedge V)$,
  3. $\ker S \cap (z \vee V) = 0$;

Proof of the classification

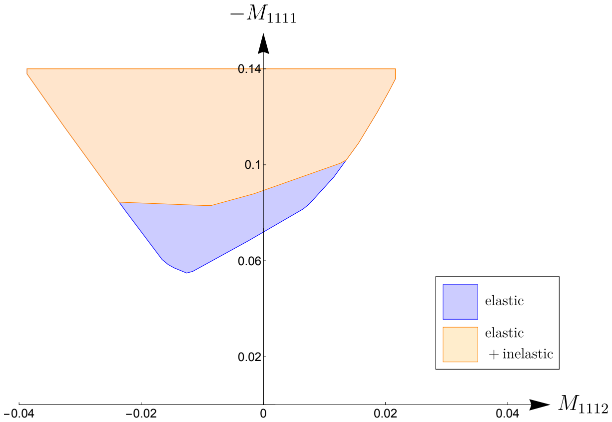
Strategy:

- establish **bounds on rank**: for $S \in \mathcal{C}_W$ extremal,

$$\dim \ker S > 3, \quad \dim \ker R(S) \geq 1, \quad \dim \ker R'(S) \geq 1;$$

- distinguish elements by the dimension of characteristic subspaces of their kernels;
- a (non-elastic) element $S \in \mathcal{C}_W$ is extremal iff
  1. $\dim(\ker S \cap \operatorname{Sym}^2 V) = 3$,
  2. $\dim(\ker S \cap \Lambda^2 V) = 2 \quad (\implies \exists\, z \in V \setminus \{0\} : \ker S \cap \Lambda^2 V = z \wedge V)$,
  3. $\ker S \cap (z \vee V) = 0$;
- for these elements we can find an explicit description.

Elastic vs full bounds



Further observations

For $G \in \{\mathrm{O}(3), \mathrm{SO}(2), \mathbb{Z}_2^3\}$ consider the cone of invariant elements:

$$\mathcal{C}_W^G := \{M \in \mathcal{C}_W : g \cdot M = M \; \forall g \in G\}.$$

Theorem: All extremal elements of $\mathcal{C}_W^G$ are projections of elastic elements in $\mathcal{C}_W$.

In particular: Elastic bounds are sufficient to bound two-to-two pion scattering in chiral perturbation theory!

What about $\dim V > 3$? (1/3)

What about $\dim V > 3$? (1/3)

Take a closer look at the case $R'(S) = 0$, i.e. $S \in \mathrm{Sym}^4 V^*$:

What about $\dim V > 3$? (1/3)

Take a closer look at the case $R'(S) = 0$, i.e. $S \in \mathrm{Sym}^4 V^*$:

Denote by:

$P_{n,2d}$: cone of homogeneous degree $2d$ **non-negative polynomials** in n variables;
$\Sigma_{n,2d} \subset P_{n,2d}$: cone of **SOS-polynomials**.

What about $\dim V > 3$**?** **(1/3)**

Take a closer look at the case $R'(S) = 0$, i.e. $S \in \mathrm{Sym}^4 V^*$:

Denote by:

$P_{n,2d}$: cone of homogeneous degree $2d$ **non-negative polynomials** in n variables;
$\Sigma_{n,2d} \subset P_{n,2d}$: cone of **SOS-polynomials**.

The cone $\mathrm{Sym}^4 V^* \cap \mathcal{C}_W$ has the following interpretation for $n = \dim V$:

$$\mathrm{Sym}^4 V^* \cap \mathcal{C}_W = \Sigma^*_{n,4}.$$

What about $\dim V > 3$? (2/3)

We have

$$
\begin{aligned}
P_{n,2d} = \Sigma_{n,2d} \quad &\iff \quad \text{all extremal elements of } \Sigma^*_{n,2d} \text{ have rank } 1 \\
&\iff \quad n = 2 \quad \text{or} \quad 2d = 2 \quad \text{or} \quad n = 3,\ 2d = 4
\end{aligned}
$$

What about $\dim V > 3$? (2/3)

We have

$$
\begin{aligned}
P_{n,2d} = \Sigma_{n,2d} \quad &\Longleftrightarrow \quad \text{all extremal elements of } \Sigma^*_{n,2d} \text{ have rank } 1 \\
&\Longleftrightarrow \quad n = 2 \quad \text{or} \quad 2d = 2 \quad \text{or} \quad n = 3,\ 2d = 4
\end{aligned}
$$

Blekherman '10: Extremal elements of $\Sigma^*_{4,4}$ have either rank 1 or rank 6.

What about $\dim V > 3$? (2/3)

We have

$$P_{n,2d} = \Sigma_{n,2d} \quad \Longleftrightarrow \quad \text{all extremal elements of } \Sigma^*_{n,2d} \text{ have rank 1}$$
$$\Longleftrightarrow \quad n = 2 \quad \text{or} \quad 2d = 2 \quad \text{or} \quad n = 3,\ 2d = 4$$

Blekherman '10: Extremal elements of $\Sigma^*_{4,4}$ have either rank 1 or rank 6.

Blekherman, Hauenstein, Ottem, Ranestad, Sturmfels '11:
The rank 6 elements in $\Sigma^*_{4,4}$ form a projective variety of dimension 24 and degree 28314.

What about $\dim V > 3$? (2/3)

We have

$$P_{n,2d} = \Sigma_{n,2d} \iff \text{all extremal elements of } \Sigma^*_{n,2d} \text{ have rank 1}$$
$$\iff n = 2 \quad \text{or} \quad 2d = 2 \quad \text{or} \quad n = 3,\ 2d = 4$$

Blekherman '10: Extremal elements of $\Sigma^*_{4,4}$ have either rank 1 or rank 6.

Blekherman, Hauenstein, Ottem, Ranestad, Sturmfels '11:
The rank 6 elements in $\Sigma^*_{4,4}$ form a projective variety of dimension 24 and degree 28314.

But: There is no parameterization of all extremal elements of $\Sigma^*_{4,4}$!

What about $\dim V > 3$? (3/3)

- It is too ambitious to aim for an extremal representation in $\dim V > 3$.

What about $\dim V > 3$? (3/3)

- It is too ambitious to aim for an extremal representation in $\dim V > 3$.
- **But:** The extremal elements in the classification for $\dim V = 3$ give rise to extremal elements in $\dim V > 3$.

What about $\dim V > 3$? (3/3)

- It is too ambitious to aim for an extremal representation in $\dim V > 3$.
- **But:** The extremal elements in the classification for $\dim V = 3$ give rise to extremal elements in $\dim V > 3$.
- Using this, one can constrain possible extremal elements in $\dim V > 3$...

Summary and Outlook

Today:

- Unitarity, causality, locality of UV theory $\implies$ positivity bounds on EFT amplitude;
- Positivity bounds $\longleftrightarrow$ extremal elements of $\mathcal{C}_W$;
- For two flavors, there are only elastic bounds, for three flavors there are additional bounds;
- When imposing additional symmetries there are only elastic bounds;
- In dimension $n > 3$ the problem of determining the extremal representation becomes significantly harder.

Open questions:

- Can we "simplify" the inequalities for $\dim V = 3$?
- What is the boundary of $\mathcal{C}_W^*$ for $\dim V = 3$?
- How large is the difference between elastic and full bounds?
- Further applications to physics
- Study the case $\dim V = 4$.

Thank you!