Quantified Constraint Satisfaction Problem and H-Free Algorithmics

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Joint work with Tala Eagling-Vose, Santiago Guzman Pro, Barnaby Martin and Yiming Qiu

Mathematics and Theoretical Physics Seminar

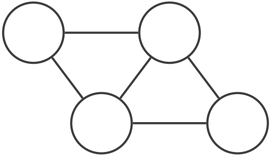
University of Hertfordshire

22 April 2026

Outline

1. **Background**
   - Constraint Satisfaction Problem (CSP)
   - Quantified Constraint Satisfaction Problem (QCSP)
   - H-freeness
2. **Our paper: Quantified Colouring and H-Free Algorithmics**
   - $\mathrm{QCSP}(K_3)$
   - P_5- and P_4-free graphs
   - The Line Graph of $K^r_{1,4}$

Colouring



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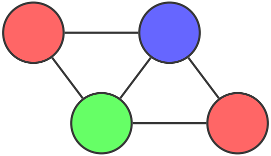
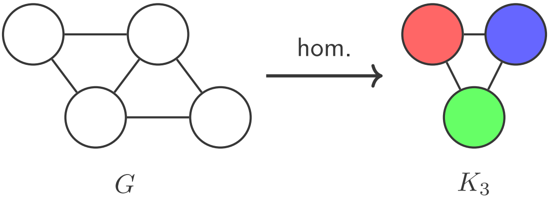
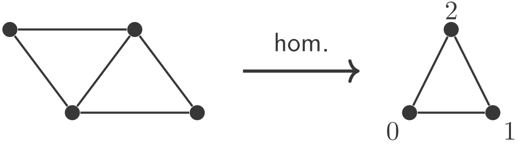


3-Colouring

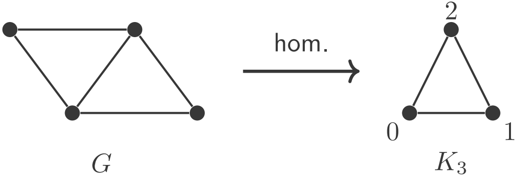


3-Colouring, **a.k.a.** $\mathrm{CSP}(K_3)$



G K_3

3-COLOURING, a.k.a. CSP(K_3)



$$\exists x_1 \exists x_2 \exists x_3 \exists x_4 (E(x_1, x_2) \wedge E(x_1, x_3) \wedge E(x_2, x_3) \wedge E(x_2, x_4) \wedge E(x_3, x_4))$$

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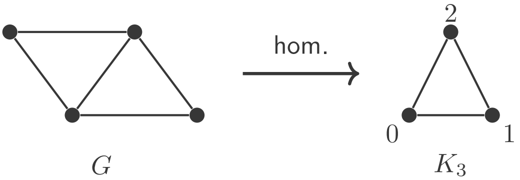

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$\mathrm{CSP}(K_3)$:

- Input: a sentence ϕ that uses $\exists, \wedge$ and the edge relation E
- Output: yes if $K_3 \vDash \phi$, no otherwise

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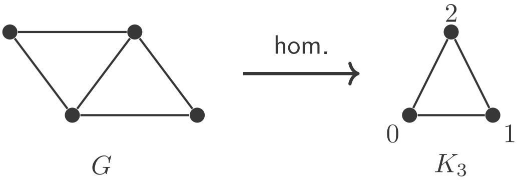

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When A is finite, $\mathrm{CSP}(A)$ is in NP.
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QUANTIFIED 3-COLOURING, a.k.a. $\mathrm{QCSP}(K_3)$

$$\forall x_1 \exists y_1 \forall x_2 \exists y_2 \ldots \Phi$$

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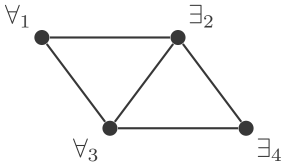

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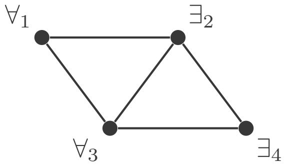

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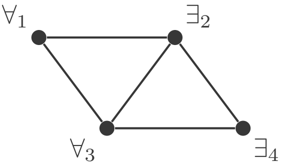

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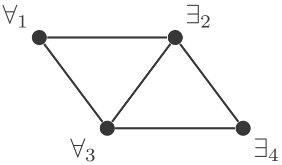

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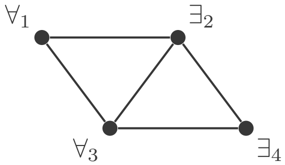

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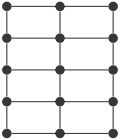
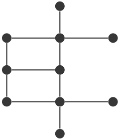
QCSP captures games: Existential vs Universal players.

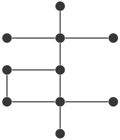
H-Free Graphs: Examples of Containment



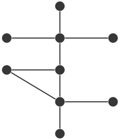
G



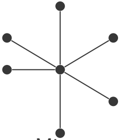
Induced subgraph
(vertex deletion)



Subgraph
(vertex deletion
and edge deletion)



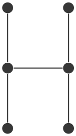
Topological minor
(vertex deletion,
edge deletion and
vertex dissolution
(degree 2 only))



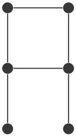
Minor
(vertex deletion,
edge deletion and
edge contraction)

If a graph G doesn't have some graph H as an induced subgraph, then we say that G is H-free.

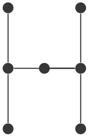
H-Free Graphs



H_1



H_1-free
but not
H_1-subgraph-free



H_1-subgraph-free
but not
H_1-minor-free

C123-framework (JMOPPSV 2025)

A graph problem Π satisfying the following conditions (the operation of *edge subdivision* replaces an edge with a path of some fixed length):

C1. Π is efficiently solvable for every graph class of bounded treewidth;
C2. Π is computationally hard for the class of subcubic graphs; and
C3. computational hardness is preserved under edge subdivision of subcubic graphs;

is called a *C123-problem*.

Let $\mathcal{S}$ be the class of disjoint unions of subdivided claws or paths.

Theorem

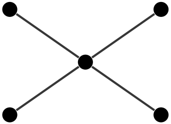
Let Π be a C123-problem. For any finite set of graphs $\mathcal{H}$, the problem Π on $\mathcal{H}$-subgraph-free graphs is efficiently solvable if $\mathcal{H}$ contains a graph from $\mathcal{S}$ (or equivalently, if the class of $\mathcal{H}$-subgraph-free graphs has bounded treewidth) and computationally hard otherwise.

3-Colouring

- $\mathrm{CSP}(K_3)$ is NP-complete.
- $\mathrm{CSP}(K_3)$ is in P for subcubic graphs (Brooks' Theorem).

subcubic = maximum degree 3 = $K_{1,4}$-subgraph-free

$K_{1,4}$



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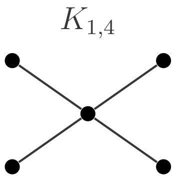

- $\mathrm{QCSP}(K_3)$ is Pspace-complete.
- $\mathrm{QCSP}(K_3)$ on subcubic graphs?

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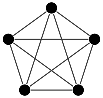
   - The Line Graph of $K^r_{1,4}$

$\mathrm{QCSP}(K_3)$: hardness results

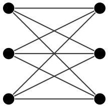
Proposition

$\mathrm{QCSP}(K_3)$ *is* Pspace*-complete on subcubic planar graphs.*

planar $= \{K_5, K_{3,3}\}$-minor-free



K_5



$K_{3,3}$

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Proposition

For $r \geq 3$ *odd,* Σ_r-$\mathrm{QCSP}(K_3)$ *is* $\Sigma^{\mathrm{P}}_{r-2}$*-hard on subcubic planar graphs.*

Theorem

For $r \geq 3$ odd, the complexity of Σ_r-$\mathrm{QCSP}(K_3)$ on $\mathcal{H}$-topological-minor-free graphs is completely classified (even when $\mathcal{H}$ is infinite). If $\mathcal{H}$ contains some planar subcubic graph, then Σ_r-$\mathrm{QCSP}(K_3)$ is solvable in polynomial time; otherwise it is $\Sigma^{\mathrm{P}}_{r-2}$-hard.

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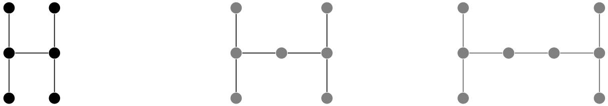
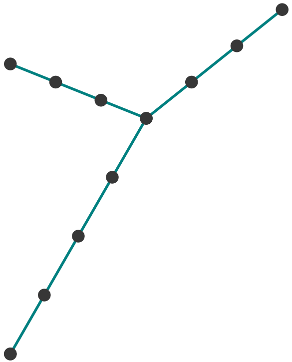
Proposition

$\mathrm{QCSP}(K_3)$ *is* Pspace*-complete on* $\mathbb{H}_1$*-subgraph-free graphs.*



H_1 H_2 H_3

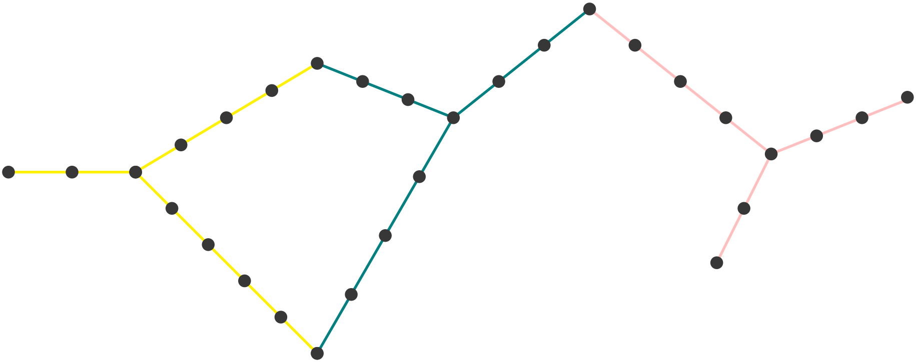
$\mathrm{QCSP}(K_3)$: tractability results



Proposition

Let ϕ be an input for subcubic $\mathrm{QCSP}(K_3)$ whose graph ϕ_G is a tree in which at most one vertex has degree 3. Then determining if $K_3 \models \phi$ is in P.

$\mathrm{QCSP}(K_3)$: tractability results



Proposition

$\mathrm{QCSP}(K_3)$ *is in polynomial time on* subcubic $\{\mathbb{H}_1, \mathbb{H}_2, \mathbb{H}_3, \mathbb{H}_4, \mathbb{H}_5, C_3, C_4\}$*-subgraph-free graphs.*

Corollary: $\mathrm{QCSP}(K_3)$ is not a C123-problem.

P_4- and P_5-free Graphs

Theorem

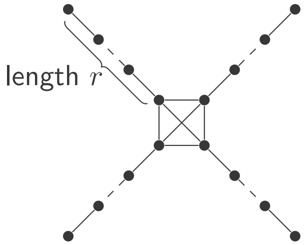
If G is a finite graph, then $\mathrm{QCSP}(G)$ restricted to inputs that are P_5-free is in NP.

P_5

Proposition

$\mathrm{QCSP}(K_3)$ is in polynomial time for inputs that are restricted to be P_4-free.

The Line Graph of $K^r_{1,4}$



$L\left(K^r_{1,4}\right)$

For $r \geq 8$:

m	The complexity of $\mathrm{QCSP}(L(K^r_{1,4}))$ on P_m-free graphs
≤ 3	P
$r-1$	NP-complete
$\geq 2r+10$	Pspace-complete

The end

Thank you for your attention!