

Quivers and BPS states in 3d and 4d

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Based on 2508.09729 with
P. Longhi, D. Noshchenko, S. Park, & P. Sułkowski

University of Hertfordshire, April 23, 2026

Outline

- 1 Motivation
- 2 Dimension door from two perspectives
 - Physics
 - Topology
- 3 Entering the dimension door: wall-crossing and unlinking
 - Chambers, dualities, and equivalent quivers
 - Structure of wall-crossing and unlinkings

So close, yet so far

BPS states in 3d $\mathcal{N} = 2$

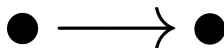
Symmetric quivers



Ekholm, P.K., Longhi,
Reineke, Stošić,
Sułkowski

BPS states in 4d $\mathcal{N} = 2$

Directed quivers



Gaiotto, Moore, Neitzke,
Alim, Cecotti, Cordova,
Espahbodi, Rastogi, Vafa...

How to travel between dimensions?

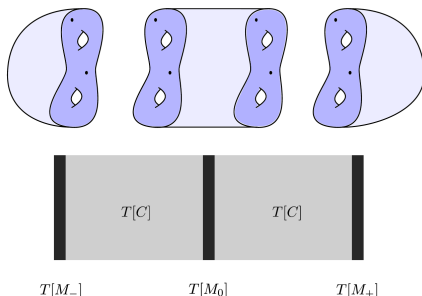


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M-theory setup ($6 = 2+4 = 3+3$)

- Spacetime: $T^*M \times S^1 \times \mathbb{R}^4$
- Two M5-branes wrap $M \times S^1 \times \mathbb{R}^2$

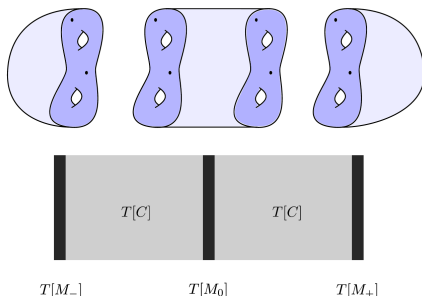


- Coupled 3d-4d system

- $C \sim \partial M_{\pm/0}$ gives 4d $\mathcal{N} = 2$ theory $T[C]$ on $S^1 \times \mathbb{R}^2 \times \text{interval}$
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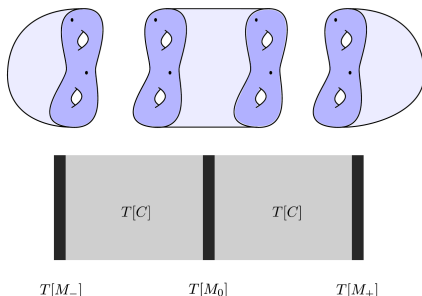


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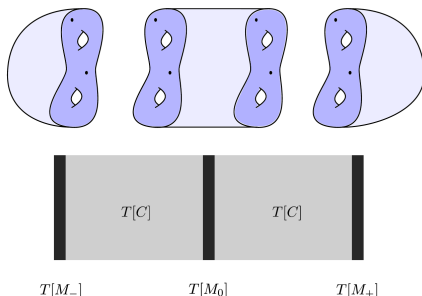
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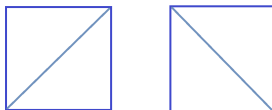
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BPS states in 4d

- Argyres-Douglas theories: $C = \mathbb{C}$ with $n+3$ roots of unity $\sim (n+3)$ -gon
- They are associated to the A_n -type quivers:



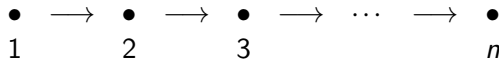
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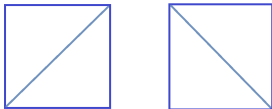
[Gaiotto-Moore-Neitzke]

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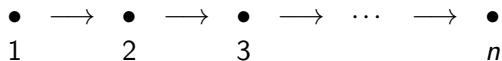
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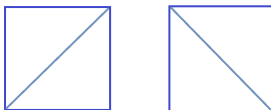
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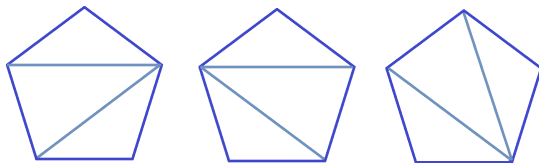
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Example: BPS states for the pentagon

- Example: Argyres-Douglas theory associated to A_2 quiver

$$\begin{array}{ccc} \bullet & \longrightarrow & \bullet \\ 1 & & 2 \end{array}$$

- $n = 2 \Rightarrow C = \mathbb{C}$ with 5 roots of unity \sim pentagon



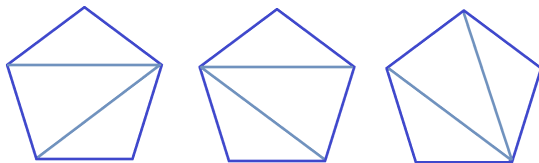
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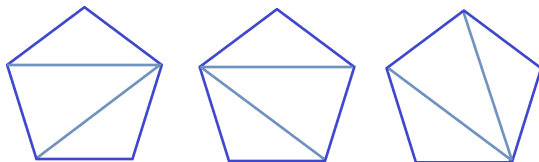
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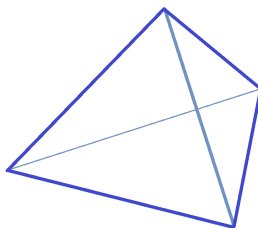
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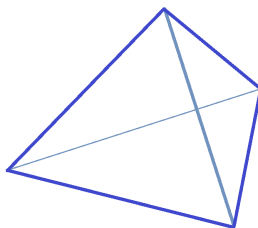
- We want M_0 such that $\partial M_0 \sim$ polygons
- BPS states are given by tetrahedra



[Cecotti-Cordova-Vafa, Dimofte-Gaiotto-Gukov, Terashima-Yamazaki]

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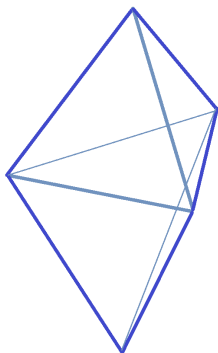
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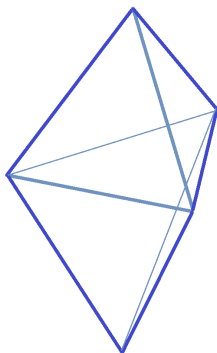


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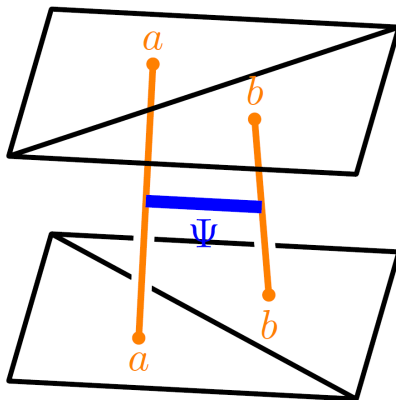
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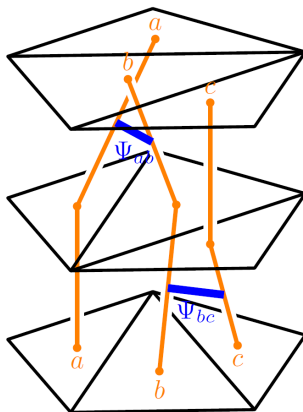
BPS states and holomorphic disks

- Consider double cover of C and M
- BPS states \sim holomorphic disks (M2 branes) stretched between branching loci



Example: two disks

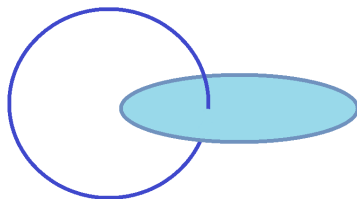
- Branching loci and disks in $C \sim$ pentagon, $M \sim$ bipyramid



[Cecotti-Cordova-Vafa]

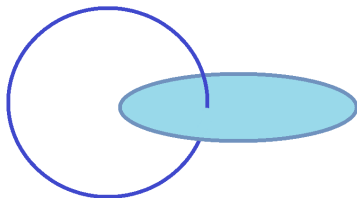
Linking

- When we close M_0 by M_{\pm} , we can define linking
- We count intersections between one disk and the boundary of the other



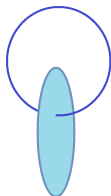
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Linked holomorphic disks and symmetric quivers

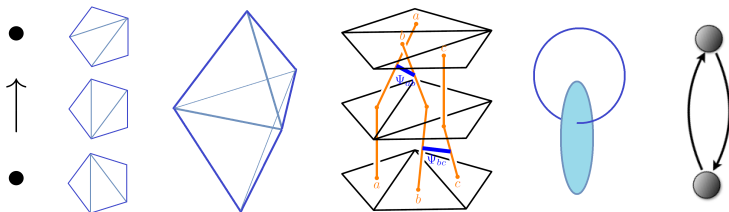
- Knot-quiver correspondence \Rightarrow linked holomorphic disks can be associated to symmetric quivers
 - holomorphic disks \sim quiver nodes
 - linking \sim quiver arrows
- We can apply this idea also to disks that do not come from knots!



[Ekholm-P.K.-Longhi]

BPS counts and quivers

Count of BPS states in 4d theory $T[C]$ for directed quiver =
 = Count of BPS states in 3d theory $T[M]$ such that $\partial M \sim C$
 = Generating series of symmetrisation of the initial quiver

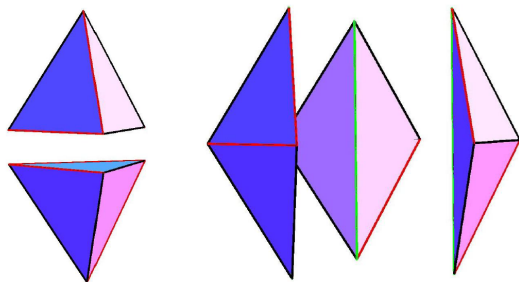


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Dual 3d theories – Pachner move

- We can divide the bipyramid in two different ways!



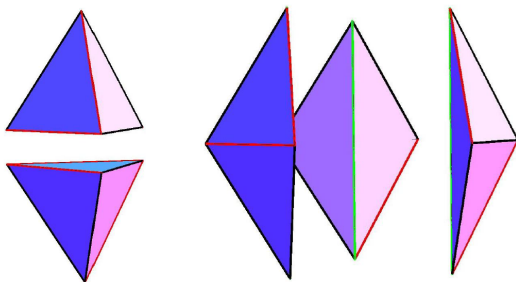
Source: Cecotti, Cordova, Vafa, 1110.2115

- We can also have 3 tetrahedra \Rightarrow 3 BPS states
- Two dual 3d theories

[Cecotti-Cordova-Vafa, Dimofte-Gaiotto-Gukov, Terashima-Yamazaki]

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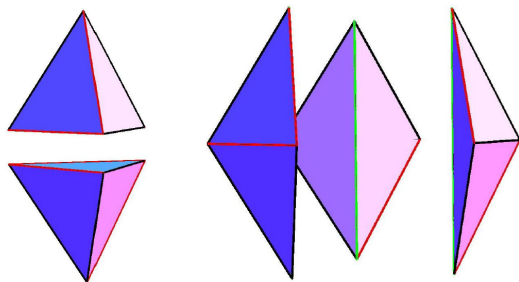
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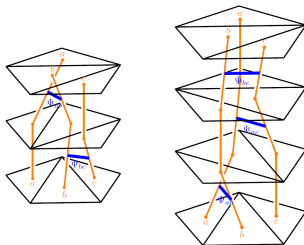
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2 and 3 holomorphic disks

- 3 BPS states correspond to 3 holomorphic disks



- Algebraic version for non-commutative operators representing holomorphic disks – 2-3 wall-crossing relation:

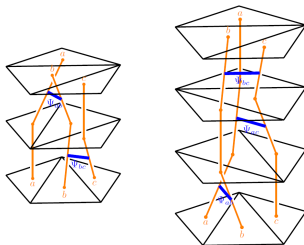
$$\Psi_{ab}\Psi_{bc} = \Psi_{bc}\Psi_{ac}\Psi_{ab}$$

$$\Psi_1\Psi_2 = \Psi_2\Psi_{1+2}\Psi_1$$

[Fadeev-Kashaev, Kontsevich-Soibelman]

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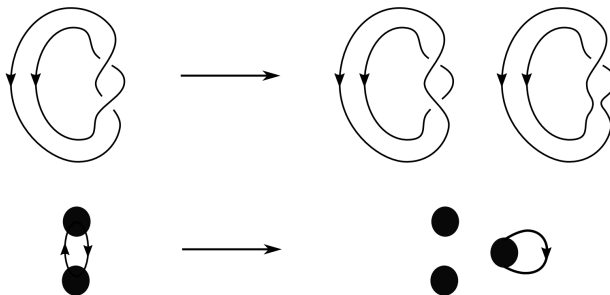
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Equivalent symmetric quivers – unlinking

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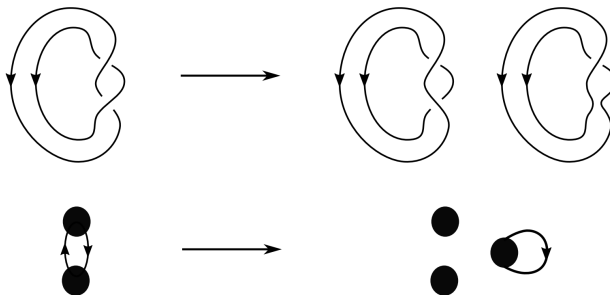


- Two symmetric quivers with the same generating series

[Ekholm-P.K.-Longhi]

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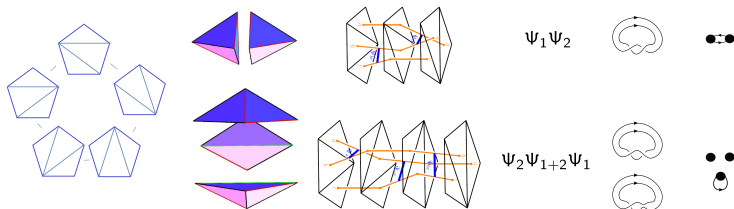


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[Ekholm-P.K.-Longhi]

2-3 move

Two different chambers with 2 and 3 BPS states in 4d theory \sim
 \sim Two dual 3d theories with 2 and 3 BPS states
 \sim Two equivalent symmetric quivers with 2 and 3 nodes



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More complicated wall-crossing: choices

- Example: Argyres-Douglas theory associated to A_3 quiver



- We can perform 2-3 move for (12) and (23)
- We have to choose the order!

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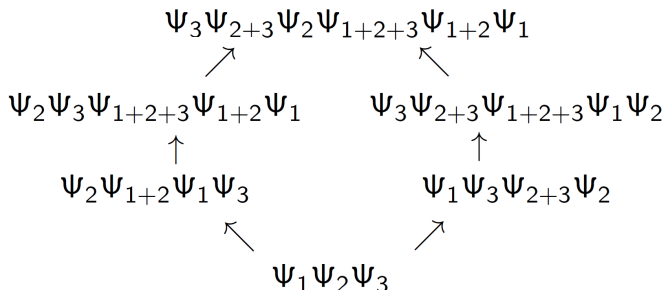
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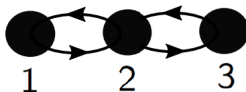
More complicated wall-crossing: choices

- Algebraic perspective:
 - operators connected by arrows satisfy 2-3 wall-crossing relation
 - operators which are not connected by arrows commute



More complicated unlinking: choices

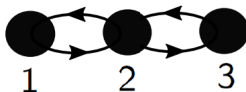
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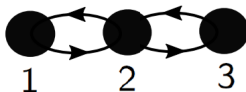
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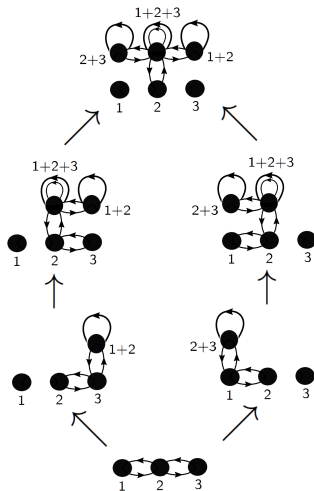
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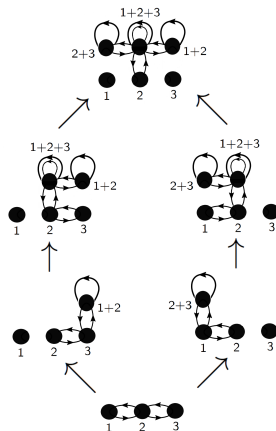
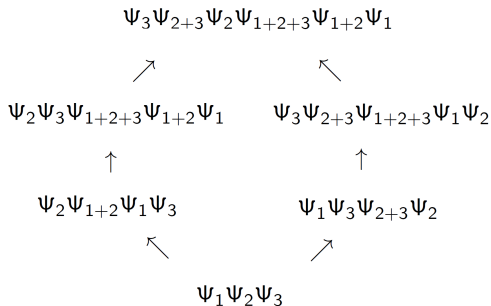


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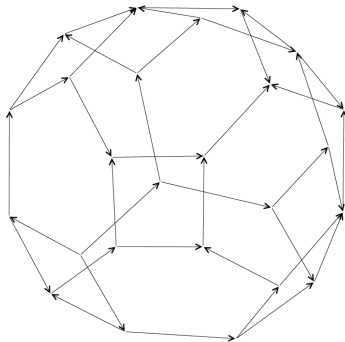
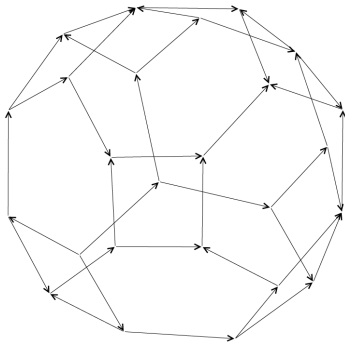
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Wall-crossing for A_3 and unlinking for its symmetrisation

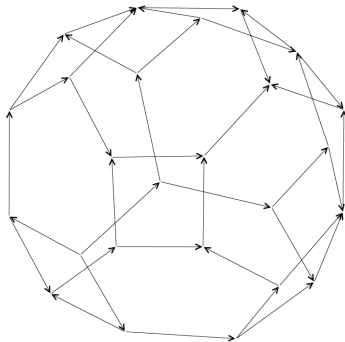
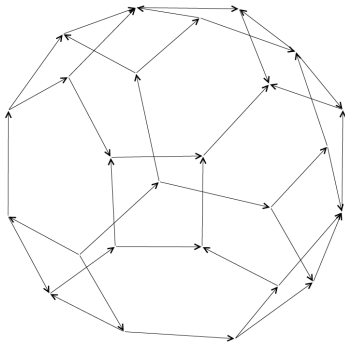


Wall-crossing for A_4 and unlinking for its symmetrisation



- The same pattern works for all A_n quivers and their symmetrisations!

Wall-crossing for A_4 and unlinking for its symmetrisation



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Summary

- Proper division of theory on M5-brane ($6d = 4d+2d = 3d+3d$) allows for relating BPS spectra in 4d and 3d theories associated to 2d and 3d manifolds
- This allows to relate directed quivers to their symmetrisations
- This relation maps the structure of wall-crossing to the structure of unlinkings

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Last messages

My work is supported by the Polish National Science Centre through Sonata grant (2022/47/D/ST2/02058)

Thank you for attention!