

Perspectives on Higher and Lower Form Symmetries

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Outline

- ▶ Introduction to Generalized Global Symmetries
- ▶ Higher Form Symmetries
- ▶ Gauging Higher Form Symmetries
- ▶ Lower Form Symmetries
- ▶ Generalized Abelian Gauge Theories
- ▶ Anomaly Field Theories for Dual Gauge Fields

Standard Symmetries in QFT

- ▶ Textbook symmetries: a global continuous symmetry on M corresponds to a conserved Noether current $J \in \Omega^1(M)$: $d * J = 0$
- ▶ Integrating J^0 over a spatial hypersurface $\Sigma \subset M$ at fixed time gives a conserved Noether charge $Q_\Sigma = \int_\Sigma d\Sigma J^0$
In quantum theory it generates the symmetry on the Hilbert space
- ▶ By relativistic invariance, it is more natural to integrate over any codimension 1 submanifold $\Sigma \subset M$: $Q_\Sigma = \int_\Sigma *J$, which is conserved in normal direction to Σ
- ▶ $\exp(i\alpha Q_\Sigma) =$ **topological operator of codimension 1**: depends only on homology class of Σ by Stokes' Theorem and $d * J = 0$
- ▶ **Generalization**: Submanifolds Σ of higher codimension, or forms J of higher degree

Generalized Symmetries

- ▶ Textbook symmetries in QFT are *group* actions on *local* operators
- ▶ Relax 'local' \iff higher form symmetries are *group* actions on **all** operators
- ▶ Relax 'group' \iff non-invertible symmetries are **categorical** actions on *local* operators
- ▶ Combinations of the two are **categorical** actions on **all** operators
- ▶ Modern paradigm: (Gaiotto, Kapustin, Seiberg & Willet '14; ...)
Symmetries of QFT are topological operators
- ▶ “Generalized symmetries” of QFT are described by **higher algebra**, whose action should be defined on objects with higher linear structures (e.g. higher vector bundles)

Higher Form Symmetries

- ▶ **Classical** symmetries that survive quantization
- ▶ p -form symmetries correspond to topological operators of codim $(p + 1)$, act on p -dimensional charged operators by linking, multiply by fusion; for $p > 0$ described by *abelian* groups (local = '0-form symmetries')
- ▶ **Example:** Pure d -dimensional Yang-Mills theory with simply-connected gauge group G : Centre $Z \subset G$ acts trivially on local gauge-invariant operators, but non-trivially on Wilson lines; finite electric 1-form symmetry
- ▶ Combinations of higher form symmetries can merge into **higher groups**; e.g. 1-form symmetry H combines with 0-form symmetry G to form a 2-group extension $1 \rightarrow BH \rightarrow \underline{\Gamma} \rightarrow G \rightarrow 1$
- ▶ **Expectation:** Topological operators of a d -dimensional QFT form a $(d - 1)$ -category (Johnson-Freyd '20; Bhardwaj, Schäfer-Nameki & Wu '22)
- ▶ p -form symmetries are p -morphisms in this higher category
- ▶ Accounts for topological operators of codimensions $1, \dots, d$, that is, p -form symmetries with $0 \leq p \leq (d - 1)$

Non-Invertible Symmetries

- ▶ **Quantum** symmetries described by (higher) rings or algebras
- ▶ Topological operators closely related to 'symmetry defects' in QFT, described using (higher) categorical algebra
(Fuchs, Runkel & Schweigert '02; ...)
- ▶ **Example:** Pure $SU(N)$ Yang-Mills theory on a surface Σ ($d = 2$):
 - ▶ For $x \in \Sigma$ and $U_0 \in SU(N)$, take path integral over gauge connections A with holonomy in conjugacy class $[U_0]$ on $\Sigma \setminus x$
 - ▶ Defines local topological symmetry operator $V[U_0]$ (analogue of Gukov-Witten operator in $d = 4$) (Nguyen, Tanizaki & Ünsal '21)
 - ▶ When $\Sigma = \text{disk}$, $W_R =$ Wilson line along $\partial\Sigma$ in irrep R :

$$\langle V[U_0] W_R \rangle = \chi_R(U_0) e^{-\frac{g^2}{4} C_2(R)}$$

Vanishes for a large set of U_0 and $R \implies$ non-invertible 1-form symmetry

- ▶ Imposing invertibility fixes $U_0 \in Z = \mathbb{Z}_N$ and recovers usual 1-form centre symmetry

SymTFT Picture

- ▶ Generalized symmetries of any d -dimensional QFT \mathcal{Q} on M define a $(d + 1)$ -dimensional TQFT $\text{Sym}(\mathcal{Q})$ on X – the **Symmetry Topological Field Theory (SymTFT)** – such that \mathcal{Q} and its p -form group symmetries can be recovered from $\text{Sym}(\mathcal{Q})$ by manipulations on ∂X
 - (Freed & Teleman '12; Pulman, Ševera & Valach '19; Gaiotto & Kulp '20; Freed, Moore & Teleman '22; Borsten, Kanakaris & Kim '25; . . .)
- ▶ **'Sandwich Construction'** on $X = M \times [0, 1]$:
 - ▶ $M \times 0$: Dynamical boundary where non-topological theory \mathcal{Q} lives
 - ▶ $M \times 1$: Topological boundary conditions (domain wall) on SymTFT fields depending solely on symmetries of \mathcal{Q}
- ▶ Insertion of topological defects in X : Since $[0, 1]$ is contractible, collapsing slab gives theory \mathcal{Q} on M
- ▶ Isolate geometric aspects by taking $X = M \times [0, \infty)$ (no effect on topological data)
- ▶ Formally, $\text{Sym}(\mathcal{Q})$ is a fully extended $(d + 1)$ -dimensional TQFT, boundary conditions are modules

Gauging Symmetries

Symmetries can be gauged, giving orbifold QFT with dual symmetry (not necessarily invertible) which is undone by reverse orbifold

- ▶ 0-form symmetry in $d = 2$ dimensions $\xrightarrow{\text{gauge}}$ dual (or 'quantum') 0-form symmetry (Vafa '89)
- ▶ p -form symmetry in d dimensions $\xrightarrow{\text{gauge}}$ dual $(d - p - 2)$ -form symmetry (Gaiotto, Kapustin, Seiberg & Willet '14; Tachikawa '17)
- ▶ p -form symmetry in d dimensions $\xrightarrow{\text{gauge}}$ dual category (Bhardwaj & Tachikawa '17; Chang, Lin, Shao, Wang & Yin '18)
- ▶ **Example:** Pure d -dimensional Yang-Mills theory with simply-connected gauge group G , centre $Z \subset G$
Electric 1-form symmetry $Z \xrightarrow{\text{gauge}}$ d -dimensional Yang-Mills theory with gauge group $PG = G/Z$ and magnetic $(d - 3)$ -form symmetry $\widehat{Z} = \text{Hom}(Z, U(1))$ acting on 't Hooft operators

Gauge Fields for Finite Symmetries

- ▶ Gauge field for a p -form symmetry $\mathbb{Z}_n \subset U(1)$ on M has periods in \mathbb{Z}_n
- ▶ Alternatively, a pair of $U(1)$ gauge fields $(A, B) \in \Omega^p(M) \times \Omega^{p+1}(M)$ with $nB = \text{curvature of } A$: Replace \mathbb{Z}_n gauge field by its $U(1)$ avatar $\frac{n}{2\pi} B$
- ▶ Pair (B, A) defines a **$(p + 1)$ -connection on a p -gerbe** on M :

$$\begin{array}{ccccccccc}
 0 & \longrightarrow & \mathbb{Z} & \longrightarrow & \mathbb{R} & \longrightarrow & \underline{U(1)} & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
 & & \mathbb{1} & & \iota & & & & \\
 0 & \longrightarrow & \mathbb{Z} & \longrightarrow & \frac{1}{n}\mathbb{Z} & \longrightarrow & \mathbb{Z}_n & \longrightarrow & 0
 \end{array}$$

- ▶ Regard \mathbb{Z}_n -gerbes as $U(1)$ -gerbes restricted to image of Bockstein homomorphism $H^{p+1}(M, \mathbb{Z}_n) \longrightarrow H^{p+2}(M, \mathbb{Z})$
- ▶ \mathbb{Z}_n gauge field is $\iota^* B$ for $(p + 1)$ -form $U(1)$ gauge field B
- ▶ Implement constraint by Lagrange multiplier in path integral

Lower Form Symmetries

What is the meaning of (invertible) **lower form** symmetries?

- ▶ There are **no** operators of dimension $p \leq -1$ to act on
- ▶ A (-1) -form symmetry **is** generated by topological operators filling the connected components of spacetime, modify defining parameters of theory
(Córdova, Freed, Lam & Seiberg '19)
- ▶ Gauging **is** 'reversible': gauging a $(d - 1)$ -form symmetry (generated by local topological operators) must produce a dual (-1) -form symmetry
(Sharpe '19; Wu '20)
- ▶ Consider full family of QFTs fibred over parameter space X of the theory: Symmetries are automorphisms of the fibration $\text{QFT} \rightarrow X$
(Gaiotto & Kulp '20; Vandermeulen '22; Heckman, Hübner & Murdia '24; Santilli & Sz '24)
- ▶ p -form symmetries act fibrewise on QFT_x for each $x \in X$ if $p \geq 0$, on base X if $p \leq -1$
- ▶ Including physical theory in larger descent structure allows for $p \leq -1$
(Antinucci & Benini '24; Borsten, Kanakaris & Kim '25)

Lower Form Symmetries

Problem: What is a gauge field for a (-1) -form symmetry?

- ▶ Useful perspective: An invertible p -form symmetry on M is a p -gerbe, gauge field is a $(p + 1)$ -connection (Santilli & Sz '24)
- ▶ Modelled by differential $(p + 2)$ -cocycle with curvature $F \in \Omega_{\mathbb{Z}}^{p+2}(M)$ and characteristic class in $H^{p+2}(M, \mathbb{Z})$
- ▶ Assemble into the structure of a $(p + 1)$ -groupoid, classes for differential cohomology $\hat{H}^{p+2}(M)$; e.g. in Deligne cohomology:
 - ▶ $p = -2$: Differential 0-cocycle is a class in group of connected components $H^0(M, \mathbb{Z})$
 - ▶ $p = -1$: Differential 1-cocycle is a function $f : M \rightarrow U(1)$, curvature $F = d \log f$ represents winding number in $H^1(M, \mathbb{Z})$
 - ▶ $p = 0$: Differential 2-cocycle is a $U(1)$ -bundle with connection A , curvature $F = dA$ represents Chern class in $H^2(M, \mathbb{Z})$
 - ▶ $p = 1$: Differential 3-cocycle is a $U(1)$ -gerbe with 2-connection (A, B) , $F = dB$ represents Dixmier-Douady class in $H^3(M, \mathbb{Z})$

Lower Form Symmetries

- ▶ **Chern-Weil symmetries:** Continuous abelian (-1) -form symmetries, background fields are couplings for characteristic classes in the theory
- ▶ **Example:** $SU(N)$ Yang-Mills theory in $d = 4$ with θ -angle

$$\frac{\theta}{8\pi^2} \text{Tr}(F \wedge F)$$

Invariance under $\theta \mapsto \theta + 2\pi$ is invariance under large background gauge transformations

- ▶ Gauging means promoting θ to spacetime dependent field $\theta(x)$ and treating it as background gauge field
- ▶ Dual theory with 3-form symmetry is formally an infinite ‘sum’ over all θ
- ▶ **Decomposition conjecture:** QFT in d dimensions with $(d - 1)$ -form symmetry is equivalent to disjoint union of theories (‘universes’) in which symmetry is gauged using local topological operators

(Hellerman, Henriques, Pantev, Sharpe & Ando '06; Sharpe '14; ...)

Lower Form Symmetries

Open problem: Cast (-1) -form symmetries in categorical setting, to get intrinsic explanation of gauging a $(d - 1)$ -form symmetry

- ▶ Category of topological operators in SymTFT (in $(d + 1)$ dimensions) will include (-1) -form symmetry generators; iterations $\text{SymTFT} \circ \text{SymTFT} \circ \dots$ allow for more general $p \leq -1$
- ▶ Non-invertible (-1) -form symmetry; e.g. gauging non-invertible 1-form symmetry of pure Yang-Mills theory in $d = 2$ (Santilli & Sz '24)
- ▶ **Modify notion of symmetry category:** View as d -category whose objects are d -dimensional QFTs, morphisms are topological interfaces between QFTs (Wu '20; Perez-Lona '25)
Symmetry category of an object \mathcal{Q} is the $(d - 1)$ -category $\text{End}(\mathbb{1}_{\mathcal{Q}})$, but does not capture (-1) -form symmetries which are more naturally embedded in the whole d -category

Generalized Abelian Gauge Theories

$$S = \frac{1}{2} \int_M F \wedge * F$$

M = closed d -dimensional manifold with metric

$F \in \Omega_{\mathbb{Z}}^{p+1}(M)$; locally $F = dA$ for $A \in \Omega^p(M)$, globally $F =$ curvature of a $U(1)$ - $(p-1)$ -gerbe

- ▶ 'Electric' p -form symmetry: $A \mapsto A + \lambda$, $\lambda \in \Omega_{\text{cl}}^p(M)$
Global $U(1)_e$ symmetry for $\lambda \in \Omega_{\text{cl}}^p(M)/d\Omega^{p-1}(M) \simeq H^p(M, \mathbb{R})$
- ▶ Conserved $(p+1)$ -form current: $J = F$, $d * J = d * F = 0$
- ▶ Codimension $(p+1)$ topological operator: $U_\alpha[\Sigma] = \exp(i\alpha \int_\Sigma * J)$
- ▶ Charge objects are 'Wilson' operators $W_\ell[L] = \exp(i\ell \int_L A)$ for $L \subset M$ closed with $\dim L = p$
Deforming symmetry defect $U_\alpha[\Sigma]$ past $W_\ell[L]$ results in a phase:

$$U_\alpha[\Sigma] W_\ell[L] = e^{i\alpha \ell \text{lk}(\Sigma, L)} W_\ell[L]$$

Generalized Abelian Gauge Theories

- ▶ Dual global 'magnetic' $(d - p - 2)$ -form symmetry $U(1)_m$ with conserved $(d - p - 1)$ -form current: $\tilde{J} = *F$, $d*\tilde{J} = dF = 0$
- ▶ Codim $(d - p - 1)$ topological operators: $U_{\tilde{\alpha}}[\tilde{\Sigma}] = \exp(i\tilde{\alpha} \int_{\tilde{\Sigma}} *\tilde{J})$
- ▶ Charged objects are 't Hooft' operators $\tilde{U}_{\tilde{\ell}}[\tilde{L}]$ for $\tilde{L} \subset M$ closed with $\dim \tilde{L} = (d - p - 2)$:

$$\tilde{U}_{\tilde{\alpha}}[\tilde{\Sigma}] \tilde{W}_{\tilde{\ell}}[\tilde{L}] = e^{i\tilde{\alpha} \tilde{\ell} \text{lk}(\tilde{\Sigma}, \tilde{L})} \tilde{W}_{\tilde{\ell}}[\tilde{L}]$$

- ▶ **Note:** Total higher form symmetry is **not** $U(1)_e \times U(1)_m$ when there is torsion in $H^\bullet(M, \mathbb{Z})$, because torsion link pairing yields noncommutative topological operators

(Freed, Moore & Segal '06; Becker, Benini, Schenkel & Sz '15)

Anomaly Field Theories

- ▶ Expectation: Any generalised global symmetry G with a 't Hooft anomaly is described by a TQFT, the **anomaly field theory** — an invertible relative QFT (Freed & Teleman '12; Freed '14)
- ▶ Remedies anomaly by “coupling” d -dimensional theory with symmetry G to $(d + 1)$ -dimensional anomaly field theory such that the contributions of gauge transformations cancel each other: *anomaly inflow* or *edge modes*
- ▶ Generalized abelian gauge theories have a **mixed 't Hooft anomaly**: it is not possible to simultaneously gauge the electric ($F \mapsto F - B$) and magnetic ($\tilde{B} \wedge F$) higher form symmetries
- ▶ Regard the original theory as sitting on the boundary of a $(d + 1)$ -dimensional manifold X with $\partial X = M$
The anomaly is then cancelled by an anomaly inflow from the bulk classical theory, such that the bulk+boundary theory is gauge invariant
- ▶ Conversely, starting from the bulk TQFT on X , the theory on $M = \partial X$ should follow by carefully imposing boundary conditions on the fields

Anomaly Field Theories for Dual Gauge Fields

- ▶ We'll do this in a slightly different setting than usual, which lifts the dichotomy between electric and magnetic descriptions/symmetries of the same theory — close relative of SymTFT (Schenkel & Sz '??)
- ▶ Lift generalized electric-magnetic duality of higher abelian gauge theory to pullback $\mathcal{C}^{p+1}(M, \mathbb{Z})$ in differential cohomology:

$$\begin{array}{ccc}
 \mathcal{C}^{p+1}(M, \mathbb{Z}) & \dashrightarrow & \hat{H}^{d-p-1}(M) \\
 \downarrow & & \downarrow * \tilde{F} \\
 \hat{H}^{p+1}(M) & \xrightarrow{F} & \Omega^{p+1}(M)
 \end{array}$$

$(F, \tilde{F}) =$ curvatures of $((p-1)$ -gerbe, $(d-p-2)$ -gerbe) such that $F = * \tilde{F}$, $dF = 0 = d\tilde{F}$; implies equation of motion $d * F = 0$

(Freed, Moore & Segal '06; Becker, Benini, Schenkel & Sz '15)

- ▶ **Illustration:** Anomaly field theory for semi-classical configuration space $\mathcal{C}^2(M, \mathbb{Z})$ of dual gauge fields for Maxwell theory ($p = 1$) in $d = 4$

2-BF Theory

$$\mathcal{S}(B, \tilde{B}) = \frac{1}{2} \int_X \tilde{B} \wedge dB - B \wedge d\tilde{B}$$

X = oriented 5-manifold with $\partial X = M = 4$ -manifold with metric
 $B, \tilde{B} \in \Omega^2(X)$; reduces to the standard higher BF action when $\partial X = \emptyset$

- ▶ **1-gauge transformations:** $(\Lambda, \tilde{\Lambda}) : (B, \tilde{B}) \longrightarrow (B + d\Lambda, \tilde{B} + d\tilde{\Lambda})$
with $(\Lambda, \tilde{\Lambda}) \in \Omega^1(X) \times \Omega^1(X)$
- ▶ **2-gauge transformations:** $(\varepsilon, \tilde{\varepsilon}) : (\Lambda, \tilde{\Lambda}) \implies (\Lambda + d\varepsilon, \tilde{\Lambda} + d\tilde{\varepsilon})$
with $(\varepsilon, \tilde{\varepsilon}) \in \Omega^0(X) \times \Omega^0(X)$
- ▶ Action is **not** gauge-invariant:

$$\mathcal{S}(B + d\Lambda, \tilde{B} + d\tilde{\Lambda}) = \mathcal{S}(B, \tilde{B}) + \frac{1}{2} \int_M d\tilde{\Lambda} \wedge B - d\Lambda \wedge \tilde{B}$$

Configuration 2-Groupoid for 2-BF Theory

$\mathcal{F}(X)$ = 2-groupoid of strictly boundary conditioned fields on X with

- ▶ Objects: $(B, \tilde{B}) \in \Omega^2(X) \times \Omega^2(X)$ such that $\tilde{B} = *B$ on M
- ▶ 1-morphisms: $(\Lambda, \tilde{\Lambda}) : (B, \tilde{B}) \rightarrow (B + d\Lambda, \tilde{B} + d\tilde{\Lambda})$ with $(\Lambda, \tilde{\Lambda}) \in \Omega^1(X) \times \Omega^1(X)$ such that $\Lambda = 0 = \tilde{\Lambda}$ on M
- ▶ 2-morphisms: $(\varepsilon, \tilde{\varepsilon}) : (\Lambda, \tilde{\Lambda}) \Rightarrow (\Lambda + d\varepsilon, \tilde{\Lambda} + d\tilde{\varepsilon})$ with $(\varepsilon, \tilde{\varepsilon}) \in \Omega^0(X) \times \Omega^0(X)$ such that $\varepsilon = 0 = \tilde{\varepsilon}$ on M

Then $\mathcal{S} : \mathcal{F}(X) \rightarrow \mathbb{R}$ is a gauge-invariant function on this 2-groupoid

Note: $\mathcal{F}(X)$ is a strict pullback (in the category of 2-groupoids), which is equivalent to the homotopy pullback $\mathcal{F}_{\text{ho}}(X)$ (Hollander '07)

\Rightarrow We can equivalently implement the boundary condition 'weakly' by a homotopy pullback of 2-groupoids

Edge Modes for 2-BF Theory

$\mathcal{F}_{\text{ho}}(X) \simeq$ 2-groupoid of weakly boundary conditioned fields on X with

- Objects: $(B, \tilde{B}, A, \tilde{A}) \in \Omega^2(X)^{\times 2} \times \Omega^1(M)^{\times 2}$ such that

$$\tilde{B} - d\tilde{A} = *(B - dA) \quad \text{on } M$$

- 1-morphisms:

$$(\Lambda, \tilde{\Lambda}, \lambda, \tilde{\lambda}) : (B, \tilde{B}, A, \tilde{A}) \longrightarrow (B + d\Lambda, \tilde{B} + d\tilde{\Lambda}, A + \Lambda - d\lambda, \tilde{A} + \tilde{\Lambda} - d\tilde{\lambda})$$

$$\text{with } (\Lambda, \tilde{\Lambda}, \lambda, \tilde{\lambda}) \in \Omega^1(X)^{\times 2} \times \Omega^0(M)^{\times 2}$$

- 2-morphisms:

$$(\varepsilon, \tilde{\varepsilon}) : (\Lambda, \tilde{\Lambda}, \lambda, \tilde{\lambda}) \Longrightarrow (\Lambda + d\varepsilon, \tilde{\Lambda} + d\tilde{\varepsilon}, \lambda + \varepsilon, \tilde{\lambda} + \tilde{\varepsilon})$$

$$\text{with } (\varepsilon, \tilde{\varepsilon}) \in \Omega^0(X) \times \Omega^0(X)$$

Edge Modes for 2-BF Theory

► Equivalence implemented by 2-functor $\Psi : \mathcal{F}(X) \xrightarrow{\simeq} \mathcal{F}_{\text{ho}}(X)$ with
 $\Psi(B, \tilde{B}) = (B, \tilde{B}, 0, 0)$, $\Psi(\Lambda, \tilde{\Lambda}) = (\Lambda, \tilde{\Lambda}, 0, 0)$, $\Psi(\varepsilon, \tilde{\varepsilon}) = (\varepsilon, \tilde{\varepsilon})$

► **Essential surjectivity:**

$$(A, \tilde{A}, 0, 0) : \Psi(B - dA, \tilde{B} - d\tilde{A}) \longrightarrow (B, \tilde{B}, A, \tilde{A})$$

► Transferring $\mathcal{S} : \mathcal{F}(X) \longrightarrow \mathbb{R}$ along equivalence $\Psi : \mathcal{F}(X) \xrightarrow{\simeq} \mathcal{F}_{\text{ho}}(X)$
 gives gauge-invariant action $\mathcal{S}^{\text{ext}} : \mathcal{F}_{\text{ho}}(X) \longrightarrow \mathbb{R}$ extended by edge modes:

$$\begin{aligned} \mathcal{S}^{\text{ext}}(B, \tilde{B}; A, \tilde{A}) &= \mathcal{S}(B - dA, \tilde{B} - d\tilde{A}) \\ &= \frac{1}{2} \int_X (\tilde{B} \wedge dB - B \wedge d\tilde{B}) - \frac{1}{2} \int_M F \wedge *(F - B) + B \wedge \tilde{F} \end{aligned}$$

where $F = dA$, $\tilde{F} = d\tilde{A}$, and $\tilde{B} - \tilde{F} = *(B - F)$ on M

► Recovers Maxwell action $-\frac{1}{2} \int_M F \wedge *F$ for $B = 0$,
 with $\tilde{F} = *F \implies (A, \tilde{A}) \in \Omega^1(M) \times \Omega^1(M)$ are dual gauge fields