On the structure of higher-dimensional integrable field theories

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Four-dimensional Chern-Simons

Useful paradigm — Costello - Yamazaki

Four dimensional Chern-Simons provides a systematic framework for studying many (perhaps all?) 2d Lax integrable field theories.

- Start with the following theory on $X_4 = \Sigma \times C$,

$$S = \frac{1}{2\pi i} \int_{X_4} \omega \wedge \mathrm{CS}(A) \, ,$$

  where $\omega = \varphi(z)dz$ is a meromorphic one-form and $\mathrm{CS}(A)$ is the Chern-Simons three form.
- The data of ω may contain poles and zeroes on C. These give rise to boundary conditions and disorder defects respectively.
- This data defines a divisor on our Riemann surface C, $D : C \to \mathbb{Z}$.

Four-dimensional Chern-Simons

- An example, for $C = \mathbb{CP}^1$, the data defining the Principal Chiral Model is given by

$$\omega = \alpha \, \frac{(z-a)(z+a)}{z^2} \, dz$$

or alternatively we can write as a divisor

$$(\omega) = \underbrace{1 \cdot a + 1 \cdot -a}_{(\omega)_0 - \text{ zeroes}} + \underbrace{-2 \cdot 0 - 2 \cdot \infty}_{(\omega)_\infty - \text{ poles}} \, .$$

- We want to make a choice of frame $\hat{g} : X_4 \to G$ that 'trivialises' the bundle over C,

$$A_{\bar{z}} = \hat{g}^{-1} \partial_{\bar{z}} \hat{g}$$

the existence of such a frame allows one to write

$$A = \hat{g}^{-1} \mathcal{L} \, \hat{g} + \hat{g}^{-1} \partial_{\bar{z}} \, \hat{g} \; d\bar{z} \; , \text{ where } \quad \mathcal{L} = \mathcal{L}_x dx + \mathcal{L}_y dy$$

Four-dimensional Chern-Simons

- The equations of motion

$$\omega \wedge \partial_{\bar{z}} \mathcal{L} = 0 \ ,$$
$$F_{\Sigma}(\mathcal{L}) = 0 \ ,$$

  i.e. $\mathcal{L}$ has meromorphic dependence on C and is flat along Σ as such one concludes the above is a Lax connection.

- Then defining $\hat{\mathbf{j}} = -d\hat{g}\hat{g}^{-1}$ one has

$$S[\mathcal{L}, \hat{g}\,] = \frac{1}{2\pi i} \int_{X_4} d\omega \wedge \left(\mathrm{tr}(\hat{\mathbf{j}} \wedge \mathcal{L}) + \frac{1}{3} \int_{[0,1]_t} \mathrm{tr}(\tilde{\mathbf{j}}^3) \right) \ ,$$

  integrating over C localises the theory to a 2d action on Σ .

Lax integrability briefly

- For the PCM example we have a Lax connection given by

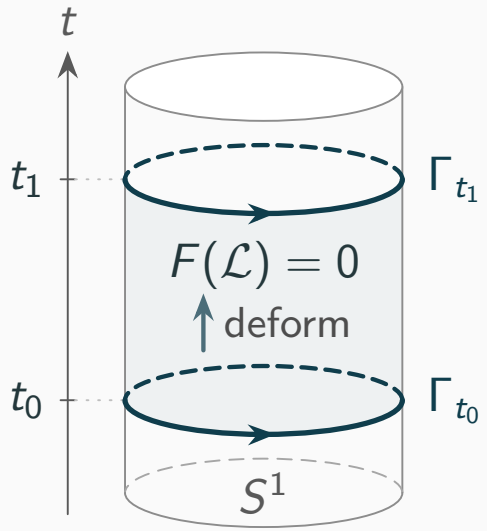
$$\mathcal{L} = \frac{j + z * j}{a^2 - z^2}$$

- We can form a quantity called the monodromy

$$\mathcal{M}(t; z) = \mathcal{P} \exp\left(\int_\Gamma \mathcal{L}\right) = \sum_n M_n z^n$$

- As $\mathcal{L}$ is flat, deforming Γ from one equal time slice to another leaves the monodromy invariant as such

$$d\mathcal{L} + \frac{1}{2}[\mathcal{L}, \mathcal{L}] = 0 \quad \longleftrightarrow \quad \partial_t M_n = 0$$



From curves to surfaces

- In $d+1$ dimensions, a local conserved charge is an integral over a closed d-dimensional spatial slice:

$$Q[\Sigma_t] = \int_{\Sigma_t} J, \qquad J \in \Omega^d(M_{d+1}), \qquad dJ = 0 \quad \text{on shell.}$$

- We seek deformation-invariant spectral parameter dependent quantities coupled to the entire spatial slice.

	$1+1$ **dimensions**	$2+1$ **dimensions**
Spatial slice	curve Γ_t	surface S_t
Transport	line holonomy	surface holonomy
Local data	$\mathcal{L}(z) \in \Omega^1$	$(\mathcal{L}(z), \mathcal{B}(z)) \in \Omega^1 \oplus \Omega^2$

Motivation for higher gauge theory

A principal 2-bundle with a 2-connection provides the geometric framework for surface transport. Higher flatness plays the role of zero curvature.

Higher gauge theories and the BV formalism

- Higher categorical generalisations of 4dCS were introduced, starting with those based on a Lie 2-algebra $\mathcal{G} := \mathfrak{h}_{-1} \xrightarrow{t_*} \mathfrak{g}_0$

[Schenkel, Vicedo; Chen, Liniado]

- The fields in 4d Chern–Simons are $\mathfrak{g}$-valued differential forms.
- The fields in 5d 2-Chern–Simons are $\mathcal{G}$-valued differential forms.
- We can define such objects consistently by totalising the Lie 2-algebra $\mathcal{G}$ with the Dolbeault–de Rham complex $\Omega^\bullet := \Omega^{\bullet,(0,\bullet)}(X_5)$ on $X_5 = \mathbb{R}^3 \times \mathbb{CP}^1$:

$$L := \begin{pmatrix} 0 & & \Omega^0(\mathfrak{g}_0) & & & & \Omega^3(\mathfrak{g}_0) & & \Omega^4(\mathfrak{g}_0) \\ \oplus & \xrightarrow{\mathrm{d}_L} & \oplus & \xrightarrow{\mathrm{d}_L} \cdots \xrightarrow{\mathrm{d}_L} & & \oplus & \xrightarrow{\mathrm{d}_L} & \oplus \\ \Omega^0(\mathfrak{h}_{-1}) & & \Omega^1(\mathfrak{h}_{-1}) & & & \Omega^4(\mathfrak{h}_{-1}) & & 0 \end{pmatrix}$$

where the totalised differential is $\mathrm{d}_L(\alpha, \beta) = \big(\mathrm{d}\alpha + (-1)^p\, t_*(\beta),\, \mathrm{d}\beta\big)$ for $\alpha \in \Omega^p(\mathfrak{g}_0)$.

Higher Gauge theories and the BV formalism

With this we can read off the protagonists of our theory

- Dynamical field - the degree one component of totalised complex

$$\mathcal{A} = A \oplus B \in \Omega^1(X_5, \mathfrak{g}_0) \oplus \Omega^2(X_5, \mathfrak{h}_{-1}) \, .$$

- Infinitesimal gauge transformations are given by the degree zero elements

$$c = \epsilon \oplus \Gamma \in \Omega^0(X_5, \mathfrak{g}_0) \oplus \Omega^1(X_5, \mathfrak{h}_{-1}) \, .$$

- A consequence of this higher categorical structure are the presence of 2-gauge transformations, these are given by degree -1 elements,

$$h \in \Omega^0(X_5, \mathfrak{h}_{-1}) \, .$$

Annoying obstacle

The collection of 2-gauge transformations fail to be well-defined unless a fake-flatness condition is satisfied:

$$F(A) - t_*(B) = 0 \, .$$

Higher Gauge theories and the BV formalism

To negate the issue posed of on-shell closure, we adopt the Batalin–Vilkovisky formalism for gauge theories.

- The formalism gives a derived geometric description of the familiar space of field configurations by introducing anti-ghosts and anti-fields.

Useful fact Costello - Gwilliam

The perturbative BV formalism is equally captured by the data of an L_∞ algebra

- As such adopting the BV formalism allows us to adopt powerful tools from homological algebra to study our theory.
- Issue for our purposes: This framework is perturbative and, as such, only describes infinitesimal gauge transformations. How do we find the edge modes of our theory?

L_∞ algebras

An L_∞-algebra is a pair (L, ℓ) consisting of

- a $\mathbb{Z}$-graded vector space $L = (L^i)_{i\in\mathbb{Z}}$ and
- a family $\ell = (\ell_n)_{n\geq 1} = (\ell_1, \ell_2, \ell_3, \dots)$ of graded antisymmetric linear maps $\ell_n : L^{\otimes n} \to L$, of degree $|\ell_n| = 2 - n$ satisfying the quadratic constraints

$$\sum_{k+l-1=n} \sum_{\sigma\in \mathrm{Sh}(l,k-1)} (-1)^{|\sigma|}\, (-1)^{k-1}\, \ell_k \circ \Big(\ell_l \otimes \mathrm{id}^{\otimes(k-1)}\Big) \circ \gamma_\sigma \;=\; 0 \quad .$$

What do these constraints tell us?

1. $\ell_1(\ell_1(x)) = 0$
2. $\ell_1(\ell_2(x, y)) = \ell_2(\ell_1(x), y) + (-1)^{|x|}\ell_2(x, \ell_1(y))$
3. $J(x, y, z) = -[\ell_1, \ell_3](x, y, z)$

L_∞ algebras will turn out to be an incredibly felicitous choice of gadget when wishing to describe topological-holomorphic theories in the BV formalism.

L_∞ algebras

Before generalising to nCS theories let us consider how this works for 4dCS

- We start with the complex given by $\mathfrak{g}$-valued Dolbeault-de Rham complex of X_4,

$$(\mathcal{E}(X_4), \mathrm{d}) \;:=\; \left(\mathrm{Tot}^{\oplus}\left(\Omega^{\bullet,(0,\bullet)}(X_4, \mathfrak{g})\right), \mathrm{d}_\Sigma + \bar{\partial}\right)$$

- This is a start, but it doesn't account for how the poles and zeroes of ω impose conditions on the fields.
- We will introduce boundary conditions and singularity structures on our complex by twisting our complex by line bundles defined by the divisor (ω)

$$\Omega^{\bullet,(0,\bullet)}_{\mathrm{sgl,bdy}}(X_4) := \left(\Omega^{0,(0,\bullet)}\left(X_4, L_{(\omega)^0_\infty}\right) \xrightarrow{d_\Sigma} \bigoplus_{i=\pm} \Omega^{i,(0,\bullet)}\left(X_4, L_{(\omega)^i_0+(\omega)^i_\infty}\right) \xrightarrow{d_\Sigma} \Omega^{2,(0,\bullet)}\left(X_4, L_{(\omega)^2_0+(\omega)^2_\infty}\right) \right).$$

Then the full 4dCS BV complex is given by the DGLA

$$(\mathcal{F}(X_4), \ell_1) \;:=\; \left(\mathrm{Tot}^{\oplus}\left(\mathfrak{g} \otimes \Omega^{\bullet,(0,\bullet)}_{\mathrm{sgl,bdy}}(X_4)\right), d_\Sigma + \bar{\partial}\right).$$

L_∞ algebras

The L_∞-algebras arising in field theory are typically equipped with a cyclic structure: a non-degenerate, graded-symmetric pairing

$$\langle\!\langle -, - \rangle\!\rangle : L \otimes L \to \mathbb{C}$$

of degree -3, compatible with the higher brackets ℓ_n,

$$\langle\!\langle \ell_n(x_1, \ldots, x_n), x_{n+1} \rangle\!\rangle = (-1)^{n+|x_1|(|x_2|+\cdots+|x_{n+1}|)} \langle\!\langle \ell_n(x_2, \ldots, x_{n+1}), x_1 \rangle\!\rangle.$$

- In the 4dCS theory case we have the following choice of cyclic structure

$$\langle\!\langle \cdot, \cdot \rangle\!\rangle_\omega : \mathcal{F}_c(X) \otimes \mathcal{F}_c(X) \longrightarrow \mathbb{C}, \quad \alpha \otimes \beta \longmapsto \int_X \omega \wedge \langle \alpha \wedge \beta \rangle$$

and the action is then given by

$$\begin{aligned} S(A) &:= \left\langle\!\!\left\langle A, \sum_{n\geq 1} \tfrac{1}{(n+1)!}\, \ell_n\big(A^{\otimes n}\big) \right\rangle\!\!\right\rangle_\omega \\ &= \int_X \omega \wedge \left(\tfrac{1}{2} \left\langle A \,\hat{,}\, (d_\Sigma + \bar{\partial}) A \right\rangle + \tfrac{1}{3!} \left\langle A \,\hat{,}\, [A \,\hat{,}\, A] \right\rangle \right). \end{aligned}$$

Homological Tools and Integrable Field Theories

A key step in the 4dCS recipe was "integrating out" C.

Useful fact de Rham, Dolbeault, etc

Integration over C can be modeled by a homological operation, namely by passing from our BV complex $\big(\mathcal{F}(X_4), \ell_1\big)$ to its Dolbeault cohomology along C.

- This operation can be done in practice using a strong deformation retract

$$(L', d') \underset{p}{\overset{i}{\rightleftarrows}} (L, d) \circlearrowright h$$

where (i, p, h) satisfy

$$p\, i \;=\; \mathrm{id}_{L'}, \qquad i\, p \;=\; \mathrm{id}_{L} + d\, h + h\, d,$$

together with the side conditions

$$h\, i = 0, \qquad p\, h = 0, \qquad h^2 = 0.$$

Homological Tools and Integrable Field Theories

- Using results from Riemann surface theory we can easily compute the $\bar{\partial}$-cohomology of $(\mathcal{F}(X_4), \ell_1)$ along C.

Useful fact

For the Riemann sphere $C = \mathbb{CP}^1$,

- If $\deg(D) < 0$, $\mathrm{H}^0\,\Omega^{0,\bullet}(C, L_D) \cong 0$ and $\mathrm{H}^1\,\Omega^{0,\bullet}(C, L_D) \cong \mathbb{C}^{-(1+\deg(D))}$.
- If $\deg(D) \geq 0$, $\mathrm{H}^0\,\Omega^{0,\bullet}(C, L_D) \cong \mathbb{C}^{1+\deg(D)}$ and $\mathrm{H}^1\,\Omega^{0,\bullet}(C, L_D) \cong 0$.

- So we want to use the following strong deformation retract[1]

$$(\mathrm{H}^\bullet\Omega^{0,\bullet}(C, L_D), 0) \underset{p_D}{\overset{i_D}{\rightleftarrows}} (\Omega^{0,\bullet}(C, L_D), \bar{\partial}) \circlearrowleft h_D$$

to construct a complex $(\mathcal{F}(\Sigma), d')$, related to $(\mathcal{F}(X_4), d + \bar{\partial})$ by

$$(\mathcal{F}(\Sigma), d') \underset{\tilde{p}_D}{\overset{\tilde{i}_D}{\rightleftarrows}} (\mathcal{F}(X_4), d + \bar{\partial}) \circlearrowleft \tilde{h}_D$$

- This can be achieved using the homological perturbation lemma giving

$$d' = pdi + pdhdi\,, \quad \tilde{i} = i + hdi\,, \quad \tilde{p} = p + pdh\,, \quad \tilde{h} = h$$

[1] For full details, with explicit formulae see [2607.12142].

Twisting our complex in such a manner that the degree one fields vanish in cohomology, the result is

$$(\mathcal{F}(\Sigma), d') := \left(\Omega^0(\Sigma, \mathfrak{g}^N) \overset{(1)}{\xrightarrow{p^2 \,\mathrm{d}_\Sigma \left(h^+ \mathrm{d}_\Sigma^+ + h^- \mathrm{d}_\Sigma^-\right) i^0}} \Omega^2(\Sigma, \mathfrak{g}^N) \overset{(2)}{} \right) \quad .$$

n.b. d' is simply the linearised equation of motion operator for a sigma model.

- So indeed $(\mathcal{F}(\Sigma), d')$ is the BV complex of a linearised sigma-model
- How do we go beyond the linear level? More homological algebra!

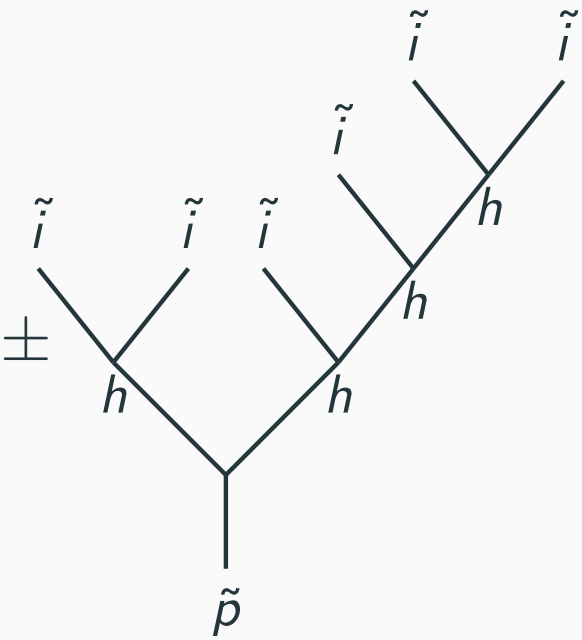
Really nice fact - Homotopy Transfer

Let (L, ℓ) be any L_∞-algebra and $(L', d') \rightleftarrows (L, \mathrm{d}) \circlearrowleft$ any strong deformation retract from its underlying cochain complex $(L, \mathrm{d} := \ell_1)$ to another cochain complex (L', d'). Then there exists a transferred L_∞-algebra structure ℓ' on L' with $\ell'_1 = \mathrm{d}'$, such that the cochain map i extends to an ∞-quasi-isomorphism $i_\infty : (L', \ell') \rightsquigarrow (L, \ell)$.

Homological Tools and Integrable Field Theories

- For our case the L_∞ structure of the 4dCS complex $(\mathcal{F}(X_4), d + \bar{\partial})$ is simply a DGLA structure.
- As such, the ℓ'_n brackets can be constructed by summing over rooted, binary, tree diagrams:

$$\ell'_n = \frac{1}{\mathrm{Aut(T)}} \sum_{\mathrm{RBT}_n} \pm$$



Example of the PCM

In the case of the PCM + WZ term summing up over all these diagrams gives

$$
\begin{aligned}
S_{2d}[\phi] &= \sum_{n\geq 0} \frac{1}{(n+1)!} \int_\Sigma \left\langle \phi, \ell'_n(\phi^{\otimes n}) \right\rangle \\
&= \sum_{n\geq 0} \frac{1}{(2n+2)!} \int_\Sigma \left\langle \phi, \ell'_{2n+1}(\phi^{\otimes(2n+1)}) \right\rangle \\
&\quad + \sum_{n\geq 0} \frac{1}{(2n+3)!} \int_\Sigma \left\langle \phi, \ell'_{2n+2}(\phi^{\otimes(2n+2)}) \right\rangle \\
&= \sum_{n\geq 0} \frac{2}{(2n+2)!} \int_\Sigma \left\langle \phi, \mathrm{d}^+_\Sigma \left(\mathsf{ad}^{2n}_\phi \ \mathrm{d}^-_\Sigma \phi \right) \right\rangle \\
&\quad + \sum_{n\geq 0} \frac{2k}{(2n+3)!} \int_\Sigma \left\langle \phi, \mathrm{d}^+_\Sigma \left(\mathsf{ad}^{2n+1}_\phi \ \mathrm{d}^-_\Sigma \phi \right) \right\rangle .
\end{aligned}
$$

This coincides exactly with the perturbative expansion of the action of the principal chiral model with WZ-term for the group valued field $g = e^\phi$.

Higher dimensional models

For higher dimensional theories we will introduce an d-term L_∞ algebra

$$\mathfrak{g} = \left(\mathfrak{g}^{1-d} \xrightarrow{\ell_1^{\mathfrak{g}}} \cdots \xrightarrow{\ell_1^{\mathfrak{g}}} \mathfrak{g}^{-1} \xrightarrow{\ell_1^{\mathfrak{g}}} \mathfrak{g}^{0} \right) \quad .$$

then look to form a BV complex with

$$\left(\mathcal{E}(X), \ell\right) := (\mathfrak{g}, \ell^{\mathfrak{g}}) \otimes \Omega^{\bullet,(0,\bullet)}(X)$$

Where X is the $d+3$-dimensional manifold $M \times C$. Again, this fails to capture the consequences of the poles and zeros of ω. This leads us to

Singularity structures and local boundary conditions

Geometric Setup

- Manifold M is assumed to be parallelisable.
- Differential forms on M decompose into rank 1 free modules: $\Omega^\bullet(M) = \bigoplus_{\underline{a}\in\{0,1\}^{d+1}} \Omega^{\underline{a}}(M)$.

Then a singularity structure D is a family of non-negative divisors $D^i_{\underline{a}} \geq 0$ on the curve C satisfying:

- **de Rham Compatibility:** $D^i_{\underline{a}} \leq D^i_{\underline{b}}$ for all successors $\underline{b} \in \mathrm{suc}(\underline{a})$.
- L_∞ **Compatibility:** $\sum_{k=1}^n D^{i_k}_{\underline{a}_k} \leq D^{2-n+\sum_{k=1}^n i_k}_{\sum_{k=1}^n \underline{a}_k}$ whenever $\ell^{\mathfrak{g}}_n \neq 0$.
- **Cyclic Compatibility:** $D^i_{\underline{a}} + D^{1-d-i}_{\underline{1}-\underline{a}} = (\omega)_0$.

Singularity structures and local boundary conditions

A local boundary condition is a family of non-positive divisors $B^i_{\underline{a}} \leq 0$ on the curve C satisfying:

- **de Rham Compatibility:** $B^i_{\underline{a}} \leq B^i_{\underline{b}}$ for all successors $\underline{b} \in \mathrm{suc}(\underline{a})$.
- L_∞ **Compatibility:** $\sum_{k=1}^n B^{i_k}_{\underline{a}_k} \leq B^{2-n+\sum_{k=1}^n i_k}_{\sum_{k=1}^n \underline{a}_k}$ whenever $\ell^{\mathfrak{g}}_n \neq 0$.
- **Cyclic Compatibility:** $B^i_{\underline{a}} + B^{1-d-i}_{\underline{1}-\underline{a}} = (\omega)_\infty$.
- **Vanishing Cohomology:** $\deg\left(D^i_{\underline{a}} + B^i_{\underline{a}}\right) = -1$
  (for $i + |\underline{a}| = 1$, assuming genus $g = 0$, i.e., $C = \mathbb{CP}^1$).

The BV complex for nCS theory is then constructed by twisting the Dolbeault complexes with line bundles defined by both the singularity structure D and the local boundary conditions B:

$$\mathcal{E}_{D+B}(X) := \bigoplus_{i,\underline{a}} \mathfrak{g}^i \otimes \left(\Omega^{\underline{a}}(M) \,\widehat{\otimes}\, \Omega^{0,\bullet}\left(C, L_{D^i_{\underline{a}} + B^i_{\underline{a}}}\right)\right) .$$

A particularly nice example: Setup

Let us consider 5d2CS on $X_5 = M_3 \times \mathbb{CP}^1$ for the case where $\ell_1^{\mathfrak{g}} = 0$ and

$$(\omega) = q_1 + q_2 + q_3 - 2p_1 - 2p_2 - p_3.$$

We will adopt the singularity structure

$$D_{\underline{a}}^i := \sum_{k=1}^{3} a_k \, q_k \ ,$$

and local boundary conditions

$$B_{\underline{a}}^i := \ -p_1 - p_2 + i \, p_0 \ .$$

This set up will give rise to a theory on M_3,

Thought

We can use the same homological framework introduced earlier to derive integrable models in 3 dimensions via passing to the Dolbeault cohomology of $\mathcal{E}_{D+B}(X)$.

A particularly nice example: Result

This gives the following BV complex on M_3

$$\mathfrak{F}(M) = \begin{pmatrix} 0 & \xrightarrow{0} & \Omega^0(M_3,\mathfrak{h}) & \xrightarrow{\ell_1'} & \Omega^2(M_3,\mathfrak{h}) & \xrightarrow{\ell_1'} & \Omega^3(M_3,\mathfrak{h}^2) \\ \oplus & & \oplus & & \oplus & & \oplus \\ \Omega^0(M_3,\mathfrak{h}^2) & \xrightarrow{\ell_1'} & \Omega^1(M_3,\mathfrak{h}) & \xrightarrow{\ell_1'} & \Omega^3(M_3,\mathfrak{h}) & \xrightarrow{0} & 0 \end{pmatrix}$$

- The differentials ℓ_1' are determined using data of the SDR and HPL
- Can use the homotopy transfer to attain the higher brackets
- Using Green's witness formalism can attain the linear differential operator of the theory, which is given by a wave equation with Lorentzian metric

$$g = \frac{1}{2}\begin{pmatrix} 0 & q_1 - q_2 & q_1 - q_3 \\ q_1 - q_2 & 0 & q_2 - q_3 \\ q_1 - q_3 & q_2 - q_3 & 0 \end{pmatrix}.$$

Conclusions and Outlook

- What does the landscape of theories attain via higher Chern-Simons theories look like?
- What are the quantum properties of these theories? ZTE?
- The framework of using SDR and HPL to study HT theories is incredibly robust - how far does it go?
- Do these higher Chern-Simons theories admit a string theoretical description much like 4dCS theory.

Thank You!