Order Parameters from Partition Functions

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Outline

Part I: generalities about how order parameters for topological phases of lattice models appear in topological field theory

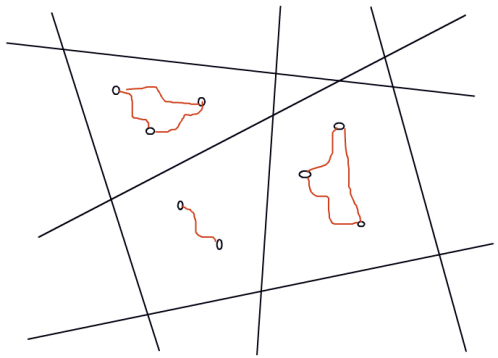
Part II: construction of a novel order parameter for distinguishing two dimensional symmetry protected topological (SPT) phases on the lattice, including in the decohered setting

Lattice models

- Lattice: State space: $\mathcal{H} = \bigotimes_v \mathcal{H}_v$.
- Local Hamiltonian: $H = \sum h_{\mathcal{R}}$.
- Translation invariance: $\mathcal{H}_v$ and $h_{\mathcal{R}}$ are independent of $v, \mathcal{R}$.
- Data $(\mathcal{H}_v, \{h_{\mathcal{R}}\})$ defines a family of systems on various lattices.
- Gapped: the energy gap between the ground and first excited states remains nonzero in the thermodynamic limit (large system size $N \gg |\mathcal{R}|$).
- On-site symmetry: $g \mapsto \otimes_v U(g)$ such that $[H, U(g)^{\otimes N}] = 0$. Take G finite.

Gapped phases

- Phase equivalence: Roughly speaking, two systems are in the same phase if they are related by smooth deformations that do not close the gap.



- Intuition: observables suffer discontinuities at degeneracies in the spectrum. A gapped phase groups microscopic systems with identical low-energy physics.

- Symmetry-enriched phase: use symmetric systems and symmetric deformations.

Classification

- Intrinsic topological orders (no symmetry):
  - 1+1D: only the trivial phase
  - 2+1D: theories of anyons / tensor categories
  - n+1D: higher categorical structures
- Symmetry-enriched topological phases: richer classification
- Symmetry-enrichments of the trivial phase are called SPT phases. These are trivial in the absence of symmetry, so symmetry "protects" them.
  - In low dimension $d+1$, classified by group cohomology

$$H^{d+1}(G, U(1)) \ .$$

    Already nontrivial in $1+1D$.

  - Despite their classification being a solved problem, it remains a challenge to distinguish SPT phases from each other in practice.

Correspondence with TFT

Belief:

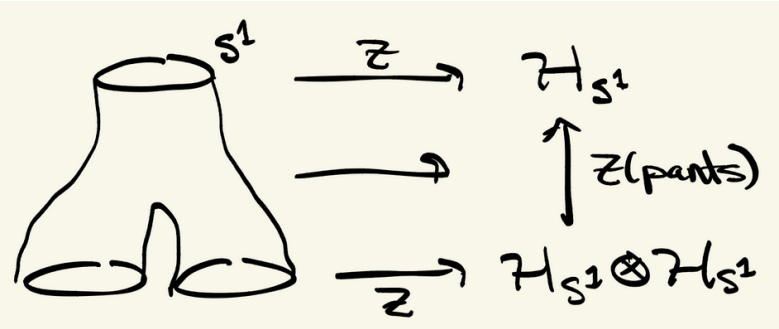
$$\text{Topological Phases} \longleftrightarrow \text{Topological Field Theories (TFTs)}$$

$$\text{SPT Phases} \longrightarrow \text{Invertible Equivariant Theories}$$

- classification of TFTs $\Longrightarrow$ conjectural classification of topological phases
- TFT partition functions $\Longrightarrow$ inspiration for order parameters that distinguish topological phases in practice

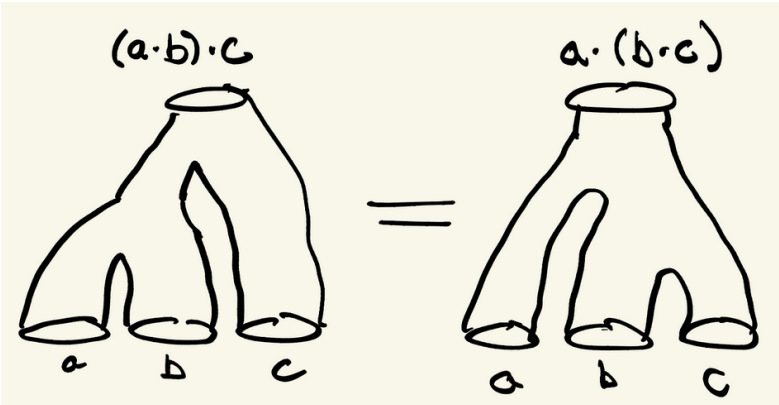
Functorial TFT, in short

- A TFT maps spaces to vector spaces and spacetimes to linear maps:



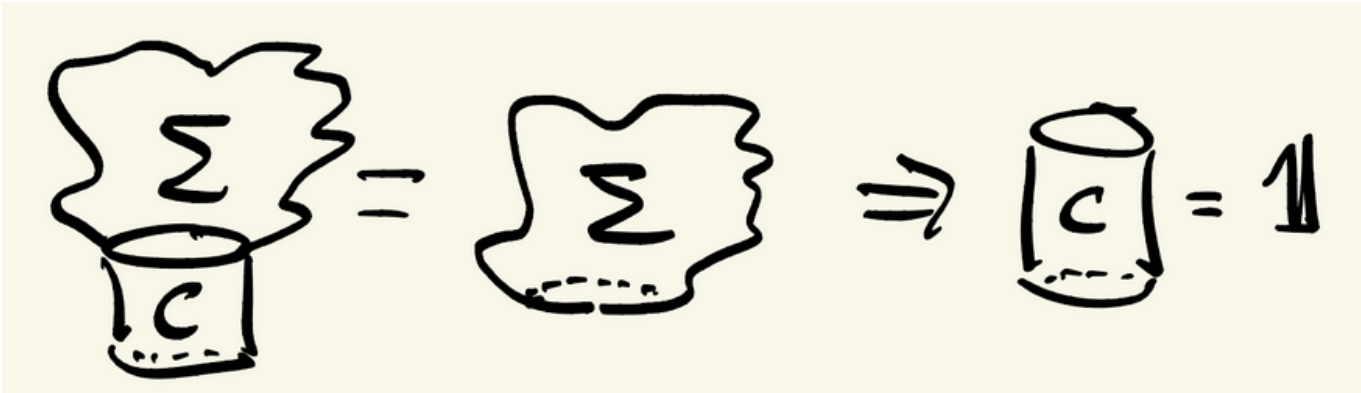
with some conditions such as...

- monoidal: $\coprod \longrightarrow \otimes$
- unit: $\emptyset \longrightarrow \mathbb{C}$
- compatibility with gluing



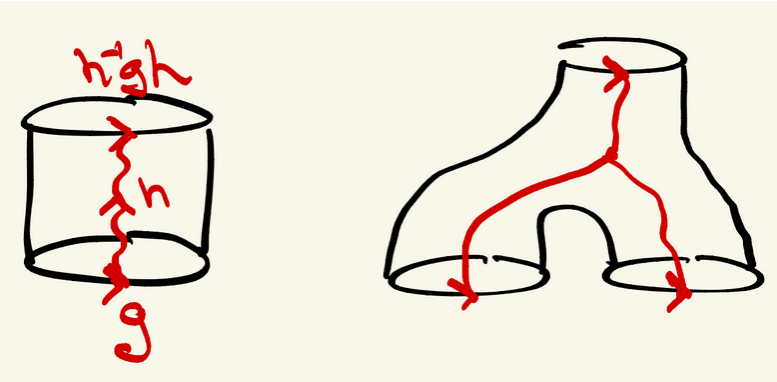
Partition function invariants

- Closed manifolds $=$ cobordisms from $\emptyset$ to $\emptyset$
- TQFT assigns them linear maps from $\mathbb{C}$ to $\mathbb{C}$, i.e. numbers!
- For example, $\mathcal{Z}(T^2) = \mathrm{Tr}[\mathcal{Z}(C)] = \mathrm{Tr}[\mathbb{1}] = \dim \mathcal{H}$.

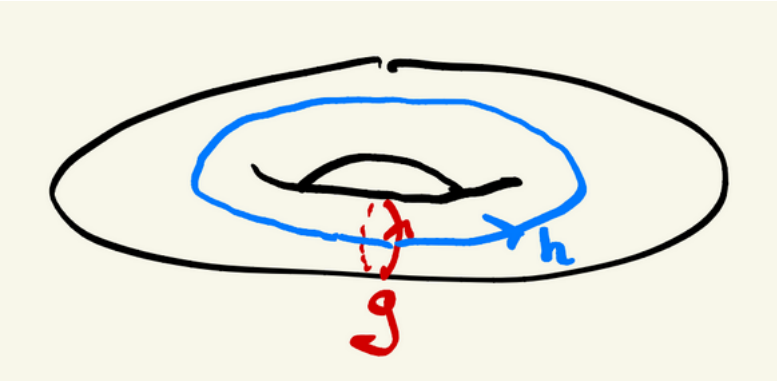


Equivariant TFT

- Spaces and spacetimes come equipped with G-bundles and connection.
  - For G finite, this is just holonomy labels:



- For example, the torus comes with two commuting labels g, h:



Invertible TFT

- Invertible TFT: $\mathcal{H}_M = \mathbb{C}$, $\forall M$ and $\mathcal{Z}(\Sigma) \neq 0$.
- Implies invertibility under stacking: $\mathcal{Z} = \mathcal{Z}_1 \otimes \mathcal{Z}_2$.
- Examples: SPT phases. Non-examples: most intrinsic topological order.

Theorem [Freed, Moore 2006]:
Invertible TFTs are completely determined by their partition functions on closed spacetimes.

Group cohomology

- Group 2-cochain: $G \times G \to U(1)$.
- 2-cocycle condition:

$$\frac{\omega(g,hk)\omega(h,k)}{\omega(g,h)\omega(gh,k)} = \delta\omega(g,h,k) = 1 \ .$$

- 2-coboundary freedom:

$$\omega(g,h) \mapsto \omega(g,h)\frac{\beta(g)\beta(h)}{\beta(gh)} = \omega(g,h)\delta\beta(g,h) \ .$$

- Quotient of ω's modulo $\delta\beta$'s is group cohomology $[\omega] \in H^2(G, U(1))$.
- It can be shown that 2D invertible G-equivariant TFTs have classification

$$H^2(G, U(1)) \ ,$$

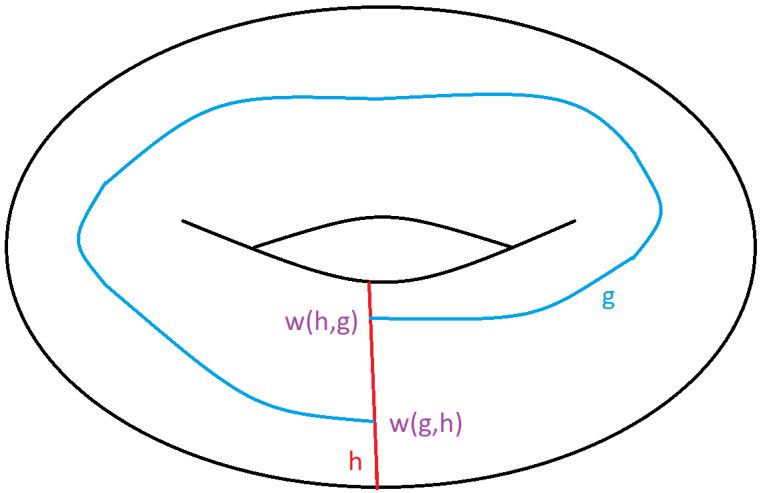
in agreement with the $1+1$D SPT classification. SSB is not invertible.

Torus order parameter for SPTs

- In the theory labeled by $[\omega]$,

$$\mathcal{Z}(T^2_{g,h}) = i_g i_h \omega = \frac{\omega(g,h)}{\omega(h,g)} \in U(1) \ .$$

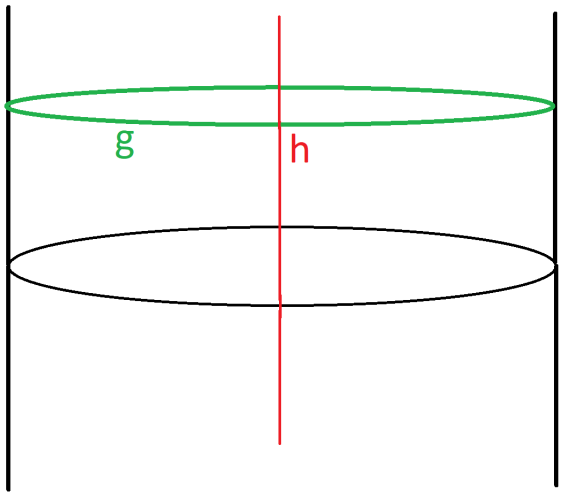
This "slant product" is independent on the representative ω of $[\omega]$.



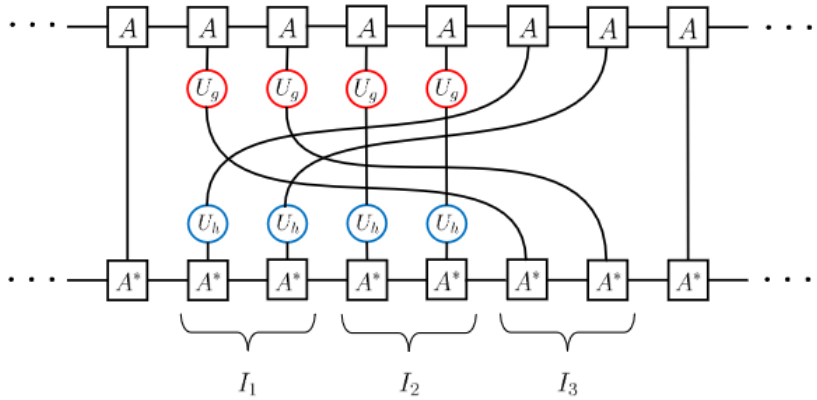
- When G is abelian, $i_g i_h \omega$ completely determines $[\omega]$ and therefore the full TFT. In general, higher genus partition functions are necessary.

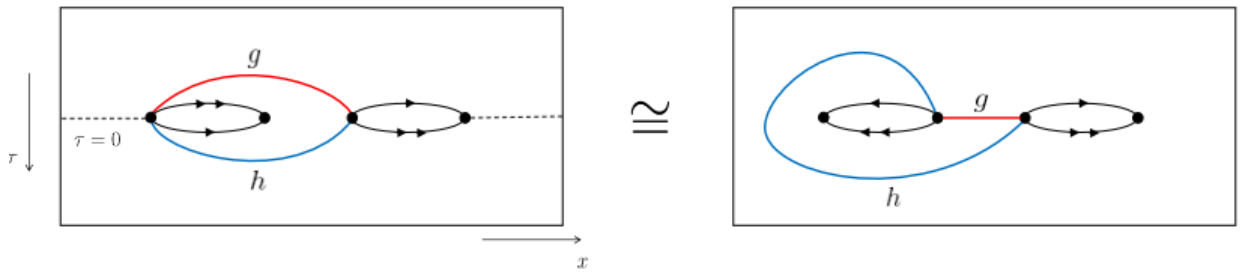
The torus as an order parameter

Flux insertion as a torus

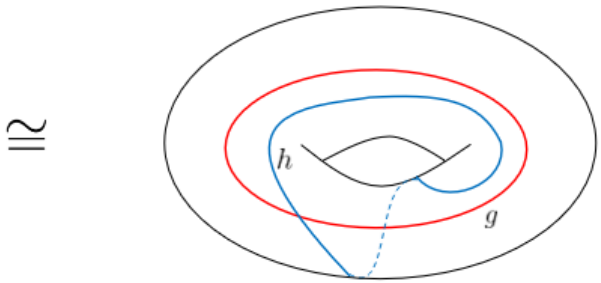


Swaps as a torus (and higher genus)





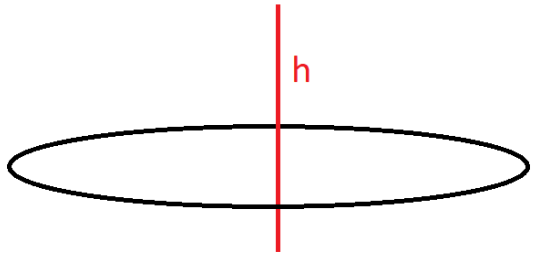
[Haegeman, Perez-Garcia, Cirac, Schuch 2012]
[Pollmann-Turner 2012]
[Shiozaki-Ryu 2017]



EXTRA: Detection of SPTO with twisted sector charges

- Twist the state by inserting an h flux:

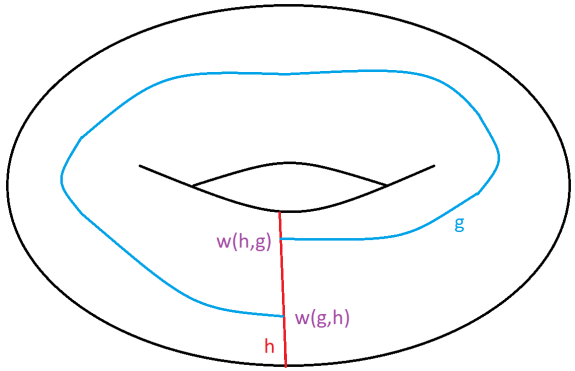
$$|\psi_h\rangle = \mathrm{Tr}\big[\, V_h\, A^{i_1} \cdots A^{i_N} \,\big]$$



- Ground state of twisted Hamiltonian H_h

- Act by U_g on each site:

$$U_g^{\otimes L} \cdot |\psi_h\rangle = \sum \mathrm{Tr}\big[\, V_g\, V_h\, V_g^{-1}\, A^{i_1} \cdots A^{i_N} \,\big]$$
$$= \frac{\omega(g,h)}{\omega(h,g)} |\psi_h\rangle$$



[Shiozaki, Ryu 17] [Kapustin, AT, You 17]

- ω/ω is the partition function of the torus $T^2_{g,h}$

Phases, invariants, and order parameters

Topological phases (definition):	**Topological invariants**:	**Order parameters**:
deformation classes of gapped systems	quantities that are constant along deformations	procedures for measuring topological invariants in concrete systems
topological orders $\mathcal{C}$	total quantum dim. D	topological entanglement entropy
1D SPTs $[\omega] \in H^2(G, U(1))$	slant products $i_g i_h \omega \in U(1)$	**various**

Lens Space Order Parameters for 2D SPTs

Detecting 2+1D SPT Order

Phase classification: $[\omega] \in H^3(G, U(1))$.

There are various ways to cut up $[\omega]$ into invariants. More on this later.

Known order parameters:

- Flux insertions [Zalatel 2014]
- SPT entanglement [Marvian 2017]
- Membrane order [Pérez-García et al 2008]

Objective

I'd like to construct an order parameter with the following features:

- Measured on a given wavefunction, without flux insertion / twisting.
  - Useful if I know little about that state, e.g. in a lab. In numerical simulation, no need to find a tensor network.
- Simply an expectation value of an operator.
  - Useful for studying evolutions using the Heisenberg picture.
- The operator is finitely supported.
  - Can be implemented with few operations.

Why should this be possible? Inspiration from TQFT...

Invariants of 2+1D SPT Phases

Lens space $L(p;q)_g$ with holonomy g around the great circle ($g^p = 1$).

$$\mathcal{Z}_\omega(L(p;q)_g) = \prod_{k=1}^{p} \omega(g, g^k, g^q) \ .$$

Torus $T^3_{g,h,k}$ with holonomies g, h, k (pairwise commuting).

$$\mathcal{Z}_\omega(T_{g,h,k}) = i_g i_h i_k \omega = \frac{\omega(g,h,k)\omega(h,k,g)\omega(k,g,h)}{\omega(h,g,k)\omega(g,k,h)\omega(k,h,g)} \ .$$

for example,

- generating $\mathbb{Z}_n$ cocycle: $L(n;1)_g = e^{2\pi i/n}$ $T^3_{g,h,k} = 1$
- Type III $\mathbb{Z}_n^3$ cocycle: $L(n;1)_g = 1$ $T^3_{g,h,k} = e^{2\pi i/n}$

Good reference: [Tantivasadakarn 1706.09769]

Lens space partition functions

$$\mathcal{Z}_\omega(L(p;q)_g) = \prod_{k=1}^{p} \omega(g, g^k, g^q)$$

satisfies

$$\mathcal{Z}_\omega(L(p;q)_g) = \mathcal{Z}_\omega(L(p;1)_g)^q$$

Determined by the invariants

$$I_{g,M} := \mathcal{Z}_\omega(L(M;1)_g) \in \mathbb{Z}_M \ , \quad g, M \text{ with } g^M = 1 \ .$$

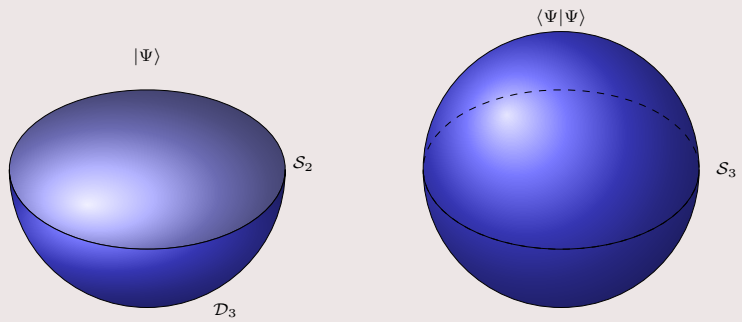
Also satisfies

$$\mathcal{Z}_\omega(L(p;1)_g)^n = \mathcal{Z}_\omega(L(pn;1)_g) \ , \quad i.e. \quad I_{g,M}^n = I_{g,nM} \ .$$

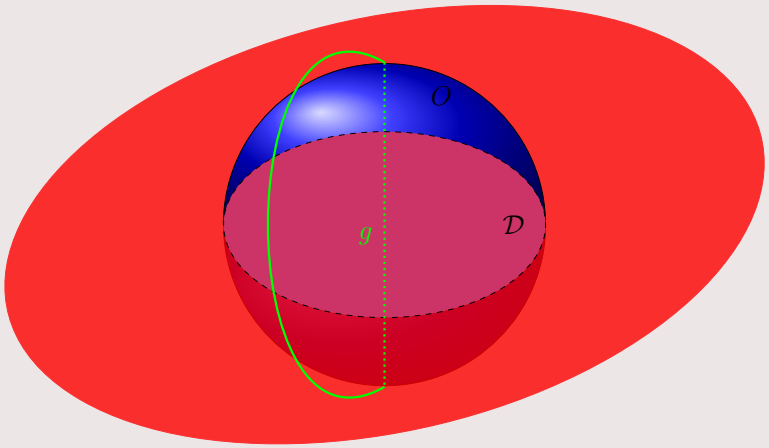
$$\mathcal{Z}_\omega(L(p;1)_g) = \mathcal{Z}_\omega(L(p/d;1)_{g^d}) \ , \quad i.e. \quad I_{g,M} = I_{g^d,M/d} \ .$$

Partial symmetry order parameter

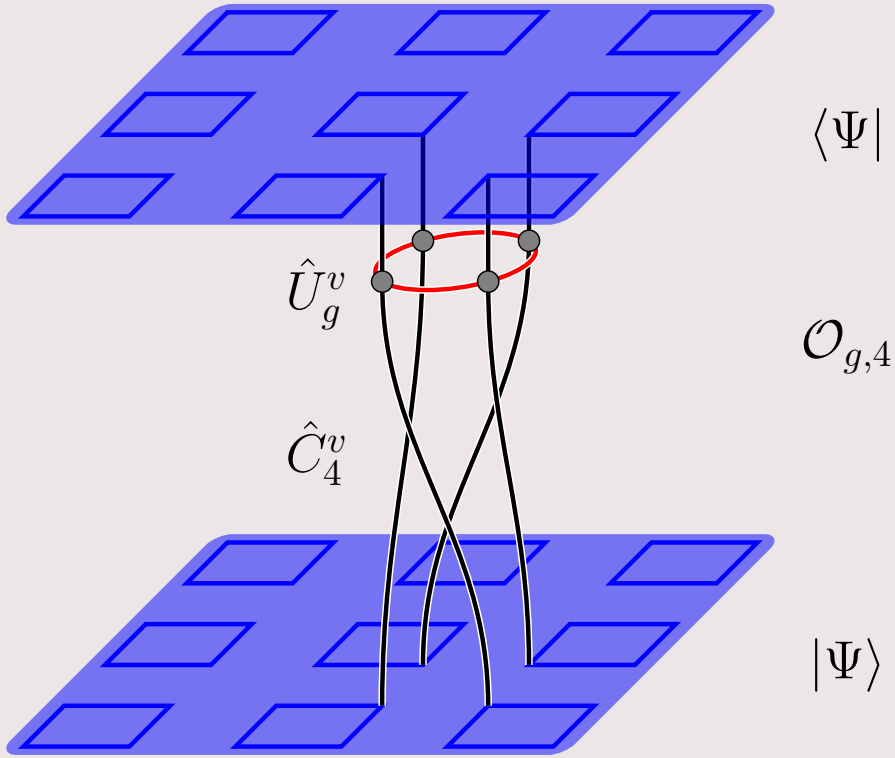
The state in TQFT:



Construction of $L(M;1)_g$:



Partial symmetry order parameter:



$$I_{g,M} = \text{phase}\left(\langle\Psi|U_g^{\mathcal{D}}\, C_M^{\mathcal{D}}|\Psi\rangle\right)$$

On the lattice

Given:

- A state $|\Psi\rangle$ in a lattice of spins. Assume finite correlation length ξ.
- A finite group symmetry G acting on-site as U_g.

Choose:

- A $\mathbb{Z}_M$ rotational symmetry of the lattice around some origin.
- A representation by C_M (may act on the on-site dofs) such that $\langle\Psi|C_M|\Psi\rangle = 1$.
- A region $\mathcal{D}$ that is large ($|\partial\mathcal{D}|/\xi \gg 1$), contractable, rotationally symmetric, and such that the restrictions of the symmetries commute:

$$U_g^{\mathcal{D}} C_M^{\mathcal{D}} = C_M^{\mathcal{D}} U_g^{\mathcal{D}}$$

Measure:

$$I_{g,M} = \text{phase}\left(\langle\Psi|U_g^{\mathcal{D}} C_M^{\mathcal{D}}|\Psi\rangle\right)$$

Comments

The region $\mathcal{D}$ must be large, i.e. $|\partial\mathcal{D}|/\xi \gg 1$.

But may still be *small in the system size*, unlike flux insertions.

e.g. 1D string operators: constant vs $e^{-L/\xi}$ decay.

Expect the phase $I_{g,M}$ to be robust and the modulus to decay in $|\partial\mathcal{D}|/\xi$.

Crystalline symmetries introduce choice:

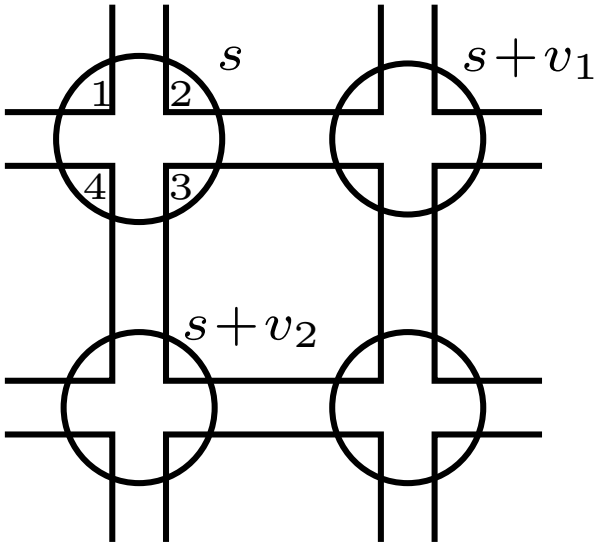
To measure $I_{g,M}$, the lattice must have $\mathbb{Z}_M$ rotations.

Which rotation operator C_M do we use?

e.g. $C'_M = C_M U_h$ permutes $I_{g,M} \mapsto I_{hg,M}$

e.g. changing the center of rotations (vertex, edge, plaquette)

Example: the CZX ground state



On-site symmetry $G = \mathbb{Z}_2$.

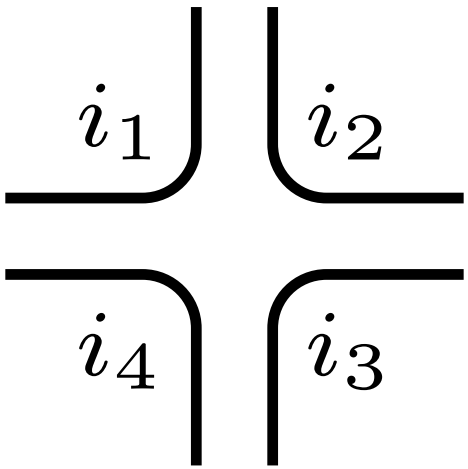
Rotations of order $M = 4, 2$ at vertices and plaquettes and order $M = 2$ at edges.

C_M also acts by rotating the subsites of each site.

Can verify (for any $\mathcal{D}$) that

$$\langle \Psi | C_M | \Psi \rangle = 1 \qquad \text{and} \qquad U_g^{\mathcal{D}} C_M^{\mathcal{D}} = C_M^{\mathcal{D}} U_g^{\mathcal{D}}$$

Example: the CZX ground state



$$\langle \Psi_{\mathrm{CZX}} | \mathcal{O}_{g,M}^{(\mathcal{D}=v)} | \Psi_{\mathrm{CZX}} \rangle$$

$$\sim \sum_{i_1,i_2,i_3,i_4} \langle i_1 i_2 i_3 i_4 | \hat{U}_g^v \hat{C}_M^v | i_1 i_2 i_3 i_4 \rangle$$

$$\sim \sum_{i_1,i_2,i_3,i_4} \langle i_1 i_2 i_3 i_4 | \hat{U}_g^v | i_{1+\frac{4}{M}} i_{2+\frac{4}{M}} i_{3+\frac{4}{M}} i_{4+\frac{4}{M}} \rangle$$

For g the generator of $\mathbb{Z}_2$, this is

$$\sum_{i_1,i_2,i_3,i_4} (-1)^{i_1 i_2 + i_2 i_3 + i_3 i_4 + i_4 i_1} \delta_{i_k = i_{k+\frac{4}{M}} + 1}$$

For $M = 4$, get $+1$. For $M = 2$, get -1.

Example: the CZX ground state

- *vertex centered rotations:*

$$I^v_{\mathbf{e},4} = 1\ , \quad I^v_{\mathbf{e},2} = 1\ , \quad I^v_{\mathbf{g},4} = 1\ , \quad I^v_{\mathbf{g},2} = -1$$

- *plaquette centered rotations:*

$$I^p_{\mathbf{e},4} = 1\ , \quad I^p_{\mathbf{e},2} = 1\ , \quad I^p_{\mathbf{g},4} = 1\ , \quad I^p_{\mathbf{g},2} = -1$$

- *edge centered rotations:*

$$I^e_{\mathbf{e},2} = 1\ , \quad I^e_{\mathbf{g},2} = 1\ .$$

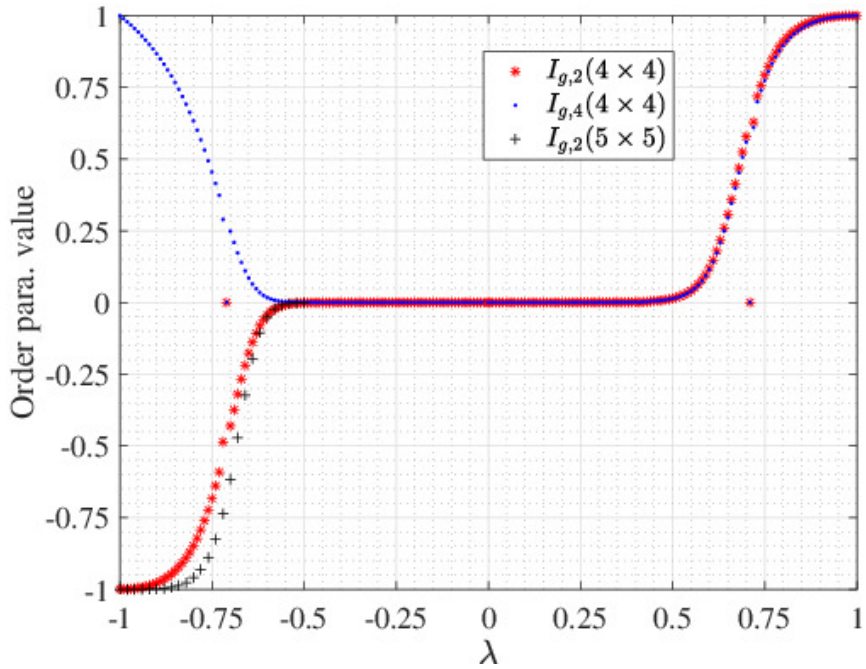
These results are independent of the size and shape of $\mathcal{D}$.

$I = -1$ distinguishes CZX from disorder.

As predicted by lens spaces, $I_{g,4} = I^2_{g,2} = +1$.

Why are some $+1$ and others -1? Why is $I_{\mathbf{g},2}$ depend on the rotation center?

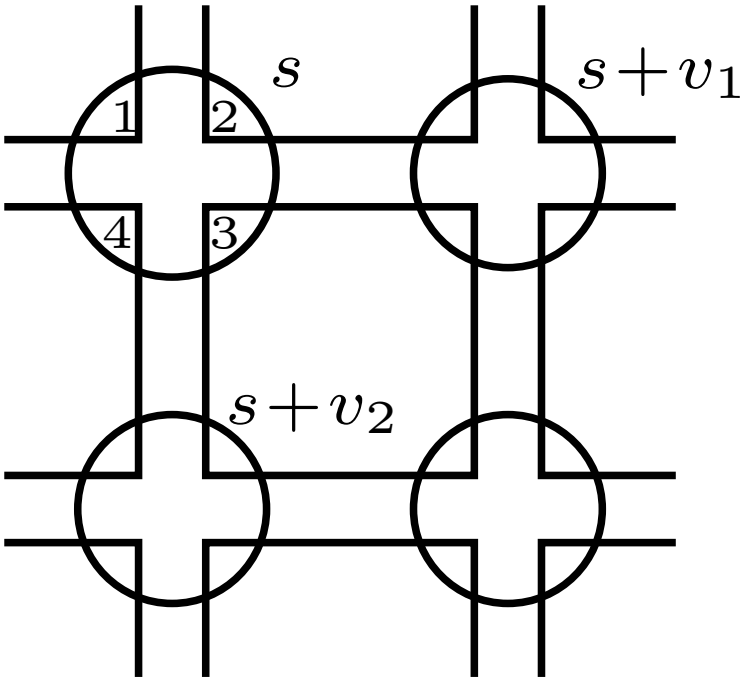
Beyond the CZX fixed point

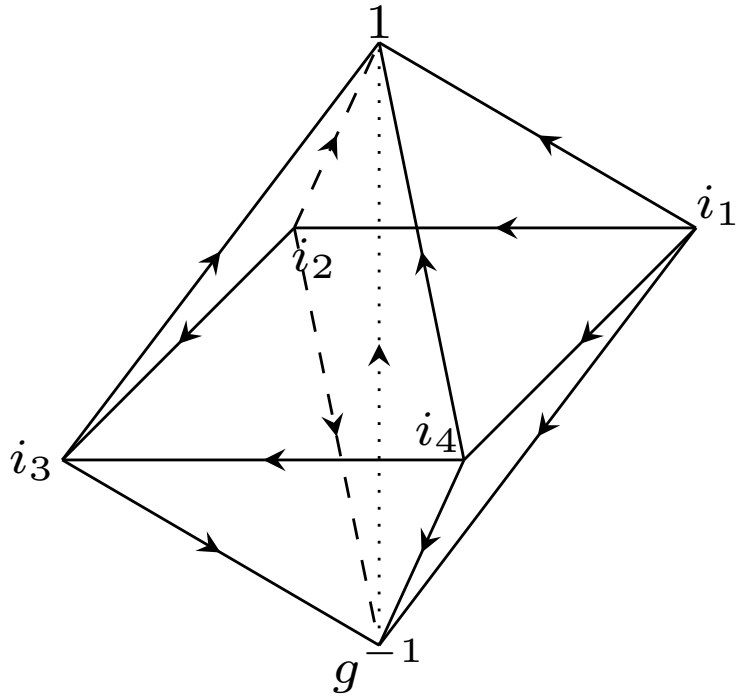


Consider a path of states from $|CZX\rangle$ to a product state $\otimes_s |GHZ_s\rangle$.

Measure $I_{g,M}$ on a single plaquette.

EXTRA: Group cohomology models





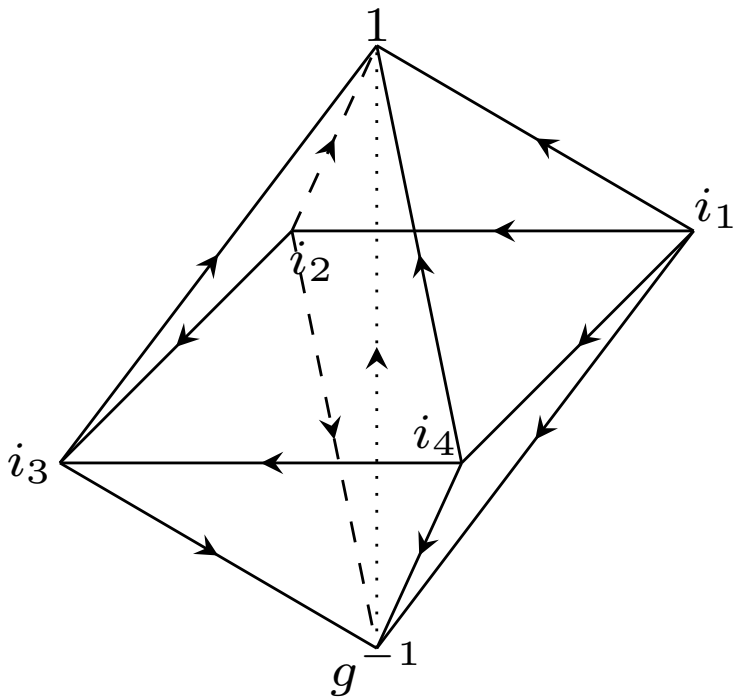
$$|\Psi_G\rangle = \bigotimes_p \sum_{\alpha_p \in G} |\alpha_p, \alpha_p, \alpha_p, \alpha_p\rangle$$

$$U_g^s |i_{s,1}, i_{s,2}, i_{s,3}, i_{s,4}\rangle = \frac{\theta^{s,g}_{1,2}\theta^{s,g}_{2,3}}{\theta^{s,g}_{4,3}\theta^{s,g}_{1,4}} |g i_{s,1}, g i_{s,2}, g i_{s,3}, g i_{s,4}\rangle$$

where $\theta^{s,g}_{r,r'}$ is shorthand for $\theta^{g}_{i_{s,r},i_{s,r'}}$, where

$$\theta^g_{i,j} = \omega(i^{-1}j, j^{-1}g^{-1}, g) \ .$$

EXTRA: Finding a rotational symmetry



$$U_g^s|i_{s,1}, i_{s,2}, i_{s,3}, i_{s,4}\rangle = \frac{\theta_{1,2}^{s,g}\theta_{2,3}^{s,g}}{\theta_{4,3}^{s,g}\theta_{1,4}^{s,g}}|gi_{s,1}, gi_{s,2}, gi_{s,3}, gi_{s,4}\rangle$$

where $\theta_{r,r'}^{s,g}$ is shorthand for $\theta_{i_{s,r},i_{s,r'}}^{g}$, where

$$\theta_{i,j}^{g} = \omega(i^{-1}j, j^{-1}g^{-1}, g) \ .$$

The naive rotation *does not* commute with U_g when restricted to $\mathcal{D}$.

Modify how C_M acts on-site:

$$\hat{C}_4^s|i_{s,1}, i_{s,2}, i_{s,3}, i_{s,4}\rangle = \Lambda(i_{s,1}, i_{s,2}, i_{s,3}, i_{s,4})|i_{s',4}, i_{s',1}, i_{s',2}, i_{s',3}\rangle$$

Must choose Λ such that

1. $U_g^{\mathcal{D}} C_M^{\mathcal{D}} = C_M^{\mathcal{D}} U_g^{\mathcal{D}}$
2. $\langle\Psi|C_M|\Psi\rangle = 1$

EXTRA: Finding a rotational symmetry

$$\hat{C}_4^s |i_{s,1}, i_{s,2}, i_{s,3}, i_{s,4}\rangle = \Lambda(i_{s,1}, i_{s,2}, i_{s,3}, i_{s,4}) |i_{s',4}, i_{s',1}, i_{s',2}, i_{s',3}\rangle$$

$$U_g^{\mathcal{D}} C_M^{\mathcal{D}} = C_M^{\mathcal{D}} U_g^{\mathcal{D}}$$

$$\langle \Psi | C_M | \Psi \rangle = 1$$

This means

$$\frac{\Lambda(g i_{s,1}, g i_{s,2}, g i_{s,3}, g i_{s,4})}{\Lambda(i_{s,1}, i_{s,2}, i_{s,3}, i_{s,4})} = \frac{\theta_{4,1}^{s,g} \theta_{1,4}^{s,g}}{\theta_{2,3}^{s,g} \theta_{3,2}^{s,g}}$$

A solution is given by

$$\Lambda(i_{s,1}, i_{s,2}, i_{s,3}, i_{s,4}) = \frac{\phi(i_{s,4}, i_{s,1})}{\phi(i_{s,2}, i_{s,3})}$$

for some ϕ with

$$\boxed{\frac{\phi(gi, gj)}{\phi(i, j)} = \theta_{ij}^g \theta_{ji}^g}$$

A solution is given by

$$\boxed{\frac{\phi(i, j)}{\phi(j, i)} = \frac{f(i)}{f(j)}}$$

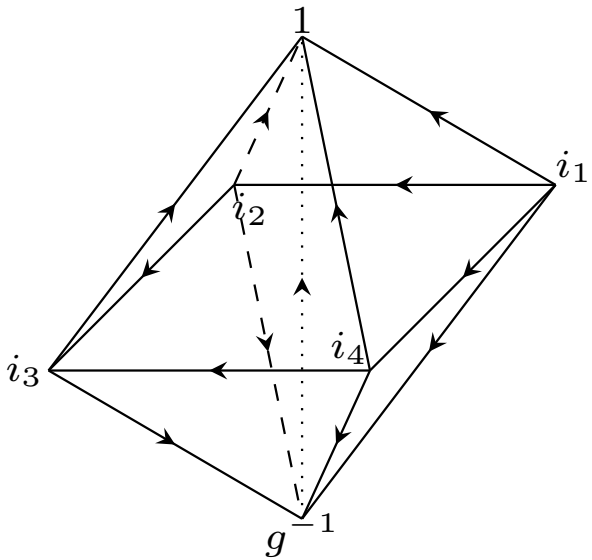
The phases cancel telescopically:

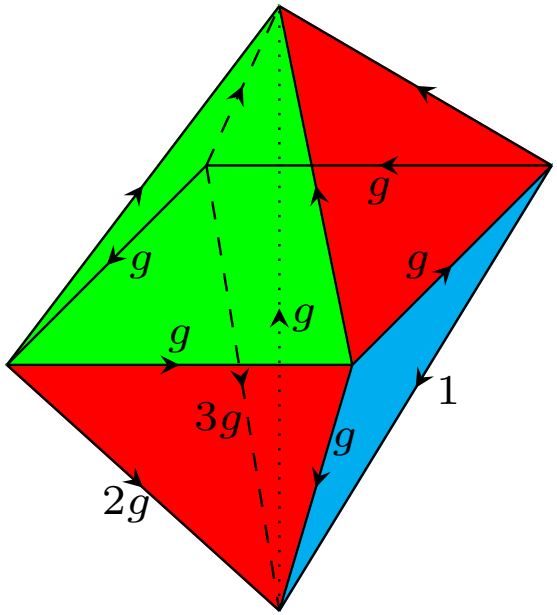
$$\prod_s \Lambda(i_{s,1}, i_{s,2}, i_{s,3}, i_{s,4})$$

$$= \prod_s \frac{\phi(i_{s,4}, i_{s,1})}{\phi(i_{s,2}, i_{s,3})} = \prod_s \frac{\phi(i_{s,4}, i_{s,1})}{\phi(i_{s+v_1,1}, i_{s+v_1,4})}$$

$$= \prod_s \frac{f(i_{s,4})}{f(i_{s,1})} = \prod_s \frac{f(i_{s+v_2,1})}{f(i_{s,1})} = 1 \, .$$

EXTRA: Digression: Modified group cohomology models





Instead of modifying C_M, modify U_g.

Use lens space triangulation:

$$\tilde{U}_g|i_{s,1}, i_{s,2}, i_{s,3}, i_{s,4}\rangle = \theta^{s,g}_{1,2}\theta^{s,g}_{2,3}\theta^{s,g}_{3,4}\theta^{s,g}_{4,1}|gi_{s,1}, gi_{s,2}, gi_{s,3}, gi_{s,4}\rangle$$

Modify $|\Psi_G\rangle$ to make G invariant:

$$|\tilde{\Psi}_G\rangle = \sum_{\alpha_p \in G} \Phi(\{\alpha_p\}) \bigotimes_p |\alpha_p, \alpha_p, \alpha_p, \alpha_p\rangle$$

$$\Phi(\{\alpha_p\}) = \prod_s \phi(i_{s,1}, i_{s,2})\phi(i_{s,2}, i_{s,3})$$

Partial symmetry order of group cohomology models

On a single vertex:

$$
\begin{aligned}
I^v_{g,4} &= \langle \Psi_G | U^s_g C^s_4 | \Psi_G \rangle \\
&= \sum_{i_1,i_2,i_3,i_4} \langle i_1,i_2,i_3,i_4 | U^s_g C^s_4 | i_1,i_2,i_3,i_4 \rangle \\
&= \sum_{i_1,i_2,i_3,i_4} \frac{\phi(i_4,i_1)}{\phi(i_2,i_3)} \langle i_1,i_2,i_3,i_4 | U^s_g | i_4,i_1,i_2,i_3 \rangle \\
&= \sum_{i_1,i_2,i_3,i_4} \frac{\phi(i_4,i_1)}{\phi(i_2,i_3)} \frac{\theta^g_{4,1}\theta^g_{1,2}}{\theta^g_{3,2}\theta^g_{4,3}} \prod_{k=1}^{4} \delta(g\cdot i_k = i_{[k+1]}) \\
&= \sum_{i_1,i_2,i_3,i_4} \theta^g_{1,2}\theta^g_{2,3}\theta^g_{3,4}\theta^g_{4,1} \prod_{k=1}^{4} \delta(g\cdot i_k = i_{[k+1]}) \\
&\quad \times \frac{\phi(i_4,i_1)}{\cancel{\phi(i_2,i_3)}} \frac{\phi(i_2,i_3)}{\phi(gi_2,gi_3)} \frac{\cancel{\phi(i_3,i_4)}}{\phi(gi_3,gi_4)} \\
&= \sum_{i_1} \prod_{j=1}^{4} \omega(i_1^{-1} g i_1, i_1^{-1} g^j, g) \\
&= |G| \times \prod_{j=1}^{4} \omega(g, g^j, g) \\
&\sim \mathcal{Z}_\omega(L(4,1)_g) \ .
\end{aligned}
$$

One arbitrary symmetric regions $\mathcal{D}$:

$$\mathcal{Z}_\omega(L(4;1)_g) \times \mathcal{Z}_\omega(L(2;1)_{g^2})^{|\partial\mathcal{D}|/4-1}$$

i.e.
on vertex-centered regions:

$$I^v_{g,4} = \mathcal{Z}_\omega(L(4;1)_g)$$

on plaquette-centered regions:

$$I^p_{g,4} = \mathcal{Z}_\omega(L(4;1)_g)\mathcal{Z}_\omega(L(2;1)_{g^2})$$

$M = 2$ rotations:

$$I_{g,2} = \mathcal{Z}_\omega(L(2;1)_g)$$

Accounting for crystalline invariants

Effective response theory

$G = \mathbb{Z}_n$. Full symmetry $\mathbb{Z}_n \times C_M$.

SPT invariants for a given origin o:

$$(k_1, k_2^o, k_3^o) \in \mathbb{Z}_n \times \mathbb{Z}_{(n,M)} \times \mathbb{Z}_M$$

Claim:

$$I_{g,M} = \exp\left(\frac{2\pi i}{M}(\alpha^2 k_1 + \alpha k_2 + k_3)\right)$$

where $\alpha = qM/n$ for $g = q \bmod n$.

Useful tool: topological response theory:

$$\mathcal{L}[A, \omega] = \frac{k_1}{2\pi} A dA + \frac{k_2}{2\pi} A d\omega + \frac{k_3}{2\pi} \omega d\omega$$

Argument:

Step 1.

$$I_{\mathbf{e},M} = \text{phase}\left(\langle\Psi| C_M^{\mathcal{D}} |\Psi\rangle\right) = \exp\left(\frac{2\pi i}{M} k_3\right)$$

Step 2.
Measuring $I_{g,M}$ is like measuring $I_{\mathbf{e},M}$ in a theory where A has flux bound to ω defects:

$$A' = A + \frac{qM}{n}\omega$$

$$\mathcal{L}[A', \omega] = \frac{k_1}{2\pi} A dA + \frac{1}{2\pi}(2\alpha k_1 + k_2) A d\omega + \frac{1}{2\pi}\left(\alpha^2 k_1 + \alpha k_2 + k_3\right)\omega d\omega \ .$$

Disentangling the invariants: $\mathbb{Z}_2 \times C_4$

SPT invariants

$$(k_1, k_2, k_3) \in \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_4$$

$$I_{g,M} = \exp\left(\frac{2\pi i}{M}(\alpha^2 k_1 + \alpha k_2 + k_3)\right)$$

$$I_{\mathbf{e},4} = e^{\frac{2\pi i}{4}k_3} ,$$
$$I_{\mathbf{g},2} = e^{\frac{2\pi i}{4}(2k_1+2k_2+2k_3)}$$
$$I_{\mathbf{g},4} = e^{\frac{2\pi i}{4}(2k_2+k_3)} .$$

$$e^{\frac{2\pi i}{2}k_1} = I_{\mathbf{g},2} I_{\mathbf{g},4} I_{\mathbf{e},4}$$
$$e^{\frac{2\pi i}{2}k_2} = I_{\mathbf{e},4}/I_{\mathbf{g},4} ,$$
$$e^{\frac{2\pi i}{4}k_3} = I_{\mathbf{e},4} .$$

Applied to the CZX model:

- *vertex centered rotations:*

$$I^v_{\mathbf{e},4} = 1 , \quad I^v_{\mathbf{e},2} = 1 , \quad I^v_{\mathbf{g},4} = 1 , \quad I^v_{\mathbf{g},2} = -1$$

- *plaquette centered rotations:*

$$I^p_{\mathbf{e},4} = 1 , \quad I^p_{\mathbf{e},2} = 1 , \quad I^p_{\mathbf{g},4} = 1 , \quad I^p_{\mathbf{g},2} = -1$$

- *edge centered rotations:*

$$I^e_{\mathbf{e},2} = 1 , \quad I^e_{\mathbf{g},2} = 1 .$$

- *vertex centered rotations:*

$$k_1 = 1 , \quad k_2^v = 0 , \quad k_3^v = 0 .$$

- *plaquette centered rotations:*

$$k_1 = 1 , \quad k_2^p = 0 , \quad k_3^p = 0 .$$

- *edge centered rotations:*

$$k_1 = 1 , \quad k_2^e = 1 , \quad k_3^e = 0 .$$

Note: edge alone only detects $k_1 + k_2^e$.

Disentangling the invariants: $\mathbb{Z}_4 \times C_4$

SPT invariants

$$(k_1, k_2, k_3) \in \mathbb{Z}_4 \times \mathbb{Z}_4 \times \mathbb{Z}_4$$

$$I_{\mathbf{e},4} = e^{\frac{2\pi i}{4} k_3} ,$$
$$I_{\mathbf{g},4} = e^{\frac{2\pi i}{4}(k_1+k_2+k_3)} ,$$
$$I_{\mathbf{g}^2,4} = e^{\frac{2\pi i}{4}(2k_2+k_3)} ,$$
$$I_{\mathbf{e},2} = I_{\mathbf{e},4}^2 , \quad I_{\mathbf{g}^2,2} = I_{\mathbf{g},4}^2 ,$$

$$e^{\frac{2\pi i}{4}(k_1+k_2)} = I_{\mathbf{g},4}/I_{\mathbf{e},4} ,$$
$$e^{\frac{2\pi i}{4} 2k_1} = I_{\mathbf{g},4}^2/I_{\mathbf{g}^2,4}I_{\mathbf{e},4}$$
$$e^{\frac{2\pi i}{4} k_3} = I_{\mathbf{g},4}$$

k_1 is only determined mod 2!

Consider the group cohomology model with ω generating $H^3(\mathbb{Z}_4, U(1))$.

Previously, we computed

$$I^v_{\mathbf{e},4} = \mathcal{Z}_\omega(L(4;1)_\mathbf{e}) = 1 ,$$
$$I^v_{\mathbf{g},4} = \mathcal{Z}_\omega(L(4;1)_\mathbf{g}) = i ,$$
$$I^v_{\mathbf{g}^2,4} = \mathcal{Z}_\omega(L(4;1)_{\mathbf{g}^2}) = 1$$

$$I^p_{\mathbf{e},4} = \mathcal{Z}_\omega(L(4;1)_\mathbf{e})\mathcal{Z}_\omega(L(2;1)_\mathbf{e}) = 1 ,$$
$$I^p_{\mathbf{g},4} = \mathcal{Z}_\omega(L(4;1)_\mathbf{g})\mathcal{Z}_\omega(L(2;1)_{\mathbf{g}^2}) = -i$$
$$I^p_{\mathbf{g}^2,4} = \mathcal{Z}_\omega(L(4;1)_{\mathbf{g}^2})\mathcal{Z}_\omega(L(2;1)_\mathbf{e}) = 1 ,$$

Disentangling the invariants: $\mathbb{Z}_4 \times C_4$

Consider the group cohomology model with ω generating $H^3(\mathbb{Z}_4, U(1))$.

Previously, we computed

$$I^v_{\mathbf{e},4} = \mathcal{Z}_\omega(L(4;1)_\mathbf{e}) = 1 \ ,$$
$$I^v_{\mathbf{g},4} = \mathcal{Z}_\omega(L(4;1)_\mathbf{g}) = i \ ,$$
$$I^v_{\mathbf{g}^2,4} = \mathcal{Z}_\omega(L(4;1)_{\mathbf{g}^2}) = 1$$

$$I^p_{\mathbf{e},4} = \mathcal{Z}_\omega(L(4;1)_\mathbf{e})\mathcal{Z}_\omega(L(2;1)_\mathbf{e}) = 1 \ ,$$
$$I^p_{\mathbf{g},4} = \mathcal{Z}_\omega(L(4;1)_\mathbf{g})\mathcal{Z}_\omega(L(2;1)_{\mathbf{g}^2}) = -i$$
$$I^p_{\mathbf{g}^2,4} = \mathcal{Z}_\omega(L(4;1)_{\mathbf{g}^2})\mathcal{Z}_\omega(L(2;1)_\mathbf{e}) = 1 \ ,$$

Therefore, we can conclude

$$k_1 + k_2^v = 1 \mod 4 \ ,$$
$$k_1 = 1 \mod 2 \ ,$$
$$k_3^v = 0 \mod 4 \ ,$$

$$k_1 + k_2^p = 3 \mod 4 \ ,$$
$$k_1 = 1 \mod 2 \ ,$$
$$k_3^p = 0 \mod 4 \ .$$

Independence of k_1 on choice of rotation

Changing the rotation operator C_M can change $I_{g,M}$.

However, k_1 is unchanged.

e.g. $G = \mathbb{Z}_2$

Redefining $C'_M = U_{\mathbf{g}} C_M$ permutes

$$I_{\mathbf{g},4} \longleftrightarrow I_{\mathbf{e},4}$$

Leaving fixed the invariant

$$(-1)^{k_1} = I_{\mathbf{g},2} I_{\mathbf{g},4} I_{\mathbf{e},4}$$

Similarly, changing origin of rotations can change $I_{g,M}$ but not k_1.

Comparison

Success:

- Found an order parameter for 2D SPTs that is an expectation value of a finitely supported operator on a given (untwisted) wavefunction.
- Analogous to 1D order parameters that have been enormously useful.

But we pay a price:

- Require an additional crystalline symmetry on top of the on-site G.
- Can only determine SPT invariants modulo some divisor of M.

Captures lens space invariants but not T^3 invariants.

- In terms of flux insertions, captures bound charges but not bound proj. dofs.
- No overlap with membrane order, which is equivalent to T^3.

SPT Order in Mixed States

Strong and weak symmetries

Strong symmetry:

$$U_g \rho = \rho$$

Weak symmetry:

$$U_g \rho U_g^\dagger = \rho$$

For pure states, these notations coincide.

Preserved by strongly and weakly symmetric channels, respectively:

$$\begin{aligned} \text{strong}: &\quad \mathcal{E}^\dagger(U_g) = U_g \ , \quad i.e. \quad U_g K_i = K_i U_g \\ \text{weak}: &\quad \text{conj}(U_g) \circ \mathcal{E} \circ \text{conj}(U_g)^\dagger = \mathcal{E} \end{aligned}$$

[Buča, Prosen 2012] [Albert, Jiang 2014]

Mixed state SPT phases

Goal: study phases of mixed states protected by symmetry

$$G = E \times A \,,$$

where E is strong (exact) and A is weak (average).

Various notions of short-range entanglement (SRE):

- Has an SRE purification.
- Has an SRE Choi doubled state (a particular purification of ρ^2).
- Is a mixture of SRE pure states.

Modulo symmetric deformations.

- Low depth circuits of local channels.
- Local channels satisfy the appropriate strong and weak symmetry conditions.

Classification

In one dimension...

- strong symmetry $G = E$: $H^2(E, U(1))$
- weak symmetry $G = A$: trivial [de Groot, Turzillo, Schuch 2021]
- both types of symmetry $G = E \times A$: [Ma et al 2023] [Guo et al 2024]

$$H^2(E \times A, U(1))/H^2(A, U(1)) = H^2(E, U(1)) \times H^1(A, H^1(E, U(1)))$$

pure SPT classification, minus those protected solely by A
second term: A domain walls decorated by E charges

In D dimensions, phases protected by $E \times A$ *are believed to have* classification

$$H^{D+1}(E \times A, U(1))/H^{D+1}(A, U(1)) = \bigoplus_{p=0}^{D} H^p(A, H^{D+1-p}(E, U(1)))$$

[Ma et al 2023], [Ma, Turzillo 2024]
interpretation: junctions of A domain walls decorated by E SPT phases

Example: $\mathbb{Z}_2 \times \mathbb{Z}_2$ in 2D

2D $G = \mathbb{Z}_2 \times \mathbb{Z}_2$ SPT phase pure state classification:

$$H^3(\mathbb{Z}_2 \times \mathbb{Z}_2, U(1)) = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$$

a "CZX" for each $\mathbb{Z}_2$ symmetry, and a phase protected jointly by the $\mathbb{Z}_2$'s.
For generators a, b, these phases are detected by I_a, I_b and $I_{ab} I_a^* I_b^*$.

Apply noise that is strongly symmetric in one $\mathbb{Z}_2$ and weakly symmetric in the other.

Now we have symmetry $E = \mathbb{Z}_2$ and $A = \mathbb{Z}_2$.

The classification reduces to

$$\mathbb{Z}_2 \times \mathbb{Z}_2 \ .$$

The purely A SPT has been washed out by the noise.

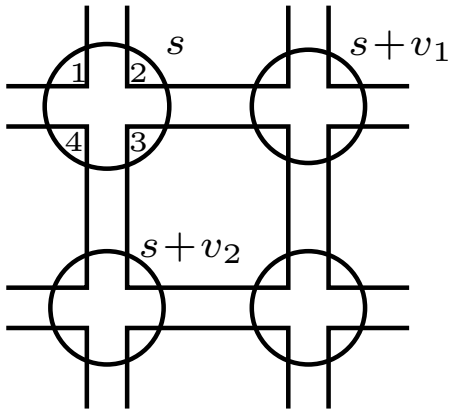
But the pure E and joint E, A protected SPT phases are robust.

Noisy CZX

Consider the CZX state

$$\rho = |\psi\rangle\langle\psi| = \sum_{\alpha,\beta} |\alpha\rangle\langle\beta| \ ,$$

where α and β are configurations of the plaquette labels.



Apply noise $K_0 = \frac{1}{\sqrt{2}}\mathbb{1}$ and $K_1 = \frac{1}{\sqrt{2}}Z$ to one subsite of each site s.

$$\mathcal{E}(\rho) = \sum_{\alpha,\beta} \frac{1}{2}\left(|\alpha\rangle\langle\beta| + Z|\alpha\rangle\langle\beta|Z\right) = \sum_{\alpha} |\alpha\rangle\langle\alpha|$$

This state now only has a weak $\mathbb{Z}_2$ symmetry.

Adapting the order parameter

Measure $I_{g,M}$ on the decohered CZX state.

Might expect to see that the invariant is trivial or ill-defined. But... the result is nonsense: the expectation value is nonzero at vertices, zero at plaquettes.

Must adapt the order parameter to states where the crystalline symmetry is weak.

Proposal:

$$J_{g,h,M} = \text{phase}\,\text{Tr}\left[U^{\mathcal{D}}_{gh} C^{\mathcal{D}}_{M} \sqrt{\rho} C^{\mathcal{D}\dagger}_{M} C^{\mathcal{D}\dagger}_{h} \sqrt{\rho}\right] .$$

Interpretation: J is a partial symmetry order parameter on the canonical purification:

$$J_{g,h,M} = \langle\langle \sqrt{\rho}|\, U^{\mathcal{D}}_{gh} C^{\mathcal{D}}_{M} \otimes U^{\mathcal{D}}_{h} C^{\mathcal{D}}_{M}\, |\sqrt{\rho}\rangle\rangle$$

Detection of mixed state SPT order

On the CZX state:

- before decoherence: $J_{\mathbf{g},\mathbf{e},2} = I_{\mathbf{g},2} = -1$. Nontrivial invariant.
- after decoherence: $J_{\mathbf{g},h,M} = 0$. Ill-defined invariant.

General result for $G = E \times A$ group cohomology models:

- before decoherence: $J_{g,h,M} = I_{gh,M} I^*_{h,M}$
- after decoherence: $J_{g,h,M} = I^{\text{pure}}_{gh,M} I^{\text{pure}*}_{h,M}$ if g is strong, and 0 otherwise.

Therefore we can recover E SPT invariant as

$$I^{\text{pure}}_{g,M} = J_{g,\mathbf{e},M}$$

and the joint E, A SPT invariants as

$$I^{\text{pure}}_{gh,M} I^{\text{pure}*}_{g,M} I^{\text{pure}*}_{h,M} = J_{g,h,M} J^*_{g,\mathbf{e},M} \ .$$

And, as expected, the A SPT invariant cannot be recovered.

Summary

Partial symmetry order parameters $J_{g,h,M}$ detect...

- mixed state SPT phases protected solely by strong symmetry
- mixed state SPT phases protected jointly by weak and strong symmetry

But they do not detect SPT phases protected solely by weak symmetry...

- consistent with the expectation that these phases are not robust.

Future Directions

Theory:

- Can we replace the C_M crystalline symmetry with a replica symmetry?
- Gauging: can we distinguish toric code from double semion?
- What can we say about phase transitions?
- Proof of robustness?
- Formal construction in infinite spin systems?

Quantum simulation:

- How to implement on a quantum simulator?
- Use to verify the simulation of exotic states such as intrinsically mixed topological phases.